BASIC TOPOLOGY IN $\mathbb{R}^n$.

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- Reminder: Definition of Euclidean norm $|\cdot|$ in $\mathbb{R}^n$ and triangle inequality.
- Definition of a ball (closed and open). Notation: $B(x,r)$ for the OPEN ball of radius r center x. The CLOSED ball is denoted by $\overline{B}(x,r)$.
- An open BOX in $\mathbb{R}^n$ is $Q = \prod_{i=1}^{n}(a_i, b_i)$. The corresponding closed box is $\overline{Q} = \prod_{i=1}^{n}[a_i, b_i]$.
- Definition: A BOUNDED set is a set that's contained in a ball centered at 0.
- If C is bounded, we define its DIAMETER $\delta(C)$ by

$$\delta(C) = \sup\{|x-y|, \quad x, y \in C\}.$$

- Definition: An OPEN set is a set such that each of its points has a closed/open ball (of positive radius) around it, contained in the set.
- Definition: A CLOSED set is such that its complement is an open set.
- Claim : The union of open sets is open, intersection of closed sets is closed.
- Claim: The intersection of finitely many open sets is open and the union of finitely many closed sets is closed.
- Definition: The CLOSURE of a set A (notation $\overline{A}$) is the minimal closed set containing A.
- Definition: The INTERIOR of a set A (notation $\mathring{A}$) is the union of all open balls contained in A.
- Definition: A point x is the limit of a sequence $\{x^{(k)}\}$ if every ball around x contains all $x^{(k)}$ for $k > K$, where K depends on the ball.
- PROPERTIES OF CONVERGENT SEQUENCES: (a) Component-wise convergence is equivalent to convergence (b) Cauchy sequences (c) Bolzano-Weierstrass theorem: Every bounded sequence contains a convergent subsequence.
- Claim : A is closed iff it contains the limits of all converging sequences (in A).
- Examples: The open ball (box) is open, closed ball (box) is closed, the set $\{(x,y), a < x < b, y < f(x)\}$ is open if $f(x)$ is continuous on (a,b)...
- Definition: A subset $B \subseteq A$ is DENSE (in A) if the closure of B CONTAINS A.
- Claim : B is dense in A if for every $x \in A$ and every $r > 0$, the ball $B(x,r)$ contains a point in B.
- Examples: The rationals are dense in an interval, points of the form $(m/2^n, p/2^q)$ are dense in $\mathbb{R}^2$ $(m, n, p, q$ integers).

- Definition: The BOUNDARY ∂A of A is the intersection of the closures of A and its complement, $\partial A = \overline{A} \cap \overline{A^c} = A \setminus \mathring{A}$.
- Conclusion: The boundary is always CLOSED.
- Conclusion: If A is open, $A \cap \partial A = \emptyset$.
- Definition: An open COVERING of a set A is a collection of open sets whose union contains A.
- A SUBSET of a given covering of A is a SUBCOVERING if it is by itself a covering.
- Definition: A set A is COMPACT if every open covering of A contains a finite subcovering (i.e., finitely many sets of the original covering are enough to cover).
- Theorem (BASIC): A is compact iff it is closed and bounded.
- Theorem (BASIC): A is compact iff it is bounded and contains all limits of converging sequences contained in A.
- CANTOR's LEMMA: If $\overline{Q_1} \supseteq \overline{Q_2} \supseteq ... \supseteq \overline{Q_j} \supseteq ..$ is a "shrinking" sequence of closed boxes such that $\delta(\overline{Q_j}) \to 0$ then the intersection $\bigcap\limits_{j=1}^{\infty} \overline{Q_j}$ consists of EXACTLY one point.
- Claim: If $F_1 \supseteq F_2 \supseteq ... \supseteq F_j \supseteq ..$ is a "shrinking" sequence of closed nonempty sets and if F_1 is bounded, then the intersection $\bigcap\limits_{j=1}^{\infty} F_j \neq \emptyset$.
- REMARK: The previous claim is sometimes called the "FINITE INTERSECTION PROPERTY" since, if the INFINITE intersection is EMPTY then already $F_j = \emptyset$ for some j (how is this related to compactness?).
- Definition: A is CONNECTED if it has the following property: If A is contained in the union of two open sets U, V such that the intersection $U \cap A$ is disjoint from the intersection $V \cap A$, then one of these intersections must be empty.
- Example: An (open or closed) interval on the line is connected.
- Example : The open ball in $\mathbb{R}^n$ is connected.

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