INTERNET PARTIAL DIFFERENTIAL EQUATIONS

EXERCISES I (HARMONIC FUNCTIONS)

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1. BOOKS


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Notation

\( \Omega \subseteq \mathbb{R}^n \) \hspace{1cm} A bounded open domain with smooth boundary.

\( G_\Omega \) \hspace{1cm} Green’s function of \( \Omega \).

\( B_R(y) \) \hspace{1cm} The open ball (in \( \mathbb{R}^n \)) of radius \( R \), centered at \( y \).

\( \frac{\partial w}{\partial \nu} \) \hspace{1cm} The outward normal derivative of \( w \) at \( \partial \Omega \).

\( dS \) \hspace{1cm} The (Lebesgue) surface measure on \( \partial \Omega \).

\( \omega_n \) \hspace{1cm} The volume (Lebesgue measure) of \( B_1(0) \).

\( n\omega_n \) \hspace{1cm} The surface (Lebesgue) measure of \( \partial B_1(0) \).

\( |x| \) \hspace{1cm} The Euclidean norm in \( \mathbb{R}^n \).

\( \Gamma(x) = \begin{cases} \frac{1}{(n(2-n)\omega_n)|x|^{2-n}} & \text{if } n > 2, \\ \frac{1}{2\pi} & \text{if } n = 2. \end{cases} \)
(1) (a) Prove that in spherical coordinates 
(defined by \((x, y, z) = (r \cos \varphi \sin \theta, r \sin \varphi \cos \theta, r \cos \theta)\) 
The Laplacian is given by,
\[
\Delta u = \frac{1}{r^2 \sin \theta} \left( \frac{\partial}{\partial r} \left( r^2 \sin \theta \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \varphi} \frac{\partial u}{\partial \varphi} \right)
\]
(b) Show that, up to an additive constant, a radially symmetric harmonic 
function is given by \(c \cdot \Gamma(x)\).

(2) Describe the solution of Dirichlet’s problem in a rectangle and in a cube by 
means of separation of variables. 
(see [W], Ch. IV, Sec. 23, p. 95 and Ch. VI, Sec. 32, p. 146). 
In particular, construct Green’s function for a rectangle. 
(see [W], Ch. V, Sec. 30, p. 137).

(3) Let \(u\) be harmonic in the open disk (i.e., \(n = 2\)) \(B_R(0)\) and continuous in 
its closure. Let \(f(\theta)\) be its boundary value (polar coordinates \(r, \theta\)). Prove 
Poisson’s formula
\[
\frac{R^2 - r^2}{2 \pi} \int_{-\pi}^{\pi} \frac{f(\phi)}{r^2 + R^2 - 2Rr \cos(\theta - \phi)} \, d\phi,
\]
valid for \(r < R\). Compare this result with the derivation by Fourier series, 
using separation of variables  
(see [W], Ch. IV, Sec. 24).

(4) (Poisson’s formula in a half-plane). Let \(D = \{(x, y) \in \mathbb{R}^2, \ y > 0\}\) and let 
g \in \mathcal{C}^0(\mathbb{R}) \cap L^\infty(\mathbb{R})\. Define in \(D\) the function \(f(x, y)\) by 
\[
f(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{yg(t)}{(t-x)^2 + y^2} \, dt, \quad (x, y) \in D.
\]
Prove that: 
(a) \(f(x, y)\) is harmonic in \(D\) and can be extended continuously to \(\overline{D}\), so 
that \(f(x, 0) = g(x)\) for \(x \in \mathbb{R}\).
(b) 
\[
\sup\{f(x, y), \ (x, y) \in D\} = \sup\{g(x), \ x \in \mathbb{R}\},
\]
\[
\inf\{f(x, y), \ (x, y) \in D\} = \inf\{g(x), \ x \in \mathbb{R}\}.
\]

(5) Let \(\Delta u = f\) in \(\Omega\). Show that the Kelvin transform of \(u\), defined by 
\[
v(x) = |x|^{2-n}u(x/|x|^2) \quad \text{for } x/|x|^2 \in \Omega
\]
satisfies 
\[
\Delta v(x) = |x|^{-n-2}f(x/|x|^2).
\]
In particular, note that \(v\) is harmonic (where?) if \(u\) is harmonic. 
(see [GT], Problem 4.7, p. 67).

