BASIC CONCEPTS IN ANALYSIS

EXERCISES II (HILBERT TRANSFORM)

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1. BOOKS

[Ru] W. Rudin, Real and Complex Analysis, McGraw-Hill Co. 1966.

[Ne] U. Neri, Singular Integrals, Springer-Verlag 1971.

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Notation

$|x|$ The Euclidean norm in $\mathbb{R}^n$.

$D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$ $\quad D = (D_1, ..., D_n)$.

$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ for every multi-index $\alpha = (\alpha_1, ..., \alpha_n)$.

$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} f(x)e^{-i\xi x}dx$ The Fourier transform of f.

$S = S(\mathbb{R}^n)$ The Schwartz space of smooth rapidly decaying functions.

$S' = S'(\mathbb{R}^n)$ The space of **tempered distributions**, i.e., continuous linear functionals on S.

$T_f \in S'$ For a function f of polynomial growth

$$T_f(\varphi) = \int_{\mathbb{R}^n} f(x)\varphi(x)dx, \qquad \varphi \in S.$$

$Jf(x) = \check{f}(x) = f(-x)$.

$\mathbb{D}(\mathbb{R}^n)$ The space of smooth compactly supported functions.

$\mathbb{D}'(\mathbb{R}^n)$ The space of **distributions**, i.e., continuous linear functionals on $\mathbb{D}(\mathbb{R}^n)$.

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(1) Prove the following theorem that deals with bounded kernels decaying at infinity.

Theorem. *Suppose that $K(x) \in L^\infty(\mathbb{R}^n)$ and that $(1+|x|^{n+1})|K(x)| \le C$ for some constant $C > 0$. Suppose also that $\int_{\mathbb{R}^n} K(x)dx = 1$ and denote $K_\varepsilon(x) = \varepsilon^{-n}K(\frac{x}{\varepsilon})$. Then, for any $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$,*

$$f \star K_\varepsilon(x) \xrightarrow[\varepsilon \to 0^+]{} f(x), \qquad \textit{for a.e.} \quad x \in \mathbb{R}^n.$$

This is Theorem 6, Chapter I of [Ne]. Here are the steps in the proof.

Proof. (a) Note that $\int_{\mathbb{R}^n} K_\varepsilon(x)dx = 1$ and $|K_\varepsilon(x)| \le \frac{C\varepsilon}{\varepsilon^{n+1}+|x|^{n+1}}$.

(b) Estimate (with $f_\varepsilon = f \star K_\varepsilon$,)

$$|f_\varepsilon(x) - f(x)| \le \int_{\mathbb{R}}^n |f(x-y) - f(x)||K_\varepsilon(y)|dy = \int_{|y|\le\varepsilon} + \int_{|y|\ge\varepsilon} = I_1 + I_2.$$

(c) Note that if x is a Lebesgue point of f,

$$I_1 \le \frac{C}{\varepsilon^n}\int_{|y|\le\varepsilon} |f(x-y) - f(x)|dy \xrightarrow[\varepsilon \to 0^+]{} 0.$$

(d) Estimate

$$I_2 \le C\varepsilon \int_{|y|\ge\varepsilon} |y|^{-(n+1)}|f(x-y)-f(x)|dy = C\varepsilon\int_{\varepsilon\le|y|\le\omega} + C\varepsilon\int_{|y|\ge\omega} = C\varepsilon[J_1+J_2],$$

for some fixed $\omega > 0$.

(e) Show that

$$\varepsilon J_2 \xrightarrow[\varepsilon \to 0^+]{} 0.$$

(Use the fact that $f \in L^p$.)

(f) Denote $F(u) = \int_{|y|\le u} |f(x-y) - f(x)|dy$ and note that

$$\varepsilon J_1 = \varepsilon \int_\varepsilon^\omega u^{-(n+1)}dF(u) = \varepsilon\Big\{\Big[\frac{F(u)}{u^{n+1}}\Big]_\varepsilon^\omega + (n+1)\int_\varepsilon^\omega u^{-(n+2)}F(u)du\Big\}.$$

(g) Note that if x is a Lebesgue point then, for any $\delta > 0$, we can find $\omega > 0$ such that $F(u) \le \delta u^n$ for $u < \omega$. Choosing such ω above, show that

$$\varepsilon J_1 \le \varepsilon\frac{F(\omega)}{\omega^{n+1}} + (n+1)\delta\varepsilon\int_\varepsilon^\omega u^{-2}du \le \varepsilon\frac{F(\omega)}{\omega^{n+1}} + (n+1)\delta.$$

