HARMONIC ANALYSIS 2016

EXERCISES IV (FOURIER TRANSFORM AND SCHWARTZ SPACE $S(\mathbb{R}^n)$)

MATANIA BEN-ARTZI

1. BOOKS

[Co] R. Courant, Differential and Integral Calculus, Vol. I ,Blackie and Son, 1934.

[Ka] Y. Katznelson, An Introduction to Harmonic Analysis, Wiley, 1968.

[Ru] W. Rudin, Real and Complex Analysis, McGraw-Hill Co. 1966.

[W] H.F.Weinberger, A First Course in Partial Differential Equations, Wiley Publ. 1965.

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Notation

$B(y, R)$ The open ball (in $\mathbb{R}^n$) of radius R, centered at y.
$S(\mathbb{R}^n)$ is the Schwartz space of rapidly decaying smooth functions. The seminorms on S are defined by

$$p_{\alpha,\beta}(f) = \max_{x\in\mathbb{R}^n} |x^\alpha D^\beta f(x)|, \quad \alpha, \beta \in \mathbb{N}^n.$$

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x)e^{-i\xi x}dx.$$

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(1) Prove that $\mathcal{F} : S(\mathbb{R}^n) \to S(\mathbb{R}^n)$ is continuous.

(2) (In this problem $n = 1$) Let $f \in S(\mathbb{R})$ and define

$$g(t) = \sum_{j=-\infty}^{\infty} f(t + 2\pi j).$$

 (a) Show that the sum converges absolutely (so the order of summation is not important).

 (b) Show that $g \in C^\infty(S^1)$, namely, it is $2\pi-$periodic and continuously differentiable of any order.

Date: January 6, 2017.

(c) Let the Fourier coefficients of g be given by

$$\widehat{g}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} g(t) e^{-ik\theta} d\theta, \quad -\infty < k < \infty.$$

Prove that

$$\widehat{g}(k) = \widehat{f}(k) = \mathcal{F}f(k), \quad -\infty < k < \infty.$$

(d) Prove **Poisson's Summation Formula:**

$$\sum_{j=-\infty}^{\infty} \widehat{f}(j) = \sqrt{2\pi} \sum_{j=-\infty}^{\infty} f(2\pi j).$$

(See [Ka], Ch. VI. Sec. 1.15)

(3) In this problem $n = 1$.
  (a) Find the Fourier transform of $f(x) = \frac{x}{x^2+1}$.
  (b) Find the Fourier transform of $f(x) = \frac{\sin x}{x}$.
  Use your result in order to evaluate $\int\limits_{\mathbb{R}} \frac{\sin^2 x}{x^2} dx$.

(See [W], Sec. 66.)

(4) In this problem $n = 1$.
  (a) Find the function $f(x)$ if $\hat{f}(\xi) = \frac{\sin a\xi}{\xi}$.
  (b) Find the function $f(x)$ if $\hat{f}(\xi) = \frac{1-\cos a\xi}{\xi}$.
  (c) Find the function $f(x)$ if $\hat{f}(\xi) = \frac{\xi}{\xi^2+1}$.
  (d) Find the function $f(x)$ if $\hat{f}(\xi) = \exp(-\xi^2)$.
  (See [W], Sec. 67.)

(5) In this problem $n = 1$.
  Define the Féjer kernel on the real line by

$$K(x) = \frac{1}{2\pi}\Big(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\Big)^2, \quad x \in \mathbb{R}.$$

  (a) Show that the family $\{K_\varepsilon\}_{\varepsilon>0}$ is a positive summability kernel, where $K_\varepsilon(x) = \varepsilon^{-1}K(\frac{x}{\varepsilon})$.
  (Compare Problem 3(b) above).
  (b) Prove that for every $f \in L^1(\mathbb{R})$,

$$f = \lim_{\varepsilon\to 0} K_\varepsilon * f, \quad \text{in the topology of } L^1(\mathbb{R}).$$

  (c) Prove that

$$K(x) = \frac{1}{2\pi}\int_{-1}^{1}(1-|\xi|)e^{i\xi x}d\xi,$$

  and use it to find the Fourier transform $\widehat{K}(\xi)$.
  (d) Prove that for every $f \in L^1(\mathbb{R})$,

$$f(x) = \lim_{\varepsilon\to 0}\frac{1}{\sqrt{2\pi}}\int_{-\varepsilon^{-1}}^{\varepsilon^{-1}}(1-\varepsilon|\xi|)\widehat{f}(\xi)e^{i\xi x}d\xi, \quad \text{in the topology of } L^1(\mathbb{R}).$$

(See [Ka] ,Chapter VI, Sec. 1.11).

(6) Let $p(x)$ be a polynomial in $\mathbb{R}^n$ (with complex coefficients). What is $\mathcal{F}(p(x)\exp(-\frac{1}{2}|x|^2))$?

(7) In this problem $n = 1$.

Let the operator $H : S(\mathbb{R}) \to S(\mathbb{R})$ be defined by

$$H\phi = \Big(-\frac{d^2}{dx^2} + x^2 \Big)\phi(x).$$

(a) Prove that if $\psi_\lambda \in S$ is an eigenfunction of H with eigenvalue λ, then $\psi_{\lambda+2} := \Big(x - \frac{d}{dx}\Big)\psi_\lambda$ is an eigenfunction of H with eigenvalue $\lambda + 2$.

(b) Let

$$H_n(x) = e^{\frac{1}{2}x^2}\Big(x - \frac{d}{dx}\Big)^n e^{-\frac{1}{2}x^2}.$$

Prove that $H_n(x)$ is a polynomial of degree n, $n = 0, 1, 2...$ (called *Hermite polynomial*) and

$$H_n(x) = e^{x^2}(-1)^n\Big(\frac{d}{dx}\Big)^n e^{-x^2}.$$

(c) Prove that $\psi_{2n+1} = H_n(x)e^{-\frac{1}{2}x^2}$ is an eigenfunction of H, with eigenvalue $2n + 1$, $n = 0, 1, 2, ...$

(d) Prove the orthogonality

$$\int_{\mathbb{R}} H_n(x)H_m(x)e^{-x^2}dx = 0, \quad n \neq m.$$

(e) Prove that , after normalization, the family $\{\psi_{2n+1}(x),\ n = 0, 1, 2...\}$ is a basis for $L^2(\mathbb{R})$.

(8) Suppose A is an invertible linear operator on $\mathbb{R}^n$, $f \in L^1(\mathbb{R}^n)$ and $g(x) = f(Ax)$. Express $\hat{g}$ in terms of $\hat{f}$.

(See [Ru] , Exercise 1, Ch. 7).

Institute of Mathematics, Hebrew University, Jerusalem 91904, Israel
E-mail address: mbartzi@math.huji.ac.il