BASIC CONCEPTS IN ANALYSIS

EXERCISES III (FUNDAMENTAL SOLUTIONS AND PARTIAL DIFFERENTIAL EQUATIONS)

MATANIA BEN-ARTZI

1. BOOKS

[Ru] W. Rudin, Functional Analysis, McGraw-Hill Co. 1973.

[E] L. C. Evans, Partial Differential Equations, Graduate Studies in Mathematics Vol. 19, AMS 1998.

[Ka] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag 1976.

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Notation

$|x|$ The Euclidean norm in $\mathbb{R}^n$.

$$\Gamma(x) = \begin{cases} \frac{1}{n(2-n)\omega_n}|x|^{2-n} & \text{if } n > 2, \\ \frac{1}{2\pi}\log(|x|), & \text{if } n = 2. \end{cases}$$

$D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$ $\qquad D = (D_1, ..., D_n)$.

$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ for every multi-index $\alpha = (\alpha_1, ..., \alpha_n)$.

$|\alpha| = \alpha_1 + \cdots \alpha_n$ for every multi-index $\alpha = (\alpha_1, ..., \alpha_n)$.

$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} f(x)e^{-i\xi x}dx$ The Fourier transform of f.

$S = S(\mathbb{R}^n)$ The Schwartz space of smooth rapidly decaying functions.

$S' = S'(\mathbb{R}^n)$ The space of **tempered distributions**, i.e., continuous linear functionals on S.

$\mathbb{D}(\mathbb{R}^n)$ The space of smooth compactly supported functions.

$\mathbb{D}'(\mathbb{R}^n)$ The space of **distributions**, i.e., continuous linear functionals on $\mathbb{D}(\mathbb{R}^n)$.

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(1) (a) Show that $\Gamma(x)$ (as defined in the "Notations") is a fundamental solution of Δ in $\mathbb{R}^n$. (See [E], Sec. 2.2).
   (b) Let $f \in C^2(\mathbb{R}^n)$ (i.e., $D^\alpha f$ is continuous if $|\alpha \leq 2)$. Assume further that f is compactly supported. Prove that $u = \Gamma \star f \in C^2(\mathbb{R}^n)$ and that $\Delta u = f$.
   Explain why this fact does not follow immediately from part (a) of the problem.

(2) (a) Let $\psi \in \mathbb{D}(\mathbb{R}^n)$ such that $\psi = 1$ in a neighborhood of 0. Prove that $\Delta(\psi\Gamma) - \delta_0 \in \mathbb{D}(\mathbb{R}^n)$, where Γ is the fundamental solution of δ. Use this to show that $\Delta(\psi\Gamma) \star u$ can be defined for any $u \in \mathbb{D}'(\mathbb{R}^n)$.
   (Remark: More generally, the convolution of two distributions in $\mathbb{D}'(\mathbb{R}^n)$ can be defined if one of them is compactly supported).
   (b) Show that if $u \in \mathbb{D}'(\mathbb{R}^n)$ and $D_j u \in L^2_{loc}(\mathbb{R}^n), \quad j = 1, \cdots, n$ then we have the "integration by parts" formula

$$\Delta(\psi\Gamma) \star u = -\sum_{j=1}^{n} D_j(\psi\Gamma) \star (D_j u).$$

   (Suggestion: While such a formula can be established in general, when one distribution is compactly supported, you can use here the fact that each $D_j(\psi\Gamma)$ is in L^1, and then approximate u by "nice" functions).
   (c) Prove that if $u \in \mathbb{D}'(\mathbb{R}^n)$ is such that $D_j u \in L^2_{loc}(\mathbb{R}^n), \quad j = 1, \cdots, n$ then $u \in L^2_{loc}(\mathbb{R}^n)$.
   (See [Ru], Ch. 8, Exercise 11).
   (d) Prove that if $u \in \mathbb{D}'(\mathbb{R}^n)$ is such that Δu (defined as a distribution) is a continuous function, then also u is a continuous function.
   (See [Ru], Ch. 8, Exercise 12).

(3) In this problem $n = 3$.

   (a) Let $f \in \mathbb{D}'(\mathbb{R}^3)$ be such that $\hat{f}(\xi) = (2\pi)^{-\frac{3}{2}}(1+\xi^2)^{-1}$. Show that in fact $f \in L^2(\mathbb{R}^3)$ and that it is a fundamental solution of $1 - \Delta$.
   (b) Show that $f(x) = (\frac{\pi}{2})^{\frac{1}{2}}|x|^{-1}\exp(-|x|)$.
   (See [Ru], Ch. 8, Exercise 5).
   (c) Find a fundamental solution for $k^2 - \Delta$ for any $k \in \mathbb{R}$.

(4) In this problem $n = 1$.

   (a) Suppose $P(D) = D^2 + aD + b$, where a, b are complex numbers. Let $f, g \in C^2(\mathbb{R})$ such that $P(D)f = P(D)g = 0$, and

$$g(0) - f(0) = 0, \qquad g'(0) - f'(0) = 1.$$

   Prove that

$$G(x) = \begin{cases} g(x) & \text{if } x > 0, \\ f(x), & \text{if } x \leq 0, \end{cases}$$

   is a fundamental solution of $P(D)$.
   (See [Ru], Ch. 8, Exercise 10).

(b) Let $k \in \mathbb{R}$. Write explicitly (in terms of a convolution kernel) the solution to $y'' + ky = \varphi$ where $\varphi \in \mathbb{D}(\mathbb{R})$.
Is this solution unique? Can you write an "optimal" one?