(h) Conclude that

$$\limsup_{\varepsilon\to 0^+} I_2 \le (n+1)\delta, \quad \text{for any} \quad \delta > 0.$$

□

In the following problems $n = 1$. Recall that for every distribution $u \in S'$,

$$\widehat{\frac{\partial u}{\partial x}}(\xi) = i\xi\hat{u}(\xi), \qquad \widehat{ixu}(\xi) = -\frac{\partial u}{\partial \xi}\hat{u}(\xi).$$

(2) Let $H(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0. \end{cases}$
Show that $T_H' = \delta_{x=0}$.
Often this is simplified to $H'(x) = \delta$.
Conclude that

$$i\xi \widehat{H}(\xi) = (2\pi)^{-\frac{1}{2}} (= T_{(2\pi)^{-\frac{1}{2}}}).$$

Does it follow that you can conclude $\widehat{H}(\xi) = -i(2\pi)^{-\frac{1}{2}}\xi^{-1}$?

(3) (a) Show that

$$\widehat{H}(\varphi) = H(\hat{\varphi}) = \int_0^\infty \hat{\varphi}(x)dx, \qquad \varphi \in S$$

and conclude that (for $\varphi \in S$)

$$\widehat{H}(\varphi) = (2\pi)^{-\frac{1}{2}} \lim_{\varepsilon \to 0^+} \int_0^\infty \int_{\mathbb{R}} \varphi(\xi) e^{-i\xi x} e^{-\varepsilon x} d\xi dx = (2\pi)^{-\frac{1}{2}} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \frac{\varphi(\xi)}{i\xi + \varepsilon} d\xi.$$

(b) Show similarly that

$$\widehat{JH}(\varphi) = (2\pi)^{-\frac{1}{2}} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \frac{\varphi(\xi)}{-i\xi + \varepsilon} d\xi.$$

(c) Conclude that

$$\widehat{T_1}(\varphi) = (\widehat{H} + \widehat{JH})(\varphi) = (2\pi)^{-\frac{1}{2}} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \frac{2\varepsilon}{\xi^2 + \varepsilon^2} \varphi(\xi) d\xi.$$

(d) (Poisson kernel) Show that (for $\varphi \in S$),

$$\varphi(\tau) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \frac{\varepsilon}{(\xi - \tau)^2 + \varepsilon^2} \varphi(\xi) d\xi, \qquad \tau \in \mathbb{R}.$$

Definition. The kernel $P(\xi) = \frac{1}{\pi} \frac{1}{\xi^2+1}$ is called the **Poisson kernel**. For every $\varepsilon > 0$ we set $P\varepsilon(\xi) = \varepsilon^{-1} P(\frac{\xi}{\varepsilon}) = \frac{1}{\pi} \frac{\varepsilon}{\xi^2+\varepsilon^2}$.

(e) Extend this result to any bounded continuous function $g(\xi)$. Give a condition on g that will ensure that the limit is attained *uniformly* in $\tau \in \mathbb{R}$.

(4) (a) For any $\varphi \in S$, show that the following limit exists

$$PV \int_{\mathbb{R}} \frac{\varphi(\xi)}{\xi} d\xi := \lim_{\eta \to 0^+} \int_{\mathbb{R} \setminus (-\eta, \eta)} \frac{\varphi(\xi)}{\xi} d\xi.$$

(b) Show that

$$\frac{1}{2}(\widehat{H} - \widehat{JH})(\varphi) = (2\pi)^{-\frac{1}{2}} i^{-1} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \frac{\xi}{\xi^2 + \varepsilon^2} \varphi(\xi) d\xi = (2\pi)^{-\frac{1}{2}} i^{-1} PV \int_{\mathbb{R}} \frac{\varphi(\xi)}{\xi} d\xi.$$

(Hint: For the last equality you need to show that

$$\int_{\mathbb{R}} \frac{\xi}{\xi^2 + \varepsilon^2} \varphi(\xi) d\xi - \int_{\mathbb{R} \setminus (-\varepsilon, \varepsilon)} \frac{\varphi(\xi)}{\xi} d\xi \xrightarrow[\varepsilon \to 0^+]{} 0.$$

Use Problem 1 with the kernel

$$K(t) = \begin{cases} \frac{t}{t^2+1}, & \text{if} \quad |t| \le 1, \\ \frac{t}{t^2+1} - \frac{1}{t}, & \text{if} \quad |t| \ge 1. \end{cases}$$

Note that you have to modify Problem 1 since here $\int_{\mathbb{R}} K(t)dt = 0$.)