(6) Prove the following stronger form of the "Mean Value Property": 
Suppose that the Dirichlet problem can be solved in \(\Omega\) for every continuous 
(given) boundary function. Let \(u\) be continuous in \(\overline{\Omega}\) and assume that for 
every interior point \(x \in \Omega\) there exists \(\eta > 0\) such that \(B_\eta(x) \subseteq \Omega\) and \(u(x)\)
is equal to the mean value of $u$ on $\partial B_r(x)$. Then $u$ is harmonic in $\Omega$.
(see [CH], Ch. IV, Sec. 3, p.279).

(7) (Schwarz reflection principle) Let $u(x_1, x_2, ..., x_n)$ be continuous in the closed half-space $x_1 \geq 0$ and harmonic in its interior $x_1 > 0$. Assume further that $u$ vanishes on the boundary $x_1 = 0$. Extend $u$ to the lower half-space by

$$u(x_1, x_2, ..., x_n) = u(-x_1, x_2, ..., x_n) \quad \text{for } x_1 < 0.$$  

Prove that the extended function is harmonic in the whole space $\mathbb{R}^n$. Generalize to the case that $u$ is harmonic only in a subdomain (with part of its boundary in $x_1 = 0$) of the upper half-space.
(see [GT], Problem 2.4, p. 28).

(8) (Harnack’s Inequality) Let $u(x) \geq 0$ be harmonic in $B_R(0)$ and continuous in the closure of the ball. Prove that, for every $y \in B_R(0)$,

$$\frac{R^{n-2}(R - |y|)}{(R + |y|)^{n-1}} u(0) \leq u(y) \leq \frac{R^{n-2}(R + |y|)}{(R - |y|)^{n-1}} u(0).$$

(see [J], Sec. 4.3, p. 111).

(9) (Liouville’s Theorem) Prove that a harmonic function defined in $\mathbb{R}^n$ and bounded above is constant.
(see [GT], Problem 2.14, p. 29).

(b) Prove that a function $u$ which is harmonic in the open half-space $x_1 > 0$, continuous in its closure and vanishes on the boundary $x_1 = 0$, is identically zero if in addition it satisfies the growth condition $u(x) = O(|x|^\alpha)$ as $|x| \to \infty$ for some exponent $\alpha < 1$.

(10) (Uniqueness by the energy method)

(a) Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Prove the identity

$$\int_{\Omega} u \Delta u \, dx = \int_{\partial \Omega} u \frac{\partial u}{\partial \nu} \, dS - \int_{\Omega} |\nabla u|^2 \, dx.$$

(b) Let $u$ be harmonic in $\Omega$ and assume that

$$u = 0 \quad \text{on } \Gamma_1, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_2,$$

where $\Gamma_1 \cup \Gamma_2 = \partial \Omega$, and $\Gamma_1$ has positive ($dS$) measure. Prove that $u \equiv 0$.
(see [W], Ch. III, Sec. 11, p. 52).

(11) (Dirichlet’s Principle) For every $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ define the ”Dirichlet form” $D(u)$ by

$$D(u) = \int_{\Omega} |\nabla u|^2 \, dx.$$  

Let $\varphi \in C^0(\partial \Omega)$ and let $A \subseteq C^2(\Omega) \cap C^1(\overline{\Omega})$ consist of all $u$ such that $u = \varphi$ on $\partial \Omega$.

(a) Prove that if $u \in A$ is harmonic then

$$D(u) \leq D(v), \quad \text{for all } v \in A.$$  

(b) Conversely, prove that if $u \in A$ satisfies (*), then it is harmonic.
(see [E], Sec. 2.2, p.42).
Let \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) satisfy the equation \( \Delta u - u^2 = 0 \). Show that \( u \) cannot attain its maximum at an interior point unless \( u \equiv 0 \).