(5) (Fundamental solution of the heat operator).
Let

$$G(x,t) = (4\pi t)^{-\frac{n}{2}} \exp(-\frac{|x|^2}{4t}), \qquad x \in \mathbb{R}^n, \quad t > 0.$$

(a) Show that the family $\{G(\cdot,t)\}_{t>0}$ is a positive summability kernel.
(b) Define

$$\Phi(x,t) = \begin{cases} G(x,t) & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

Prove that Φ is a fundamental solution for the operator $\frac{\partial}{\partial t} - \Delta$ in $\mathbb{R}^{n+1}$ (where Δ is the Laplacian with respect to the $x-$coordinates).
(See [E], Sec. 2.3).
(c) Is $\Phi \in S'(\mathbb{R}^{n+1})$?
(d) Let $\psi \in \mathbb{D}(\mathbb{R}^{n+1})$. Show that $u(x,t) = \Phi \star \psi$ is a smooth function satisfying $(\frac{\partial}{\partial t} - \Delta)u = \psi$ and such that for some $\tau \in \mathbb{R}$ we have $u(x,t) = 0$ if $t < \tau$.
(e) (Solution of the initial-value problem for the heat equation).
Let $g(y)$ be a bounded continuous function in $\mathbb{R}^n$. Define the function

$$v(x,t) = G(\cdot,t) \star g = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp(-\frac{|x-y|^2}{4t}) g(y)dy, \qquad x \in \mathbb{R}^n, \quad t > 0.$$

Show that
(i) $v \in C^\infty(\mathbb{R}^n \times (0,\infty))$.
(ii) $(\frac{\partial}{\partial t} - \Delta)v = 0$ in $\mathbb{R}^n \times (0,\infty)$.
(iii)

$$\lim_{\substack{(x,t)\to(x_0,0)\\ x\in\mathbb{R}^n, t>0}} v(x,t) = g(x_0), \qquad \text{for each point } x_0 \text{ in } \mathbb{R}^n.$$

(See [E], Theorem 1, Sec. 2.3).

(6) (Fundamental solution of the free Schrödinger operator). In this problem $n = 1$.
Let

$$K(x,t) = (4i\pi t)^{-\frac{1}{2}} \exp(-\frac{|x|^2}{4it}), \qquad x \in \mathbb{R}, \quad t > 0.$$

(a) Show that the family

$$\mathcal{P} = \{ \exp[-b(x-a)^2], \qquad b > 0, \quad a \in \mathbb{R} \},$$

is dense in $L^2(\mathbb{R})$.
(b) Show that for every fixed $t > 0$ the convolution operator, which is first defined on elements of $\mathcal{P}$,

$$\mathbb{K}_t g(x) = \int_{\mathbb{R}} K(x-y,t) g(y)dy, \qquad x \in \mathbb{R},$$

can be extended to a unitary operator on $L^2(\mathbb{R})$.
(See [Ka], Sec. IX.1.8, also for the next parts).

(c) Define

$$\Psi(x,t) = \begin{cases} K(x,t) & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

Prove that Ψ is a fundamental solution for the free Schrödinger operator $\frac{1}{i}\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$.

(d) Is $\Psi \in S'(\mathbb{R}^2)$?

(e) Let $\theta \in \mathbb{D}(\mathbb{R}^2)$. Show that $u(x,t) = \Psi \star \theta$ is a smooth function satisfying $(\frac{1}{i}\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2})u = \theta$ and such that for some $\tau \in \mathbb{R}$ we have $u(x,t) = 0$ if $t < \tau$.

(f) (Solution of the initial-value problem for the free Schrödinger equation).
Let $g(y) \in \mathbb{D}(\mathbb{R})$. Define the function

$$v(x,t) = \mathbb{K}_t g(x), \qquad x \in \mathbb{R}, \quad t > 0.$$

Show that
 (i) $v \in C^\infty(\mathbb{R}^n \times (0,\infty))$.
 (ii) $(\frac{1}{i}\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2})v = 0$ in $\mathbb{R}^n \times (0,\infty))$.
 (iii)

$$\lim_{\substack{(x,t)\to(x_0,0) \\ x\in\mathbb{R}, t>0}} v(x,t) = g(x_0), \qquad \text{for each point } x_0 \text{ in } \mathbb{R}.$$

(g) Try to generalize the above results to $\mathbb{R}^n$.

(7) (The Wave Equation).

(a) Let $f(x,t), g(x,t)$ be two locally integrable functions in $\mathbb{R}^2$, and let $c \in \mathbb{R}$. Show that the function $u(x,t) = f(x+ct)+g(x-ct)$, considered as an element of $\mathbb{D}'(\mathbb{R}^2)$, satisfies the (homogeneous) wave equation

$$(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2})u = 0.$$

(b) Define the function

$$\Sigma(x,t) = \begin{cases} -\frac{1}{2} & \text{if } t > 0 \text{ and } -ct < x < ct, \\ 0, & \text{if } t \leq 0. \end{cases}$$

Prove that Σ is a fundamental solution for the wave operator (in one space dimension) $L(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}) = \frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}$.
(Note that Σ is supported in the "forward light cone").

(c) Let $\phi \in \mathbb{D}(\mathbb{R}^2)$ and suppose that *supp* $\phi \subseteq \mathbb{R} \times (t_0, \infty)$. Show that there exists a solution u to the wave equation $Lu = \phi$ that "propagates only into the future", namely, *supp* $u \subseteq \mathbb{R} \times (t_0, \infty)$.

(d) Furthermore, show that the solution u from the previous part is compactly supported (in x) at every fixed time level t.
(We say that it satisfies the "finite propagation speed" property).

(e) Formulate similar results for the "backward light cone".

Institute of Mathematics, Hebrew University, Jerusalem 91904, Israel
E-mail address: `mbartzi@math.huji.ac.il`