(c) Show that

$$\widehat{H}(\varphi) = (2\pi)^{-\frac{1}{2}} i^{-1} PV \int_{\mathbb{R}} \frac{\varphi(\xi)}{\xi} d\xi + (2\pi)^{-\frac{1}{2}} \frac{\varphi(0)}{2}.$$

Use this equation to compute $i\xi\widehat{H}$ and compare it with the formula in Problem 2.

(5) (a) For $\varphi \in S$ and $\tau \in \mathbb{R}$ define $\psi \in S$ by $\hat{\varphi}(x) = e^{i\tau x}\hat{\psi}(x)$. Show that

$$(2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} (H - JH)\hat{\varphi}(x)dx = \frac{1}{\pi i} PV \int_{\mathbb{R}} \frac{\psi(\xi)}{\xi - \tau} d\xi.$$

(b) For $\psi \in S$ define the transformation

$$\mathbb{H}\psi(\tau) = \frac{1}{\pi i} PV \int_{\mathbb{R}} \frac{\psi(\xi)}{\xi - \tau} d\xi, \quad \tau \in \mathbb{R}.$$

Prove that $\mathbb{H}$ is a linear isometry from S into $L^2(\mathbb{R})$ (with respect to the L^2 norm) and hence can be extended as an isometry to all of $L^2(\mathbb{R})$.

(c) Keep the notation $\mathbb{H}$ for the extended isometry and show that $\mathbb{H}^2 = I$, the identity operator. Conclude that $\mathbb{H}$ is in fact an *isomorphism* on $L^2(\mathbb{R})$.

Definition. The isomorphism $\mathbb{H}$ is called the **Hilbert transform**.

(6) Consider the kernel $\widetilde{P}_\varepsilon(t) = \frac{1}{\pi}\frac{t}{t^2+\varepsilon^2}, \quad \varepsilon > 0$ (which is sometimes called the "conjugate Poisson kernel").

(a) Show that if $\varphi \in S$ then $\widetilde{P}_\varepsilon \star \varphi \in L^2(\mathbb{R})$ and

$$\widehat{\widetilde{P}_\varepsilon \star \varphi}(\xi) = -i(H - JH)(\xi)e^{-\varepsilon|\xi|}\hat{\varphi}(\xi) = -i\, sgn(\xi)e^{-\varepsilon|\xi|}\hat{\varphi}(\xi), \qquad \xi \in \mathbb{R}.$$

Conclude that $\widetilde{P}_\varepsilon \star g \in L^2(\mathbb{R})$ for any $g \in L^2(\mathbb{R})$.

(b) Prove that

$$\widetilde{P}_\varepsilon \star g \xrightarrow[\varepsilon \to 0^+]{} i\mathbb{H}g, \quad \text{for any} \quad g \in L^2(\mathbb{R}).$$

(c) Prove that

$$\widetilde{P}_\varepsilon \star g = iP_\varepsilon \star \mathbb{H}g, \quad \text{for any} \quad g \in L^2(\mathbb{R}).$$

(Suggestion: Look at Fourier transforms).

For the following problems, you can consult Chapter III of [Ne].

(7) Let $f \in L^p(\mathbb{R})$ for some $p \in (1, \infty)$.

(a) Show that

$$F(z) = \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(t)}{t - z} dt$$

is an analytic function of $z = x + iy$ in the upper (resp. lower) half-plane $y > 0$ (resp. $y < 0$).

(b) Prove the following theorem of M. Riesz.

Theorem. *Write $F(z) = U(x,y) + iV(x,y)$. Then there exists a constant $A_p > 0$, depending only on p, such that for every $y > 0$,*

$$(*) \int_{\mathbb{R}} |U(x,y)|^p dx \leq \int_{\mathbb{R}} |f(x)|^p dx,$$

$$(**) \int_{\mathbb{R}} |V(x,y)|^p dx \leq A_p \int_{\mathbb{R}} |f(x)|^p dx.$$

This is Theorem 2 in Chapter III of [Ne]. Note that for the case $p = 2$ it follows from Problem 6 (with $A_2 = 1$). Here are the steps in the proof.

Proof. (i) The estimate (*) follows from the fact that

$$U(x,y) = f \star P_y(x),$$

where P_y is the Poisson kernel.

