PLANAR NAVIER-STOKES EQUATIONS VORTICITY APPROACH

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1. Introduction

In this survey we review the existence, uniqueness and regularity theory of solutions to the Navier–stokes equations when they are formulated in "vorticity form". We also discuss the large-time asymptotic behavior of solutions for sufficiently small initial data. In fact, the three-dimensional case has hardly been studied (we refer to the article by P. Constantin in this book), and we shall therefore concentrate on the two-dimensional case.

We recall the basic equations [26], [11]. Throughout the paper, we use bold-face notation for vectors and vector–functions (in $\mathbb{R}^2$ or $\mathbb{R}^3$). Their components are labeled as $\mathbf{w} = (w^1, \ldots, w^n)$ $(n = 2, 3)$ and $|\mathbf{w}|^2 = \sum_{i=1}^n (w^i)^2$. The scalar product is denoted by $\mathbf{a}\cdot\mathbf{b} = \sum_{i=1}^n a^i \cdot b^i$. If $\boldsymbol{\alpha} \in \mathbb{Z}_+^n$ is a multi–index, we let $\nabla^{\boldsymbol{\alpha}} = \prod_{i=1}^n \partial_{x^i}^{\alpha^i}$ and $|\boldsymbol{\alpha}| = \sum_{i=1}^n \alpha^i$.

Denoting the velocity by $\mathbf{u}(\mathbf{x}, t)$, the pressure by $p(\mathbf{x}, t)$ and the (constant) coefficient of viscosity by ν $(\nu > 0)$, the Navier–Stokes equations in a domain $\Omega \subseteq \mathbb{R}^n$ are,

$$
\begin{aligned}
&\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \nu\Delta\mathbf{u}, \qquad \partial_t = \frac{\partial}{\partial t}, \\
&\nabla\cdot\mathbf{u} = 0.
\end{aligned}
\tag{1.1}
$$

The equations are supplemented by an initial condition

$$
\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}),
\tag{1.2}
$$

and, if $\Omega \neq \mathbb{R}^n$, by boundary conditions (such as $\mathbf{u} = 0$, the "no–slip" condition) on the boundary $\partial\Omega$, for all $t \geq 0$. If $\Omega = \mathbb{R}^n$, growth (or, rather, decay) condition must be imposed on $\mathbf{u}$ at infinity.

In the case that $\mathbf{u}_0 \in L^2(\Omega)$ (or $\mathbf{u}_0 \in H^1(\Omega)$) the well–posedness of the problem with suitably defined weak solutions (strong for $H^1(\Omega)$) is well–known since the pioneering work of Leray [27] (see also [29] for the case of the full plane) . The strong well–posedness is only local in time if $n = 3$. We refer to [12], [25], [36] for full accounts of this theory. In what concerns well–posedness of the system (1.1) beyond

Date: August 25, 2002.

the L^2–framework, we refer to [24] and references therein, as well as earlier works by Kato and Ponce using commutator estimates in various Sobolev spaces [19], [21], [22], [23], [34].

Our interest here is to study well–posedness of the flow, in "rough" spaces, by using the vorticity formulation. We recall this formulation in the general three-dimensional case. Taking the curl of the first equation in (1.1), and denoting by $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ the vorticity, we get

$$\partial_t \boldsymbol{\omega} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} = \nu \Delta \boldsymbol{\omega}. \tag{1.3}$$

The connection between $\mathbf{u}$ and $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is given by the "vector potential" $\mathbf{A}$,

$$\mathbf{u} = \nabla \times \mathbf{A}, \qquad \Delta \mathbf{A} = -\boldsymbol{\omega}. \tag{1.4}$$

Under mild growth assumptions, one can take

$$\mathbf{A} = -G * \boldsymbol{\omega}, \tag{1.5}$$

where G is the fundamental solution of Δ. Note that the fact that $\nabla \cdot \boldsymbol{\omega} = 0$ for all $t \geq 0$ (a "structural assumption" that must be verified for any solution of (1.3)) implies that $\nabla \cdot \mathbf{A} = 0$, hence, indeed, from (1.4),

$$\nabla \times \mathbf{u} = -\Delta \mathbf{A} = \boldsymbol{\omega}. \tag{1.6}$$

Remark that when $\mathbf{u}$ is given by (1.4), then automatically $\nabla \cdot \mathbf{u} = 0$, so that (1.3)–(1.4) is equivalent to (1.1), at least in the case of sufficiently regular solutions. The system is supplemented by the initial condition

$$\boldsymbol{\omega}(\mathbf{x}, 0) = \boldsymbol{\omega}_0(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3. \tag{1.7}$$

From the point of view of hydrodynamical phenomena, an interesting case is that of the evolution of vorticity (and its associated velocity field) when it is initially given by isolated vortices, vortex filaments or sheets . Since, in the "zero viscosity limit" (i.e., $\nu = 0$, leading to the Euler equations) the circulation is preserved (Kelvin's theorem), the use of vorticity in numerical methods has become very popular. In particular, in "vortex methods" ([13]), even smooth initial data are replaced by a distribution of singular "vortical objects". Mathematically speaking, we need to study the system (1.3)–(1.4), (1.7), when $\boldsymbol{\omega}_0(\mathbf{x})$ is a measure. This will be the main focus of this article. Indeed, since very little is known in the three dimensional case, we shall deal here with the two–dimensional situation. We refer to the article by P. Constantin in this volume, concerning approximate solutions to the vorticity equation in the three-dimensional case. Also for simplicity, we have avoided adding a source term (external force) in the case of Eq. (1.1) or (1.3). In fact, for issues considered here such as existence, uniqueness and regularity, the results can be extended to the non–homogeneous case in a rather standard way. When $\Omega \neq \mathbb{R}^n$ the system (1.3)–(1.4) must be supplemented with boundary conditions on $\boldsymbol{\omega}(\mathbf{x}, t)$, $\mathbf{x} \in \partial\Omega$, $t \geq 0$. The most common physically plausible boundary conditions are stated in terms of $\mathbf{u}$ (such as the "non–slip" condition). Casting these conditions in terms of $\boldsymbol{\omega}$ is quite involved, and in fact, has hardly been treated in theoretical studies. On the other hand, in numerical works, the methods used for the implementation of vorticity boundary conditions (or, in the hydrodynamical language, "generation of vorticity") are quite diverse. Some of them could perhaps prove instrumental in the rigorous treatment of the problem. However, in this survey we shall not touch upon this topic, and refer the reader to the book [13] and to [2] for more details.

In order to avoid the boundary problem, we shall concentrate in this survey on the case of the full plane, $\Omega = \mathbb{R}^2$.

The velocity $\mathbf{u}(\mathbf{x}, t)$ is obtained from (1.6). In fact, in the two-dimensional case we can easily obtain a convolution integral connecting $\mathbf{u}$ to ω as follows.

The velocity field is now two–dimensional $\mathbf{u}(\mathbf{x}, t) = (u^1(x^1, x^2, t), u^2(x^1, x^2, t))$ and the vorticity is given by $\boldsymbol{\omega}(\mathbf{x}, t) = \omega(\mathbf{x}, t)\mathbf{k}$,

$$\omega(\mathbf{x}, t) = \partial_{x^1} u^2 - \partial_{x^2} u^1. \tag{1.8}$$

Furthermore, the term $(\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$ vanishes identically, so that Eq. (1.3) reduces to a (nonlinear) convection–diffusion equation for the scalar vorticity ω,

$$\partial_t \omega + (\mathbf{u} \cdot \nabla)\omega = \nu \Delta \omega, \quad \omega(\mathbf{x}, 0) = \omega_0(\mathbf{x}). \tag{1.9}$$

Carrying out the operations in (1.4)–(1.5) we obtain

$$\mathbf{u}(\mathbf{x}, t) = (\mathbf{K} * \omega)(\mathbf{x}, t) = \int_{\mathbb{R}^2} \mathbf{K}(\mathbf{x} - \mathbf{y})\omega(\mathbf{y}, t) d\mathbf{y}, \tag{1.10}$$

where the "Biot–Savart" kernel $\mathbf{K}$ is given by

$$\mathbf{K}(\mathbf{x}) = \frac{1}{2\pi} |\mathbf{x}|^{-2} (-x^2, x^1). \tag{1.11}$$

Note that $\nabla \cdot \mathbf{K} = 0$, implying (by (1.7)) the incompressibility condition $\nabla \cdot \mathbf{u} = 0$.

