HARMONIC ANALYSIS 2016

EXERCISES VII (FUNDAMENTAL SOLUTION OF THE LAPLACIAN, HEAT EQUATION)

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1. BOOKS


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NOTATION

$B(y,R)$ The open ball (in $\mathbb{R}^n$) of radius $R$, centered at $y$.

$S(\mathbb{R}^n)$ is the Schwartz space of rapidly decaying smooth functions. The seminorms on $S$ are defined by

$$p_{\alpha,\beta}(f) = \max_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)|, \quad \alpha, \beta \in \mathbb{N}^n.$$ 

$S'(\mathbb{R}^n)$ is the space of continuous linear functionals on $S(\mathbb{R}^n)$. Such functionals are called tempered distributions.

If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, then $T_f \in S'(\mathbb{R}^n)$ is defined by $T_f(\phi) = \int_{\mathbb{R}^n} f(x)\phi(x)dx$, $\phi \in S(\mathbb{R}^n)$.

Same definition for $f \in L^1_{loc}(\mathbb{R}^n)$, if $f$ grows only polynomially when $|x| \to \infty$.

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x)e^{-ix\xi}dx.$$ 

For $T \in S'(\mathbb{R}^n)$, the Fourier transform is defined by

$$\mathcal{F}T(\phi) = T(\mathcal{F}\phi), \quad \phi \in S(\mathbb{R}^n).$$ 

The distribution $\delta_0$ is defined by $\delta_0(\phi) = \phi(0)$. Its Fourier transform:

$$\mathcal{F}\delta_0(\phi) = \delta_0(\hat{\phi}) = \hat{\phi}(0) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \phi(x)dx \Rightarrow \mathcal{F}\delta_0 = (2\pi)^{-\frac{n}{2}} T_1.$$ 

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(1) Let
\[ E(x) = \begin{cases} \frac{1}{2} \log |x|, & n = 2, \\ \frac{1}{n(n-2)\omega_n} |x|^{-(n-2)}, & n > 2, \end{cases} \]
where \( \omega_n \) is the (surface) volume of the unit sphere \( S^{n-1} \subseteq \mathbb{R}^n \).

It was proved in the lecture that \( \Delta T_E = \delta_0 \), hence for every \( \phi \in S(\mathbb{R}^n) \),
\[ \int_{\mathbb{R}^n} \Delta \phi(x) E(y-x) dx = \phi(y), \quad y \in \mathbb{R}^n. \]

For any \( \psi \in C^\infty_0(\mathbb{R}^n) \) define the “Newtonian potential”
\[ \Psi(y) = \int_{\mathbb{R}^n} \psi(x) E(y-x) dx. \]

Prove that \( \Delta \Psi(y) = \psi(y), \quad y \in \mathbb{R}^n. \)

**Hint:** Write \( \Psi(y) = \int_{\mathbb{R}^n} \Delta \psi(y-x) E(x) dx, \) and differentiate once under the integral. To differentiate again you need to exclude a small ball around \( y \). Or see (e), Sec.2.2.

(2) Use the facts \( \Delta T_E = \delta_0 \), and \( F \delta_0 = (2\pi)^{-\frac{n}{2}} T_1 \) to prove that
\[ -|\xi|^2 F(T_E) = (2\pi)^{-\frac{n}{2}} T_1. \]

What is the meaning of this equality (apply it to a function in \( S(\mathbb{R}^n) \))?

Prove that \( F(T_E) = -(2\pi)^{-\frac{n}{2}} T_1 \frac{|\xi|^2}{n\pi}. \)

(3) Let \( f \in L^1(\mathbb{R}^n) \) be radially symmetric, i.e., \( f \) depends only on \( |x| \). Prove that \( \hat{f}(\xi) \) is radially symmetric, i.e., depends only on \( |\xi| \).

(4) Let \( G(x) \) be the characteristic function of the ball \( B(0,R) \), i.e.
\[ G(x) = \begin{cases} 1, & |x| < R, \\ 0, & |x| > R. \end{cases} \]

(a) Compute the partial derivatives \( \frac{\partial}{\partial x_j} T_G, \quad j = 1, 2, ... \)

(b) Compute \( \Delta T_G \).

(5) In this problem \( n = 1 \).

Let \( f(\xi) \in C^2_0(\mathbb{R}) \), i.e., twice continuously differentiable and compactly supported. Show that there exists a function \( g(x) \in L^1(\mathbb{R}) \) such that \( \hat{g}(\xi) = \hat{f}(\xi), \quad \xi \in \mathbb{R}. \)

(See [Ka], Ch. VI, Exercises for Section 1, p. 129).

(6) Let \( f, g, h \in L^1(\mathbb{R}^n) \). Prove that the convolution is associative:
\[ (f * g) * h = f * (g * h). \]

(7) (Heat Equation in one dimension) The problem is the following: Given a function \( u_0(x) \in C_0^\infty(\mathbb{R}) \), find a function \( u(x,t) \in C^\infty(\mathbb{R} \times (0,\infty)) \), so that
- \( u(x,t) \) satisfies the equation
\[ \frac{\partial}{\partial t} u(x,t) = \left( \frac{\partial}{\partial x} \right)^2 u(x,t), \quad x \in \mathbb{R}, t > 0. \]
- \[ \lim_{t \to 0^+} u(x,t) = u_0(x). \]
Show that the function
\[
u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} u_0(y)e^{-\frac{(x-y)^2}{4t}} dy, \quad x \in \mathbb{R}, \ t > 0,
\]
is indeed a solution. Use two methods:
• Direct computation (yes, differentiation under the integral sign...).
• Use Fourier transform to rewrite the equation as
\[
\frac{\partial}{\partial t} \hat{u}(\xi, t) = -|\xi|^2 \hat{u}(\xi, t), \quad \xi \in \mathbb{R}, \ t > 0,
\]
where \( \hat{u}(\xi, t) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} u(x, t)e^{-i\xi x} dx \) is the Fourier transform with respect to \( x \in \mathbb{R} \).

See [W], Chapter X, Section 70.

(8) Prove that the family
\[
\left\{ \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \right\}_{t > 0},
\]
is a positive summability kernel.

**Definition:** This kernel is called the heat kernel (in one space dimension).

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