(ii) Note that $V(x,y) = f \star \widetilde{P}_y(x)$ where as above $\widetilde{P}_y(t) = \frac{1}{\pi}\frac{t}{t^2+y^2}$. Hence conclude that $\|V(\cdot,y)\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^p(\mathbb{R})}\|\widetilde{P}_y\|_{L^q(\mathbb{R})}$ where $q = \frac{p}{p-1}$. In particular $\|V(\cdot,y) - V_j(\cdot,y)\|_{L^\infty(\mathbb{R})} \xrightarrow[j\to\infty]{} 0$, where $V_j(x,y) = f_j \star \widetilde{P}_y(x)$ and $f_j = \chi_{[-j,j]}f$ is the truncation of f.

(iii) Use Fatou's lemma to conclude that it is sufficient to prove (**) for a compactly supported f. Furthermore, by splitting into positive and negative parts you can assume $f \geq 0$ so that $U(x,y) > 0$ (except for the trivial case $f \equiv 0$).

(iv) Take $1 < p < 3$. Let $\zeta = \alpha + i\beta \in \mathbb{C}$ such that $Re(\zeta) = \alpha > 0$. Show that there exist constants $c_1, c_2 > 0$ such that

$$|\beta|^p \leq c_1\alpha^p - c_2 Re(\zeta^p).$$

(It suffices to prove

$$|\sin\theta|^p \leq c_1(\cos\theta)^p - c_2\cos(p\theta), \qquad 0 \leq |\theta| < \frac{\pi}{2}).$$

(v) Conclude that

$$\int_{\mathbb{R}} |V(x,y)|^p dx \leq c_1 \int_{\mathbb{R}} |U(x,y)|^p dx - c_2 \int_{\mathbb{R}} Re(F(x+iy)^p) dx.$$

(vi) Show that

$$\int_{\mathbb{R}} (F(x+iy)^p) dx = 0$$

using the following arguments: $F(z)^p$ is analytic in the upper half-plane ($U > 0$) and $|z||F(z)|$ is bounded. Then use Cauchy's theorem.

(vii) This takes care (in particular) of the proof if $1 < p \leq 2$. If $p > 2$ use the *"duality method"*: Let $q = \frac{p}{p-1} \in (1,2)$ and write:

$$\Big(\int_{\mathbb{R}} |V(x,y)|^p dx\Big)^{\frac{1}{p}} = \sup_{\|g\|_{L^q(\mathbb{R})=1}} \int_{\mathbb{R}} V(x,y)g(x)dx,$$

where you can assume that g is compactly supported. Now

$\int_{\mathbb{R}} V(x,y)g(x)dx = \int_{\mathbb{R}} g(x)\int_{\mathbb{R}} f(t)\tilde{P}_y(x-t)dtdx$

$= -\int_{\mathbb{R}} f(t)\int_{\mathbb{R}} g(x)\tilde{P}_y(t-x)dxdt = \qquad -\int_{\mathbb{R}} W(t,y)f(t)dt,$

where by the preceding part

$$\|W(\cdot,y)\|_{L^q(\mathbb{R})} \le A_q\|g\|_{L^q(\mathbb{R})}.$$

□

(8) Prove the following theorem of M. Riesz.

Theorem. *Let $f \in L^p(\mathbb{R})$ for some $p \in (1,\infty)$. Define*

$$\tilde{f}_\varepsilon(x) = \frac{1}{\pi}\int_{|x-t|>\varepsilon} \frac{f(t)}{x-t}dt.$$

Then

$$(*) \qquad \|\tilde{f}_\varepsilon\|_{L^p(\mathbb{R})} \le A_p\|f\|_{L^p(\mathbb{R})}, \qquad \varepsilon > 0.$$

$(**)$ *There exists a function* $\tilde{f} \in L^p(\mathbb{R})$ *such that* $\|\tilde{f}_\varepsilon - \tilde{f}\|_{L^p(\mathbb{R})} \xrightarrow[\varepsilon\to 0^+]{} 0.$

This is Theorem 3 in Chapter III of [Ne]. In the case $p = 2$ we have $\tilde{f} = i\mathbb{H}f$. Here are the steps of the proof.