In what follows we shall study the well–posedness of (1.9) in various functional spaces X. This means (at the least), that given the initial vorticity ω_0 the solution evolves along a continuous trajectory in X. The paper is organized as follows. In Section 2 we recall the derivation of solutions for smooth initial data. As is appropriate for parabolic equations, the "maximum principle" plays a fundamental role. In Section 3 we derive space-time estimates for smooth solutions. They are the main tools used in the extension of the solution operator to initial vorticities in $L^1(\mathbb{R}^2)$, as is done in Section 4. In Section 5 we discuss the further extension to measure-valued initial data. We shall see that uniqueness is still an open problem (for measures with large atomic part). In Section 6 we discuss the asymptotic behavior of the vorticity for large time. We conclude in Section 7 with remarks concerning various related open problems.

Notation

The norm in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, is denoted by

$$\|\psi\|_p = \Big[\int_{\mathbb{R}^n} |\psi(\mathbf{x})|^p d\mathbf{x}\Big]^{1/p}$$

with the usual (ess–sup) modification for $p = +\infty$.

The space $W^{s,p}(\mathbb{R}^n)$ (s positive integer) is the L^p Sobolev space, normed by

$$\|\psi\|_{W^{s,p}} = \sum_{k=0}^{s} \sum_{|\boldsymbol{\alpha}|=k} \|\nabla^{\boldsymbol{\alpha}}\psi\|_p.$$

If X is a Banach space, normed by $\|\cdot\|_X$, and $I \subseteq \overline{\mathbb{R}}_+$ is a finite or infinite interval, we define the following spaces of X–valued functions $f : I \to X$.

$C(I,X)$	Continuous functions (not necessarily bounded), topologized by uniform convergence over compact subintervals of I.
$L^p(I,X)$	Strongly measurable functions, normed by $(\int_I \|f(t)\|_X^p dt)^{1/p}$, $1 \leq p < \infty$, with the usual modification for $p = \infty$.
$L^p_{loc}(I,X)$	Strongly measurable functions such that $\phi f \in L^p(I,X)$ for all $\phi \in C_0^\infty(I)$.

If X_1, X_2 are Banach spaces, then $X = X_1 \cap X_2$ is normed by $\|\cdot\|_X = \|\cdot\|_{X_1} + \|\cdot\|_{X_2}$.

2. The case of smooth initial data

Our first theorem is a theorem of McGrath [31]. It imposes rather strong regularity assumptions on $\omega_0(\mathbf{x})$. Set, for $0 < \lambda < 1$,

$$\begin{aligned} C^\lambda(\mathbb{R}^2) = \ & \{f \in C(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2), f \text{ is uniformly } (\lambda\text{–}) \text{ Hölder continuous}, \\ & |f(\mathbf{x}) - f(\mathbf{y})| \le M_f |\mathbf{x} - \mathbf{y}|^\lambda, \mathbf{x}, \mathbf{y} \in \mathbb{R}^2 \Big\}. \end{aligned}$$

$$C^{k,\lambda}(\mathbb{R}^2) = \ \{f \in C^k(\mathbb{R}^2), \nabla^{\boldsymbol{\alpha}} f \in C^\lambda(\mathbb{R}^2), |\boldsymbol{\alpha}| \le k\}.$$

Theorem 2.1 (McGrath). *Assume that for some $0 < \lambda < 1$, $\omega_0(\mathbf{x}) \in L^1(\mathbb{R}^2) \cap C^{2,\lambda}(\mathbb{R}^2)$. Then there exists a solution to (1.9)–(1.10) such that*

(a) The solution is classical; all derivatives appearing in (1.9) are continuous in $\mathbb{R}^2 \times (0, \infty)$.

(b) $\omega(\mathbf{x}, t)$, $\mathbf{u}(\mathbf{x}, t)$ are continuous and uniformly bounded in $\mathbb{R}^2 \times [0, \infty)$.

(c) $\omega(\mathbf{x}, \cdot) \in L^\infty([0, \infty), L^1(\mathbb{R}^2))$.

(d) For every $T > 0$,

$$\sup_{0 \le t \le T, |\mathbf{x}| > R} |\mathbf{u}(\mathbf{x}, t)| \to 0 \quad as \ \ R \to \infty.$$

Under conditions (a)–(d) the solution is unique.

Proof *(outline).* Fix $T > 0$ and let

$$\begin{aligned} & Q_T = \mathbb{R}^2 \times [0, T], \\ (2.1) \qquad & \\ & X_T = C(Q_T) \cap L^\infty(Q_T) \cap L^\infty([0, T], L^1(\mathbb{R}^2)), \end{aligned}$$

where $\|\omega(\mathbf{x}, t)\|_{X_T} = \|\omega\|_{L^\infty(Q_T)} + \sup_{0 \le t \le T} \|\omega(\cdot, t)\|_1$. Let $B_0 \subseteq X_T$ be the ball

$$B_0 = \{\omega \in X_T, \|\omega\|_{X_T} \le \|\omega_0\|_1 + \|\omega_0\|_\infty\}.$$

For $\xi \in B_0$, one defines the map $\mathbf{A}_1 \xi = \mathbf{v}$ by means of (1.10), i.e., $\mathbf{v} = \mathbf{K} * \xi$, $0 \le t \le T$. In particular it is easily seen that $\mathbf{v} \in C(Q_T) \cap L^\infty(Q_T)$ and by standard facts concerning linear parabolic equations the equation

$$(2.2) \qquad \partial_t \theta + (\mathbf{v} \cdot \nabla)\theta = \nu \Delta \theta, \quad \theta(\mathbf{x}, 0) = \omega(\mathbf{x}),$$

has a unique classical solution in Q_T, and in particular $\theta \in X_T$. We let $A : B_0 \to X_T$ be the map $\theta = A\xi$ (where $\mathbf{v} = \mathbf{A}_1 \xi$ in (2.2)). Using the maximum principle and its dual statement in L^1 (note that $\nabla \cdot \mathbf{A}_1 \xi = 0$) it follows that $AB_0 \subseteq B_0$. Now the assumptions on elements of X_T imply that $\{\mathbf{v} = \mathbf{A}_1 \xi, \ \xi \in B_0\}$ is uniformly bounded and equicontinuous in Q_T. The regularity hypothesis on ω_0 (and its decay at infinity) imply therefore that AB_0, the set of all solutions of (2.2) with $\mathbf{v} \in \mathbf{A}_1 B_0$, is uniformly bounded and equicontinuous (in fact, $AB_0 \subseteq W^{1,\infty}(Q_T)$). Furthermore, the elements of AB_0 vanish uniformly as $|x| \to \infty$, $0 \le t \le T$. Thus AB_0 is compactly imbedded in B_0 and , since A is continuous, the Schauder fixed point theorem yields $\omega \in B_0$ such that $\omega = A\omega$. This ω is a solution to (1.9) with $\mathbf{u} = \mathbf{A}_1\omega$. The uniqueness is shown by a similar argument. $\square$

Remark 2.2. *The maximum principle can be applied to Eq. (2.2) and its dual (since $\nabla \cdot \mathbf{v} = 0$). We can therefore conclude (for the solution of (1.9)) that $\|\omega(\cdot,t)\|_1 \le \|\omega_0\|_1$ and $\|\omega(\cdot,t)\|_\infty \le \|\omega_0\|_\infty$, $t \ge 0$ and by interpolation,*

$$\|\omega(\cdot,t)\|_p \le \|\omega_0\|_p, \quad 1 \le p \le \infty. \tag{2.3}$$

Observe that the interpolation argument used above is based on the linear theory. Indeed, Once the solution to (1.9) is obtained, the velocity field $\mathbf{u}(\mathbf{x},t)$ is "frozen" and Eq. (1.9) is treated as a linear convection-diffusion equation. The L^p estimate (2.3) is then obtained for all solutions of equation (1.9), including the original vorticity ω. A similar reasoning is applied to justify the duality argument, and will be used also in the sequel (see the proofs of Eq. (3.4) and Theorem 6.1).