Let \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) satisfy the equation \( \Delta u - u^3 = 0 \). Show that if \( u \equiv 0 \) on the boundary \( \partial \Omega \) then \( u \equiv 0 \) in \( \Omega \).

Consider the function \( u(x, y) = \frac{1 - (x^2 + y^2)}{(1 - x^2)(1 + y^2)} \). Show that it is harmonic and positive in the unit disk \( \{ x^2 + y^2 < 1 \} \) and vanishes on its boundary except for one point. Is it a contradiction to the maximum principle?

We assume here that \( \Omega \subseteq \mathbb{R}^2 \). Let \( u \) be harmonic in \( \Omega \) and continuous in \( \overline{\Omega} \) except possibly at a point \( (x_0, y_0) \in \partial \Omega \). Let \( R > 0 \) be such that the disk of radius \( R \) centered at \( (x_0, y_0) \) contains \( \overline{\Omega} \). Suppose that:

(i) \( u \leq M \) on \( \partial \Omega \setminus (x_0, y_0) \).

(ii) \( \frac{2R^2}{u(x, y)/\log (x - x_0)^2 + (y - y_0)^2} \to 0, \quad \text{as} \quad (x, y) \to (x_0, y_0). \)

Show that \( u \leq M \) in \( \Omega \).

(See [W], Ch. IV, Sec. 25, p.112).

(a) (The Hadamard Three-Circle Theorem) Here \( \Omega \subseteq \mathbb{R}^2 \) is an annular domain given by

\[
\Omega = \{ (x, y), \quad 0 < a^2 < x^2 + y^2 < b^2, \}. 
\]

Let \( u \in C^2(\Omega) \) be subharmonic, i.e., \( \Delta u \geq 0 \) and denote

\[
M(r) = \max_{x^2 + y^2 = r^2} u(x, y), \quad a < r < b. 
\]

Prove that \( M(r) \) is a convex function of \( \log r \), i.e., that for \( a < r_1 < r < r_2 < b \),

\[
M(r) \leq \frac{M(r_1) \log(r_2/r) + M(r_2) \log(r/r_1)}{\log(r_2/r_1)}. 
\]

(b) (The Hadamard Three-Sphere Theorem) Here \( \Omega \subseteq \mathbb{R}^n, \quad n \geq 3 \), is a "spherical shell" between concentric spheres given by

\[
\Omega = \{ x \in \mathbb{R}^n, \quad 0 < a < |x| < b, \}. 
\]

Let \( u \in C^2(\Omega) \) be subharmonic, i.e., \( \Delta u \geq 0 \) and denote

\[
M(r) = \max_{|x|=r} u(x), \quad a < r < b. 
\]

Prove that for \( a < r_1 < r < r_2 < b \),

\[
M(r) \leq \frac{M(r_1)(r_2^n - r_2^{-n}) + M(r_2)(r_1^n - r_2^{-n})}{r_1^n - r_2^{-n}}. 
\]

(c) (Liouville’s Theorem –stronger version in \( \mathbb{R}^2 \)). Let \( u \in C^2(\mathbb{R}^2 \setminus (0, 0)) \) be subharmonic and uniformly bounded above. Then \( u \) is a constant.

(d) Consider the radial function \( u \) in \( \mathbb{R}^3 \) given by,

\[
u(r) = \begin{cases} 
-\frac{1}{6}(15 - 10r^2 + 3r^4), & \text{for } r \leq 1, \\
-\frac{1}{7}, & \text{for } r \geq 1. 
\end{cases}
\]

Show that it is subharmonic in all of \( \mathbb{R}^3 \) and is uniformly bounded. Thus the theorem of the previous part cannot be generalized to higher
dimensions.

(See [PW], Ch. 2, Sec. 12, p. 128).

(17) Show that if $u$ is harmonic in $\Omega$ then it is analytic there.

(see [E], Sec. 2.2, p. 31).

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