Proof. (a) Let $V(x,\varepsilon)$ be as in the previous problem and note that

$$V(x,\varepsilon) - \tilde{f}_\varepsilon = f \star K_\varepsilon$$

where K_ε is the integrable kernel introduced in Problem 4. Hence

$$\|V(x,\varepsilon) - \tilde{f}_\varepsilon\|_{L^p(\mathbb{R})} \le C\|f\|_{L^p(\mathbb{R})}$$

so that (*) follows from (**) in the previous problem.

(b) In view of (*) it suffices to prove (**) when $f \in \mathbb{D}(\mathbb{R})$. Write

$$\tilde{f}_\varepsilon(x) = \frac{1}{\pi}\int_{\varepsilon<|x-t|<3a} \frac{f(t)-f(x)}{x-t}dt$$

where $supp(f) \subseteq (-a,a)$ and $|x| < 2a$. Combined with the behavior at infinity conclude that $|\tilde{f}_\varepsilon(x)| \le C(1+|x|)^{-1}$ for $0 < \varepsilon < \varepsilon_0$. Use the Lebesgue Dominated Convergence theorem to conclude that $\{\tilde{f}_\varepsilon\}$ is a Cauchy sequence (in $L^p(\mathbb{R})$) as $\varepsilon \to 0^+$.

□

(9) Prove the following theorem .

Theorem. *Let $f \in L^p(\mathbb{R})$ for some $p \in (1,\infty)$. Define*

$$\tilde{f}_\varepsilon(x) = \frac{1}{\pi}\int_{|x-t|>\varepsilon} \frac{f(t)}{x-t}dt.$$

Let $\tilde{f} \in L^p(\mathbb{R})$ be the limit function from the previous problem. Then

$$\tilde{f}_\varepsilon(x) \xrightarrow[\varepsilon\to 0^+]{} \tilde{f}(x), \quad \text{for a.e.} \quad x \in \mathbb{R}.$$

This is Theorem 4 in Chapter III of [Ne]. Here are the steps of the proof.

Proof. (a) Prove that

$$\widetilde{P}_\varepsilon \star g = P_\varepsilon \star \tilde{g}, \quad \text{for any} \quad g \in L^p(\mathbb{R}).$$

(You can start with the result of Problem 6(c) and extend by the continuity property in Problem 7).

(b) Prove that for any $h \in L^p(\mathbb{R})$,

$$P_\varepsilon \star h(x) \xrightarrow[\varepsilon \to 0^+]{} h(x), \quad \text{for a.e.} \quad x \in \mathbb{R}.$$

(You can use the result in Problem 1).

(c) With $V(x,\varepsilon) = \widetilde{P}_\varepsilon \star f$ show that

$$V(x,\varepsilon) - \tilde{f}_\varepsilon(x) \xrightarrow[\varepsilon \to 0^+]{} 0, \quad \text{for a.e.} \quad x \in \mathbb{R}.$$

(You can use the kernel K in Problem 4).

□

Definition. The operator $\mathbb{H} : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ defined by $\mathbb{H}f = -i\tilde{f}$ is called the **Hilbert transform** in $L^p(\mathbb{R})$.

(10) (a) Prove that $\mathbb{H}^2 = I$ (the identity operator) for all $p \in (1,\infty)$.
(b) Prove that $\mathbb{H}^* = \mathbb{H}$, where $\mathbb{H}^*$ is the adjoint of $\mathbb{H}$ in $L^q(\mathbb{R})$, $\quad q = \frac{p}{p-1}$.

(11) Let $f = \chi_{[a,b]}$, the characteristic function of the closed interval $[a,b]$. Show that $\tilde{f}(x) = \frac{1}{\pi}\log|\frac{x-a}{x-b}|$, hence conclude that the above results do not hold in the cases $p = 1, \infty$.

(12) Returning to the case $p = 2$, prove that the function $F(z)$ in Problem 7 is given by

$$F(z) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{0} e^{-i\xi z} g(\xi) d\xi, \quad Im(z) > 0,$$

where $g(\xi) = 2\hat{f}(-\xi)$. Prove also the converse:

Paley-Wiener Theorem. *A necessary and sufficient condition for a function* $F(z), \quad z = x + iy, \quad y > 0$, *to be analytic in the upper half-plane and satisfy the condition*

$$\sup_{y>0} \int_{\mathbb{R}} |F(x+iy)|^2 dx < \infty,$$

is that there exists a function $g \in L^2(-\infty, 0)$ *such that*

$$F(z) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{0} e^{-i\xi z} g(\xi) d\xi, \quad Im(z) \geq 0.$$

(See Th. 19.2 in [Ru]).

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