If we limit further $\omega_0 \in C_0^\infty(\mathbb{R}^2)$ the solution $\omega \in C^\infty(\mathbb{R}^2 \times \overline{\mathbb{R}}_+)$ can be obtained as a limit of a sequence of solutions to linear convection–diffusion equations. We refer to [1] for details. In fact, certain basic estimates are easily derived in this case and then extended to more general spaces. We designate by

$$S : C_0^\infty(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2 \times \overline{\mathbb{R}}_+) \tag{2.4}$$

the solution operator to (1.9)–(1.10), $\omega = S\omega_0$. The corresponding velocity field is given by (1.10), and we denote it by

$$\begin{aligned} &\mathbf{U} : C_0^\infty(\mathbb{R}^2) \to \mathbf{C}^\infty(\mathbb{R}^2 \times \overline{\mathbb{R}}_+), \\ &\mathbf{u}(\cdot,t) = \mathbf{U}\omega_0(t) = \mathbf{K} * (S\omega_0)(t) \end{aligned} \tag{2.5}$$

(when there is no risk of confusion we shall write $\omega(t)$ instead of $\omega(\cdot,t)$).

3. Some estimates for smooth solutions

It is convenient to establish some of the basic estimates for the solution operators S, $\mathbf{U}$, assuming that $\omega_0 \in C_0^\infty(\mathbb{R}^2)$.

Multiplying (1.9) by ω and integrating over $\mathbb{R}^2$ we obtain

$$\partial_t \|\omega(\cdot,t)\|_2^2 = -2\nu \|\nabla\omega(\cdot,t)\|_2^2, \tag{3.1}$$

since $\int_{\mathbb{R}^2} \omega(\mathbf{u}\cdot\nabla)\omega d\mathbf{x} = 0$ by $\nabla\cdot\mathbf{u} = 0$. Recall that by the Nash inequality [14], [8] , if ϕ is a smooth decaying function in $\mathbb{R}^2$, then, for some $\eta > 0$,

$$\|\phi\|_2^2 \le \eta^{-1}\|\phi\|_1\|\nabla\phi\|_2.$$

Using this inequality in (3.1) and noting (2.3) with $p = 1$ we get,

$$\partial_t \|\omega(\cdot,t)\|_2^2 \le -2\nu\eta\|\omega_0\|_1^{-2}\|\omega(\cdot,t)\|_2^4, \tag{3.2}$$

hence

$$\|\omega(\cdot,t)\|_2 \le (2\nu\eta t)^{-1/2}\|\omega_0\|_1. \tag{3.3}$$

By duality (using again $\nabla\cdot\mathbf{u} = 0$),

$$\|\omega(\cdot,t)\|_\infty \le (2\nu\eta t)^{-1/2}\|\omega_0\|_2,$$

so that

$$\|\omega(\cdot,t)\|_\infty \le (\nu\eta t)^{-1}\|\omega_0\|_1. \tag{3.4}$$

To estimate $\|\mathbf{u}(\cdot,t)\|_\infty$, note that $|\mathbf{K}(\mathbf{y})| \le (2\pi)^{-1}|\mathbf{y}|^{-1}$, so that,

$$\begin{aligned} |\mathbf{u}(\mathbf{x},t)| &\le \int_{|y|\le(\frac{\nu\eta t}{2\pi})^{1/2}} + \int_{|y|\ge(\frac{\nu\eta t}{2\pi})^{1/2}} |\mathbf{K}(\mathbf{y})\omega(\mathbf{x}-\mathbf{y},t)|d\mathbf{y} \\ &\le \left(\frac{\nu\eta t}{2\pi}\right)^{1/2}\|\omega(\cdot,t)\|_\infty + (2\pi)^{-1/2}(\nu\eta t)^{-1/2}\|\omega(\cdot,t)\|_1 \\ &\le (\tfrac{2}{\pi})^{1/2}(\nu\eta t)^{-1/2}\|\omega_0\|_1. \end{aligned} \tag{3.5}$$

Note that in view of the Hardy-Littlewood-Sobolev inequality [28, Chapter 4] or the fact that $\nabla\mathbf{K}$ is a Calderon–Zygmund kernel , we also have,

$$\|\mathbf{u}(\cdot,t)\|_q \le C\|\omega(\cdot,t)\|_p, \quad 1 < p < 2, \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{2}, \tag{3.6}$$

with $C = C_p$.

We shall now refine these estimates by looking more closely at Eq. (1.9). Using the heat kernel

$$G_\nu(\mathbf{x},t) = (4\pi\nu t)^{-1}\exp\Big(-\frac{|\mathbf{x}|^2}{4\nu t}\Big),$$

the solution $\omega(\mathbf{x},t)$ can be written as,

$$\begin{aligned} \omega(\mathbf{x},t) = &\int_{\mathbb{R}^2} G_\nu(\mathbf{x}-\mathbf{y},t)\omega_0(\mathbf{y})d\mathbf{y} \\ &-\int_0^t\int_{\mathbb{R}^2}\nabla_{\mathbf{y}}G_\nu(\mathbf{x}-\mathbf{y},t-s)\cdot\mathbf{u}(\mathbf{y},s)\omega(\mathbf{y},s)d\mathbf{y}ds. \end{aligned} \tag{3.7}$$

Our first aim is to derive uniform estimates for solutions having initial data $\omega_0 \in K \subseteq C_0^\infty(\mathbb{R}^2)$, where K is precompact in the $L^1(\mathbb{R}^2)$ topology. We use the following notational convention. The constant $C > 0$ stands for a generic positive constant and $\delta(t)$ stands for a monotone nondecreasing, uniformly bounded, generic function defined for $t \geq 0$, such that $\lim_{t\to 0} \delta(t) = 0$. Both C and $\delta(t)$ may depend on various parameters $(p, \nu, \ldots)$ but not on the solution functions. However, they may depend on certain subsets of initial data. We sometimes indicate specific dependencies by adding parameters, e.g., $C(p)$ or $\delta(t; K)$.

Since $t^{1-1/p} G_\nu(\cdot, t)*$ is a bounded operator from $L^1(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$, $1 \leq p \leq \infty$, and $t^{1-1/p} G_\nu(\cdot, t) * \omega_0 \to 0$ as $t \to 0$, in $L^p(\mathbb{R}^2)$, $1 < p \leq \infty$, for every smooth ω_0, we conclude that if $K \subseteq C_0^\infty(\mathbb{R}^2)$ is precompact in $L^1(\mathbb{R}^2)$ then, for $1 < p \leq \infty$,

$$t^{1-1/p} \|G_\nu(\cdot, t) * \omega_0\|_p \leq \delta(t; K), \qquad \omega_0 \in K \tag{3.8}$$

($\delta(t; K)$ depends on p, ν).

Next we note that

$$\|\nabla G_\nu(\cdot, t)\|_r = C t^{-\frac{3}{2}+\frac{1}{r}}, \quad 1 \leq r \leq \infty. \tag{3.9}$$

Inserting (3.6), (3.8)–(3.9) in (3.7) and using the Young and Hölder inequalities we get,

(3.10)

$$\begin{aligned} \|\omega(\cdot, t)\|_p &\leq \delta(t; K) t^{-1+\frac{1}{p}} + C \int_0^t (t-s)^{-\frac{3}{2}+\frac{1}{r}} \|\omega(\cdot, s)\mathbf{u}(\cdot, s)\|_p ds \\ &\leq \delta(t; K) t^{-1+\frac{1}{p}} + C \int_0^t (t-s)^{-\frac{3}{2}+\frac{1}{r}} \|\omega(\cdot, s)\|_p^2 ds \end{aligned}$$

where $\frac{1}{q} + \frac{1}{r} = \frac{1}{p} + 1$, $\frac{1}{p} + \frac{1}{r} = \frac{3}{2}$, $1 < p < 2$. Setting $M_p(t) = \sup_{0 \leq \tau \leq t} \tau^{1-\frac{1}{p}} \|\omega(\cdot, t)\|_p$ for $\omega_0 \in K$ and noting that since ω_0 is smooth $M_p(t)$ is continuous, $M_p(0) = 0$, we infer from (3.10),

$$M_p(t) \leq \delta(t; K) + C M_p(t)^2,$$

hence $M_p(t) \leq \delta(t; K)$ $(1 < p < 2)$ and, interpolating with (3.4) we have

$$\|\omega(\cdot, t)\|_p \leq \delta(t; K) t^{-1+\frac{1}{p}}, \quad \omega_0 \in K, \quad 1 < p \leq \infty. \tag{3.11}$$

(The case $p = \infty$ is obtained by duality as in (3.4)).

Finally we note that the estimate (3.5) can be strengthened to yield

$$\|\mathbf{u}(\cdot, t)\|_\infty \leq \delta(t; K) t^{-\frac{1}{2}}, \quad \omega_0 \in K, \tag{3.12}$$

($K \subseteq C_0^\infty(\mathbb{R}^2)$, precompact in the $L^1(\mathbb{R}^2)$ topology). Indeed, this follows by replacing in (3.5) the term $(\frac{\nu\eta t}{2\pi})^{1/2}$ by $(\frac{\nu\eta t}{2\pi\delta(t;K)})^{1/2}$ and using (3.11).

Remark 3.1. *Note the similarity of the estimates (3.8) for the heat (linear) equation and (3.11) for the vorticity (nonlinear) equation. In what concerns the L^1–L^∞ decay estimate, we have (3.4), where $\eta > 0$ is the "best constant" in the Nash inequality. As pointed out in* [8], *$\eta \approx 3.67\pi$ whereas the corresponding estimate for the heat kernel is $\|G_\nu(\cdot,t)\|_\infty = (4\pi\nu t)^{-1}$. The estimate (3.4) was derived in* [1], [20] *and was improved by Carlen and Loss* [9]*, replacing η by 4π. Thus,quite surprisingly, in spite of the nonlinearity, the L^∞ estimate for $\omega(\cdot,t)$ (in terms of $\|\omega_0\|_1$) is identical to that of the linear heat solution. Observe,however, that* radial *solutions of (1.9) are also solutions of the heat equation, since the nonlinear term vanishes identically. It follows also* [9, Th.2]*, that η in (3.5) can be replaced by 4π.*

4. Extension of the solution operator

We shall now study the extension of the solution operator S, $\mathbf{U}$ (see (2.4)–(2.5)) to initial data in $L^1(\mathbb{R}^2)$. Our goal is to show that the system (1.9)–(1.10) is well-posed in $L^1(\mathbb{R}^2)$.

As in the case of the heat equation, the solution "regularizes" for positive time. Thus, estimates over time intervals $[\epsilon,\infty)$, $\epsilon > 0$, are easy to obtain, using data at $t = \epsilon$. It is convenient to introduce an "intermediate" space

$$Y = L^1(\mathbb{R}^2) \cap C_0(\mathbb{R}^2)$$

(where $C_0(\mathbb{R}^2)$ consists of continuous functions tending to zero at infinity, normed by $\|\cdot\|_\infty$). The space Y has actually been used in the study of vorticity by Marchioro and Pulvirenti [30] in their treatment of "diffusive vorticies" (approximation by finite–dimensional diffusion processes). In addition to the interest in Y as a "persistence" space for vorticity, some basic estimates in this space serve in the study of "zero viscosity" limit, being independent of $\nu > 0$ [1].

It is easy to see that the convolution operator $\mathbf{K}* : Y \to \mathbf{C}_0(\mathbb{R}^2)$ is bounded. We have the following lemma.

Lemma 4.1. *(a) (Existence). The operators S, $\mathbf{U}$ can be extended continuously as*

$$\begin{aligned} &S : Y \to C(\overline{\mathbb{R}}_+, Y) \\ &\mathbf{U} : Y \to C(\overline{\mathbb{R}}_+, \mathbf{C}_0(\mathbb{R}^2)). \end{aligned} \tag{4.1}$$

Indeed, the maps ∇S and $\nabla\mathbf{U}$ can be extended continuously as

$$(4.2)\qquad \begin{array}{l} \nabla S : Y \to C(\overline{\mathbb{R}}_+, Y) \cap L^p_{loc}(\overline{\mathbb{R}}_+, Y) \\ \nabla\mathbf{U} : Y \to C(\overline{\mathbb{R}}_+, \mathbf{C}_0(\mathbb{R}^2)) \cap L^p_{loc}(\overline{\mathbb{R}}_+, \mathbf{C}_0(\mathbb{R}^2)) \end{array}$$

for any $1 \le p \le 2$. Furthermore, the functions $\omega = S\omega_0$, $\mathbf{u} = \mathbf{U}\omega_0 = \mathbf{K} * S\omega_0$ give a weak solution to (1.9)-(1.10).

(b) (Uniqueness). Let $\theta(\mathbf{x}, t)$, $\mathbf{v}(\mathbf{x}, t) = \mathbf{K} * \theta$ be a weak solution in $\mathbb{R}^2 \times \mathbb{R}_+$, of

$$(4.3)\qquad \partial_t\theta + (\mathbf{v}\cdot\nabla)\theta = \nu\Delta\theta$$

where, for some $1 < p < 2$,

$$\begin{array}{c} \theta(\cdot, t) \in C(\overline{\mathbb{R}}_+, Y) \cap C(\mathbb{R}_+, W^{1,1} \cap W^{1,\infty}) \cap L^p_{loc}(\mathbb{R}_+, W^{1,1} \cap W^{1,\infty}), \\ \theta(\mathbf{x}, 0) = \omega_0(\mathbf{x}) \in Y. \end{array}$$

Then $\theta(\cdot, t) = S\omega(t)$ for all $t \ge 0$.

(c) (Regularity). For every $\omega_0 \in Y$ the functions $\omega(\mathbf{x}, t) = S\omega_0(t)$, $\mathbf{u}(\mathbf{x}, t) = \mathbf{U}\omega_0(t)$ are in $C^\infty(\mathbb{R}^2 \times \mathbb{R}_+)$ and Eq. (1.9) is satisfied in the classical sense. Furthermore, for every integer k and double–index $\boldsymbol{\alpha}$, the maps

$$\begin{array}{l} \partial_t^k \nabla^{\boldsymbol{\alpha}} S : Y \to C(\mathbb{R}_+, Y), \\ \partial_t^k \nabla^{\boldsymbol{\alpha}} \mathbf{U} : Y \to C(\mathbb{R}_+, \mathbf{C}_0(\mathbb{R}^2)) \end{array}$$

are continuous.

Proof *(outline ,see* [1] *for details).* Differentiating (3.7) we obtain

$$(4.4)\qquad \begin{aligned} \nabla\omega(\mathbf{x}, t) = & \int_{\mathbb{R}^2} \nabla_{\mathbf{x}} G_\nu(\mathbf{x} - \mathbf{y}, t)\omega_0(\mathbf{y})d\mathbf{y} \\ & + \int_0^t \int_{\mathbb{R}^2} \nabla_{\mathbf{x}} G_\nu(\mathbf{x} - \mathbf{y}, t - s) \cdot (\mathbf{u}(\mathbf{y}, s) \cdot \nabla)\omega(\mathbf{y}, s)d\mathbf{y}ds. \end{aligned}$$

In view of (2.3) and the boundedness of $\mathbf{K}*$ we have

$$(4.5)\qquad A := \sup_{0 \le t \le T} \|\mathbf{u}(\cdot, t)\|_\infty \le C\|\omega_0\|_Y, \quad C = C(T),$$

so that, using (3.9) in (4.4), and denoting $N(t) = \sup_{0<\tau\le t} \|\nabla\omega(\cdot, \tau)\|_\infty$,

$$(4.6)\qquad N(t) \le C\Big[t^{-1/2}\|\omega_0\|_\infty + A\int_0^t (t - s)^{-1/2} N(s)ds\Big].$$

A similar inequality is obtained for $\|\nabla\omega(\cdot, t)\|_1$. We deduce

$$(4.7)\qquad \|\nabla\omega(\cdot, t)\|_Y \le Ct^{-1/2}, \quad C = C(\nu, T, \|\omega_0\|_Y).$$

If $\theta(\mathbf{x}, t)$ is another solution to (1.9), $\theta(\mathbf{x}, 0) = \theta_0(x) \in C_0^\infty(\mathbb{R}^2)$, a similar derivation yields

$$(4.8)\qquad \|\omega(\cdot, t) - \theta(\cdot, t)\|_Y \le C\|\omega_0 - \theta_0\|_Y,$$

$$\begin{aligned}\|\nabla\omega(\cdot,t)-\nabla\theta(\cdot,t)\|_Y &\le Ct^{-1/2}\|\omega_0-\theta_0\|_Y,\\ C&=C(\nu,T,\|\omega_0\|_Y,\|\theta_0\|_Y).\end{aligned} \tag{4.9}$$

The conclusion of the proof of (a) is now standard. For $\omega_0\in Y$ we take a sequence $\{\omega_0^{(j)}\}_{j=1}^{\infty}\subseteq C_0^\infty(\mathbb{R}^2)$ converging to ω_0 in Y, and using (4.8)–(4.9) we obtain $\omega(t)=\lim_{j\to\infty}S\omega_0^{(j)}(t)$, $\mathbf{u}(t)=\lim_{j\to\infty}\mathbf{U}\omega_0^{(j)}(t)$.

To prove the uniqueness assertion, we note that $\mathbf{v}=\mathbf{K}*\theta\in\mathbf{C}_0(\mathbb{R}^2)$, so that (3.7) holds, with ω, ω_0, $\mathbf{u}$ replaced by θ, θ_0, $\mathbf{v}$. We can then derive an estimate analogous to (4.8).

Finally the regularity claim follows from standard arguments concerning parabolic equations [35]. □

We may now proceed to the main result of this section.

Theorem 4.2. *(a) (Existence). The operators S, $\mathbf{U}$ can be extended continuously as*

$$\begin{aligned}&S:L^1(\mathbb{R}^2)\to C(\overline{\mathbb{R}}_+,L^1(\mathbb{R}^2))\cap C(\mathbb{R}_+,W^{1,1}\cap W^{1,\infty}),\\ &\mathbf{U}:L^1(\mathbb{R}^2)\to C(\overline{\mathbb{R}}_+,\mathbf{C}_0(\mathbb{R}^2)).\end{aligned} \tag{4.10}$$

For every $\omega_0\in L^1(\mathbb{R}^2)$ and $t>0$ we have

$$\mathbf{u}=\mathbf{K}*(S\omega_0)(t),$$

and ω, $\mathbf{u}$ give a weak solution to (1.9). Furthermore, the estimates (3.4)–(3.5) are valid.

(b) (Uniqueness). Let $\theta(\mathbf{x},t)$, $\mathbf{v}(\mathbf{x},t)=\mathbf{K}*\theta$ be a weak solution to (4.3) in $\mathbb{R}^2\times\mathbb{R}_+$. Assume that

$$\begin{aligned}&\theta(\cdot,t)\in C(\overline{\mathbb{R}}_+,L^1(\mathbb{R}^2))\cap C(\mathbb{R}_+,L^\infty(\mathbb{R}^2)),\\ &\theta(\mathbf{x},0)=\omega_0(\mathbf{x})\in L^1(\mathbb{R}).\end{aligned} \tag{4.11}$$

Then $\theta(\cdot,t)=S\omega_0(t)$ for all $0\le t<\infty$.

(c) (Regularity). For every $\omega_0\in L^1(\mathbb{R}^2)$ the functions $\omega(\mathbf{x},t)=S\omega_0(t)$, $\mathbf{u}(\mathbf{x},t)=\mathbf{U}\omega_0(t)$ are in $C^\infty(\mathbb{R}^2\times\mathbb{R}_+)$ and Eq. (1.9) is satisfied in the classical sense. Furthermore, for every integer k and double–index $\boldsymbol{\alpha}$, the maps

$$\begin{aligned}&\partial_t^k\nabla^{\boldsymbol{\alpha}}S:L^1(\mathbb{R}^2)\to C(\mathbb{R}_+,L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)),\\ &\partial_t^k\nabla^{\boldsymbol{\alpha}}\mathbf{U}:L^1(\mathbb{R}^2)\to C(\mathbb{R}_+,\mathbf{C}_0(\mathbb{R}^2))\end{aligned}$$

are continuous.

Proof. Let $K\subseteq C_0^\infty(\mathbb{R}^2)\cap L^1(\mathbb{R}^2)$ be precompact (in the L^1 topology). We first show that the family of maps

$$t\to S\omega_0(t),\quad t\ge 0,\omega_0\in K \tag{4.12}$$

is equicontinuous. Indeed, it follows from (3.7), using (3.9) (with $r = 1$) and (2.3),(3.12) that

$$\|S\omega_0(t) - \omega_0\|_1 \le \|G_\nu * \omega_0 - \omega_0\|_1 + \delta(t; K), \tag{4.13}$$

which converges to 0 (as $t \to 0$) uniformly in $\omega_0 \in K$. Next it follows from (3.11) and Lemma 4.1(c) that, for any $\epsilon > 0$, $\boldsymbol{\alpha} = (\alpha^1, \alpha^2)$,

$$\sup_{\epsilon \le t < \infty} \sup_{\omega_0 \in K} \{\|\nabla^{\boldsymbol{\alpha}} S\omega_0(t)\|_Y\} < \infty, \tag{4.14}$$

which implies, by (3.5) and (1.9),

$$\sup_{\epsilon \le t < \infty} \sup_{\omega_0 \in K} \{\|\mathbf{U}\omega_0(t)\|_\infty + \|\partial_t S\omega_0(t)\|_1\} < \infty. \tag{4.15}$$

The estimates (4.13)–(4.15) imply the equicontinuity of (4.12) in $L^1(\mathbb{R}^2)$.

Now let $\{\omega_0^{(n)}(\mathbf{x})\}_{n=1}^\infty \subseteq C_0^\infty(\mathbb{R}^2)$ converge to $\omega_0 \in L^1(\mathbb{R}^2)$ in L^1. Taking $K = \{\omega_0^{(n)}(\mathbf{x})\}_{n=1}^\infty$ the foregoing argument yields the equicontinuity (in L^1) of the trajectories $\omega^{(n)}(t) = S\omega_0^{(n)}(t)$. In what follows we prove the uniform convergence of these trajectories. Writing $\mathbf{u}^{(n)}(t) = \mathbf{U}\omega_0^{(n)}(t)$ we have,

$$\begin{aligned}
&\omega^{(n)}(t) - \omega^{(m)}(t) = G_\nu(\cdot, t) * (\omega_0^{(n)} - \omega_0^{(m)}) \\
&+ \textstyle\int_0^t \nabla G_\nu(\cdot, t-s) * \mathbf{u}^{(n)}(s)(\omega^{(n)}(s) - \omega^{(m)}(s))ds \\
&+ \int_0^t \nabla G_\nu(\cdot, t-s) * (\mathbf{u}^{(n)}(s) - \mathbf{u}^{(m)}(s))\omega^{(m)}(s)ds \\
&= I_1(\cdot, t) + I_2(\cdot, t) + I_3(\cdot, t).
\end{aligned} \tag{4.16}$$

Let $p \in (1, 2)$. Clearly,

$$\|I_1(\cdot, t)\|_p \le Ct^{-1+\frac{1}{p}} \|\omega_0^{(n)} - \omega_0^{(m)}\|_1. \tag{4.17}$$

In view of (3.12) we obtain in I_2, for $0 < s \le t$,

$$\begin{aligned}
\|\mathbf{u}^{(n)}(\cdot, s)(\omega^{(n)}(\cdot, s) - \omega^{(m)}(\cdot, s))\|_p &\le \|\mathbf{u}^{(n)}(\cdot, s)\|_\infty \|(\omega^{(n)}(\cdot, s) - \omega^{(m)}(\cdot, s))\|_p \\
&\le \delta(t; K) \cdot s^{-1/2} \|(\omega^{(n)}(\cdot, s) - \omega^{(m)}(\cdot, s))\|_p,
\end{aligned} \tag{4.18}$$

and using (3.6), (3.11), we have in I_3,

$$\begin{aligned}
\|(\mathbf{u}^{(n)}(s) - \mathbf{u}^{(m)}(s))\omega^{(m)}(s)\|_p &\le C\|\mathbf{u}^{(n)}(s) - \mathbf{u}^{(m)}(s)\|_q \|\omega^{(m)}(s)\|_2 \\
&\le \delta(t, K) \cdot s^{-1/2} \|\omega^{(n)} - \omega^{(m)}(s)\|_p.
\end{aligned} \tag{4.19}$$

Inserting (4.17)–(4.19) in (4.16) we have

$$\begin{aligned}(4.20)\qquad &\|\omega^{(n)}(t)-\omega^{(m)}(t)\|_p \le Ct^{-1+\frac{1}{p}}\|\omega_0^{(n)}-\omega_0^{(m)}\|_1\\ &+\delta(t;K)\int_0^t (t-s)^{-1/2}s^{-1/2}\|\omega^{(n)}(s)-\omega^{(m)}(s)\|_p ds.\end{aligned}$$

Denoting $N(t) = \sup_{0\le\tau\le t}\tau^{1-\frac{1}{p}}\|\omega^{(n)}(\tau)-\omega^{(m)}(\tau)\|_p$, (4.20) can be rewritten as,

$$N(t) \le C\|\omega_0^{(n)}-\omega_0^{(m)}\|_1 + \delta(t;K)N(t)$$

which implies, for $0 < t \le t^* = t^*(K)$,

$$(4.21)\qquad \|\omega^{(n)}(t)-\omega^{(m)}(t)\|_p \le C_p\|\omega_0^{(n)}-\omega_0^{(m)}\|_1\cdot t^{-1+\frac{1}{p}},\quad 1<p<2.$$

Turning back to (4.16) we now obtain,

$$\begin{aligned}(4.22)\qquad \|\omega^{(n)}(t)-\omega^{(m)}(t)\|_1 \le C\Big\{&\|\omega_0^{(n)}-\omega_0^{(m)}\|_1\\ &+\int_0^t (t-s)^{-1/2}\|\mathbf{u}^{(n)}(s)(\omega^{(n)}(s)-\omega^{(m)}(s))\|_1 ds\\ &+\int_0^t (t-s)^{-1/2}\|(\mathbf{u}^{(n)}(s)-\mathbf{u}^{(m)}(s))\omega^{(m)}(s)\|_1 ds\Big\}.\end{aligned}$$

Take $p=\frac{4}{3}$ and use (3.6), (3.11), (4.21), to estimate,

$$\begin{aligned}\|\mathbf{u}^{(n)}(s)(w^{(n)}(s)-\omega^{(m)}(s))\|_1 &\le \|\mathbf{u}^{(n)}(s)\|_4\|\omega^{(n)}(s)-\omega^{(m)}(s)\|_{\frac{4}{3}}\\ &\le Cs^{-\frac{1}{4}}\|\omega_0^{(n)}\|_1\cdot s^{-\frac{1}{4}}\|\omega_0^{(n)}-\omega_0^{(m)}\|_1,\end{aligned}$$

$$\begin{aligned}\|(\mathbf{u}^{(n)}(s)-\mathbf{u}^{(m)}(s))\omega^{(m)}(s)\|_1 &\le \|\mathbf{u}^{(n)}(s)-\mathbf{u}^{(m)}(s)\|_4\|\omega^{(m)}(s)\|_{\frac{4}{3}}\\ &\le Cs^{-\frac{1}{4}}\|\omega_0^{(n)}(s)-\omega_0^{(m)}(s)\|_1\cdot s^{-\frac{1}{4}}\|\omega_0^{(m)}\|_1\end{aligned}$$

Inserting these inequalities in (4.22) yields, for $0<t<t^*$,

$$(4.23)\qquad \|\omega^{(n)}(t)-\omega^{(m)}(t)\|_1 \le C\|\omega_0^{(n)}-\omega_0^{(m)}\|_1.$$

We can now conclude the proof of the theorem. In view of (4.14)–(4.15), (4.23) and Lemma 4.1 the sequence $\omega^{(n)}(t) = S\omega_0^{(n)}(t)$ converges in $C(\overline{\mathbb{R}}_+, L^1(\mathbb{R}^2))$ to a function $\omega(t)\in C(\overline{\mathbb{R}}_+, L^1(\mathbb{R}^2))$. Lemma 4.1 now implies that the sequence converges also in $C(\mathbb{R}_+, Y)$ (in fact, with all derivatives), hence the regularity claim.

To establish the uniqueness claim, we note first that in view of the equicontinuity (4.12) the estimate (3.11) extends to

$$(4.24)\qquad \|S\omega_0(t)\|_p \le \delta(t;K)t^{-1+\frac{1}{p}},\quad \omega_0\in K,\quad 1<p\le\infty,$$

when $K \subseteq L^1(\mathbb{R}^2)$ is precompact. Assume first that $\theta_0 \in L^\infty(\mathbb{R}^2)$. Then $\theta(\mathbf{x}, t)$ satisfies (3.7) (with ω, ω_0, $\mathbf{u}$ replaced respectively by θ, θ_0, $\mathbf{v}$) and repeating the argument leading up to (3.11) we get,

$$\|\theta(\cdot, t)\|_\infty \leq \delta(t) t^{-1}, \quad t > 0, \tag{4.25}$$

$$\|\mathbf{v}(\cdot, t)\|_\infty \leq \delta(t) t^{-1/2}, \quad t > 0. \tag{4.26}$$

Setting $\tilde{\theta}(\cdot, t) = S\theta_0(t)$, we get, as in (4.20), for $1 < p < 2$,

$$\|\tilde{\theta}(\cdot, t) - \theta(\cdot, t)\|_p \leq \delta(t) \int_0^t (t-s)^{-1/2} s^{-1/2} \|\tilde{\theta}(\cdot, s) - \theta(\cdot, s)\|_p ds,$$

which implies that $\tilde{\theta}(\cdot, t) = \theta(\cdot, t)$, $0 \leq t \leq t^*$, for some $t^* > 0$. One can then proceed stepwise in time to obtain $\tilde{\theta}(\cdot, t) = \theta(\cdot, t)$, $t > 0$. This proves uniqueness if $\theta_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Dropping the assumption $\theta_0 \in L^\infty(\mathbb{R}^2)$, we still have by hypothesis, for any $s > 0$, that $\theta(\cdot, s) \in L^\infty(\mathbb{R}^2)$. Invoking the foregoing argument (with $\theta(\cdot, s)$ as initial data) we obtain

$$S\theta(\cdot, s)(t) = \theta(\cdot, t+s), \quad s > 0, \ t \geq 0. \tag{4.27}$$

Also, since $\theta(\cdot, t) \in C(\overline{\mathbb{R}}_+, L^1(\mathbb{R}^2))$, the set $K = \{\theta(\cdot, s), 0 < s \leq 1\} \subseteq L^1(\mathbb{R}^2)$ is precompact. Hence, combining (4.24) and (4.27)

$$\|\theta(\cdot, t+s)\|_p \leq \delta(t; K) t^{-1+\frac{1}{p}}, \quad , 0 < s \leq 1, \qquad , 1 < p \leq \infty. \tag{4.28}$$

Letting $s \to 0$ in (4.28) we have, with $t > 0$,

$$\|\theta(\cdot, t)\|_p \leq \delta(t; K) t^{-1+\frac{1}{p}}, \quad 1 < p \leq \infty, \tag{4.29}$$

and, in particular, we obtain (4.25) and (4.26). We can now repeat the first part of the proof to obtain $\theta(\cdot, t) = \tilde{\theta}(\cdot, t) = S\theta_0(t)$, $t > 0$. $\square$

Remark 4.3. *The existence of a solution to the vorticity equation (1.9)–(1.10), when $\omega_0 \in L^1(\mathbb{R}^2)$, was first proved by Giga, Miyakawa and Osada* [16]*,using a delicate estimate for Green's function of a perturbed heat equation. The constants appearing in their treatment are unspecified and depend nonlinearly on $\|\omega_0\|_1$, in contrast to the linear dependence in (3.4)–(3.5). The proof given here follows* [1] *and the uniqueness part relies also on* [6]*. Observe that only the classical estimates for the heat kernel have been used. A similar approach has been used by Kato* [20]*, using also the classical heat kernel but different functional spaces. Kato derives (3.4), but not (3.5), (3.11)–(3.12), which are essential in the uniqueness proof here. We refer to the implications of this uniqueness proof to nonlinear parabolic equations in Remark 5.3 below.*

5. Measures as initial data

Let $\mathcal{M}$ be the Banach space of finite (signed) measures on $\mathbb{R}^2$, normed by total variation $\|\cdot\|_{\mathcal{M}}$, so that naturally $L^1(\mathbb{R}^2) \subseteq \mathcal{M}$. Following Kato [20], we shall now extend the solution operators S, $\mathbf{U}$ to $\mathcal{M}$. The estimates obtained in Theorem 4.2 and the weak density of L^1 in $\mathcal{M}$ lead to a straightforward result concerning existence and regularity. However, uniqueness remains partially open.

For simplicity we henceforth assume $\nu = 1$ and write $G = G_1$ for the heat kernel. A measure $\eta \in \mathcal{M}$ can be decomposed as

$$\eta = \eta_c + \eta_a = \eta_c + \sum_{j=1}^{\infty} b_j \delta(\mathbf{x} - \mathbf{x}_j), \tag{5.1}$$

where η_c is continuous and η_a is the atomic part of η. The decomposition is "orthogonal",

$$\|\eta\|_{\mathcal{M}} = \|\eta_c\|_{\mathcal{M}} + \|\eta_a\|_{\mathcal{M}} = \|\eta_c\|_{\mathcal{M}} + \sum_{j=1}^{\infty} |b_j|. \tag{5.2}$$

In what follows we write $b = (b_1, b_2, \ldots)$ and

$$\|b\|_p = \Big(\sum_{j=1}^{\infty} |b_j|^p \Big)^{1/p}, \quad 1 \le p \le \infty. \tag{5.3}$$

We can now state the extension theorem for initial data in $\mathcal{M}$.

Theorem 5.1. *Let $\omega_0 = (\omega_0)_c + (\omega_0)_a \in \mathcal{M}$. Then the system (1.9)–(1.10) has a solution $\omega(\cdot, t) \in C(\mathbb{R}_+; W^{1,1} \cap W^{1,\infty})$, $\mathbf{u}(\cdot, t) = \mathbf{K} * \omega(\cdot, t)$ such that*

(a) $\omega(\cdot, t) \to \omega_0$ as $t \to 0$, in the weak* topology of $\mathcal{M}$.

(b) For every $1 \le p \le \infty$, $\|\omega(\cdot, t)\|_p$ is a decreasing function of $t \in \mathbb{R}_+$, and

$$\sup_{0<t<\infty} t^{1-\frac{1}{p}} \|\omega(\cdot, t)\|_p < \infty. \tag{5.4}$$

Furthermore, if $(\omega_0)_a = 0$, then

$$\lim_{t \to 0} t^{1-\frac{1}{p}} \|\omega(\cdot, t)\|_p = 0, \quad 1 < p \le \infty \qquad (\textit{compare (3.11)}). \tag{5.5}$$

(c) Let $(\omega_0)_a = b = (b_1, b_2, \cdots)$. For each $\frac{4}{3} < p < 2$ there are constants δ_p, $\epsilon_p > 0$, such that if $\|b\|_p < \delta_p$ then

$$\limsup_{t \to 0} t^{1-\frac{1}{p}} \|\omega(\cdot, t)\|_p < \epsilon_p, \quad \frac{4}{3} \le p < 2. \tag{5.6}$$

This condition (and (a)) determines uniquely the solution ω. In particular, if $(\omega_0)_a = 0$ then the condition (5.5) (for any single $\frac{4}{3} < p < 2$) determines ω uniquely.

Remark 5.2. *Since for $\tau > 0$, $\omega(\cdot,\tau) \in L^1(\mathbb{R}^2)$, Theorem 4.2 can be applied to $t \geq \tau$. Thus $\omega(\mathbf{x},t) \in C^\infty(\mathbb{R}^2 \times \mathbb{R}_+)$ and estimates like (5.4) follow for $t \geq \tau$ (see (3.11)) and need to be established only in $(0,\tau)$.*

Proof. Using a standard mollifier, we construct a sequence $\{\omega_0^{(j)}\}_{j=1}^\infty \subseteq L^1(\mathbb{R}^2)$, $\|\omega_0^{(j)}\|_1 \leq \|\omega_0\|_{\mathcal{M}}$ and $\omega_0^{(j)} \to \omega_0$ (in the weak* topology of $\mathcal{M}$). Let $\omega^{(j)}$ be the solution given by Theorem 4.2, $\omega^{(j)}(\cdot,0) = \omega_0^{(j)}$. Using the estimates (2.3), (3.4) we see that there exists a subsequence, which we relabel as $\{\omega^{(j)}\}_{j=1}^\infty$, such that, for any fixed $\tau > 0$, $\omega^{(j)}(\cdot,\tau)$ converges to a function $\omega(\cdot,\tau)$ in Y (in fact, with all derivatives, see Lemma 4.1). In particular, we have also $\omega^{(j)}\mathbf{u}^{(j)} \to \omega\mathbf{u}$, $\mathbf{u} = \mathbf{K} * \omega$. It is easy to verify that $(\omega,\mathbf{u})$ constitutes a solution to (1.9)–(1.10) and satisfies (2.3) (with $p = 1$) and (3.4), with $\|\omega_0\|_1$ replaced by $\|\omega_0\|_{\mathcal{M}}$. The estimate (5.4) follows by interpolation. To prove (a) it is clearly sufficient to show (compare (4.12)) that the family

$$t \to \omega^{(j)}(\cdot,t), \quad j = 1,2,\ldots, 0 \leq t \leq T, \tag{5.7}$$

is equicontinuous in the weak* topology of $\mathcal{M}$. Taking $\psi \in C_0^\infty(\mathbb{R}^2)$,

$$\partial_t < \omega^{(j)}(\cdot,t),\psi >=< \omega^{(j)}(\cdot,t),\Delta\psi > + < \omega^{(j)}\mathbf{u}^{(j)}(\cdot,t),\nabla\psi > \tag{5.8}$$

($<\ ,\ >$ is the $(\mathcal{M}, C_0(\mathbb{R}^2))$ pairing). Using the estimate

$$\|\mathbf{u}^{(j)}(\cdot,t)\omega^{(j)}(\cdot,t)\|_1 \leq C\|\omega_0\|_{\mathcal{M}}^2 t^{-1/2}$$

(see the derivation preceding (4.23)) we obtain the equicontinuity of (5.7) from the uniform integrability of $\partial_t < \omega^{(j)}(\cdot,t),\psi >$ in $[0,T]$.

Finally, it remains to prove (5.5) and the uniqueness part (c). We use the integral equation (3.7). Note first that the heat kernel G satisfies, for any $\eta \in \mathcal{M}$,

(5.9)
$$\sup_{0<t<\infty} t^{1-\frac{1}{p}}\|G(\cdot,t)*\eta\|_p \leq c_p\|\eta\|_{\mathcal{M}}, \quad c_p = (4\pi)^{-(1-\frac{1}{p})}p^{-1/p}, \quad 1 \leq p \leq \infty,$$

and, with the atomic part $b = \eta_a$,

$$\limsup_{t\to 0} t^{1-\frac{1}{p}}\|G(\cdot,t)*\eta\|_p = c_p\|b\|_p, \quad 1 < p \leq \infty. \tag{5.10}$$

With $\epsilon_p > 0$ to be determined, take $\delta_p = \frac{\epsilon_p}{2c_p}$. The hypothesis $\|b\|_p < \delta_p$ then implies, by (5.10), that there exists $T > 0$ such that for the sequence $\omega_0^{(j)}$,

$$\sup_{0<t\leq T} t^{1-\frac{1}{p}} \|G(\cdot,t) * \omega_0^{(j)}\|_p \leq \rho < \frac{\epsilon_p}{2}, \tag{5.11}$$

where we have used the fact that $\omega_0^{(j)}$ are obtained from ω_0 by mollification (which commutes with $G*$). Arguing as in (3.10), with $\delta(t;K)$ replaced by a function $h(t) \leq \rho$, we obtain a solution to the integral equation (3.7) for $0 \leq t \leq T$ and $\frac{4}{3} \leq p < 2$, if ϵ_p is sufficiently small. Since $\omega_0^{(j)}$ is smooth, the solution is necessarily $\omega^{(j)}(\cdot,t)$, as constructed above. As in the derivation of (3.11), we have, for some $\rho' < \epsilon_p$,

$$\sup_{0<t\leq T} t^{1-\frac{1}{p}} \|\omega^{(j)}(\cdot,t)\|_p \leq \rho', \quad j = 1,2,\ldots, \quad \frac{4}{3} \leq p < 2,$$

hence also

$$\sup_{0<t\leq T} t^{1-\frac{1}{p}} \|\omega(\cdot,t)\|_p \leq \rho', \tag{5.12}$$

where $\omega(\cdot,t)$ is the solution constructed in the first part of the proof. Now if $\theta(\cdot,t)$ is another solution of Eq. (3.7), satisfying (5.12), we may proceed as in the uniqueness part in the proof of Theorem 4.2 (the argument following (4.26)) to obtain

$$\theta(\cdot,t) = \omega(\cdot,t) \qquad \text{for} \qquad t \leq T,$$

provided that ϵ_p satisfies

$$\epsilon_p \int_0^1 s^{-\frac{3}{2}+\frac{1}{p}} (1-s)^{-\frac{1}{2}} ds < 1. \tag{5.13}$$

We can then prove the identity $\theta = \omega$ for all time by proceeding stepwise.

Finally (5.5) follows readily from (5.12). □

Remark 5.3. *The smallness condition (5.6) has been extensively used in proving uniqueness for solutions of nonlinear parabolic equations (see "note added in proof" in* [6]*), and is commonly referred to as the "Kato–Fujita" condition. In Theorem 4.2 (i.e, for initial data in $L^1(\mathbb{R}^2)$) we have avoided it by assuming that the solution θ is in $C(\overline{\mathbb{R}}_+, L^1(\mathbb{R}^2)) \bigcap C(\mathbb{R}_+, L^\infty(\mathbb{R}^2))$, thus obtaining (4.28). The requirement $\theta(\cdot,t) \in L^\infty$ for $t>0$ can be considerably relaxed, still avoiding a "Kato-Fujita" condition. We refer to* [4], [7] *where similar uniqueness arguments have been used in the study of nonlinear parabolic equations.*

When the atomic part of the initial measure is not small, a suitable uniqueness condition is still unknown.

Remark 5.4. *We refer the reader to* [5] *where the results of this section are extended to well–posedness for initial data in functional spaces beyond* $\mathcal{M}(\mathbb{R}^2)$*. In fact, these functional spaces are defined by suitable restrictions of the action of the heat kernel on the initial data.*

6. Asymptotic behavior for large time

We assume now that $\omega_0 \in L^1(\mathbb{R}^2)$. Then, for any $t > 0$, the solution $\omega(\mathbf{x}, t)$ to (1.9)-(1.10) satisfies

$$\int_{\mathbb{R}^2} \omega(\mathbf{x}, t) d\mathbf{x} = \int_{\mathbb{R}^2} \omega_0(\mathbf{x}) d\mathbf{x}.$$

Thus, in general, there is no decay (for large time) in L^1 norm. On the other hand, by (3.4), the vorticity decays in all L^p norms, $p \in (0, \infty]$. As mentioned earlier (see Remark 3.1), the constant η in (3.4) (and the subsequent inequalities) can be replaced by 4π, thus equalizing the $L^1 - L^p$ estimates for vorticity with those of the heat equation. The proof of this improvement (see [9, Theorem 5]) is obtained by using a logarithmic Sobolev inequality instead of the Nash inequality used in (3.2).

As in the case of uniqueness arguments (see Remark 5.3), the methods used in the study of the vorticity equation (1.9) can be successfully applied in the study of various types of nonlinear parabolic equations (and vice versa). This is certainly true in what concerns large-time decay estimates. The study of such estimates for Navier-Stokes equations is well established ([10], [15], [33] and references there). We refer to [3], [9] for L^1 decay estimates of solutions to "viscous" Hamilton-Jacobi and conservation equations. In particular, in the two-dimensional case, the fact that equation (1.9) is scalar renders the vorticity a convenient object of study.

A solution to the heat equation in $\mathbb{R}^n$ decays in L^1 norm if the integral of the initial value vanishes. It is remarkable that a similar fact holds for vorticity in the two-dimensional case.

Theorem 6.1. *Consider the system (1.9)-(1.10) and assume that* $\omega_0 \in L^1(\mathbb{R}^2)$ *and* $\int_{\mathbb{R}^2} \omega_0(\mathbf{x}) d\mathbf{x} = 0$. *Then*

(a) $\lim_{t\to\infty} \|\omega(\cdot, t)\|_1 = 0.$

(b) Suppose in addition that ω_0 *is compactly supported. Then*

$$\lim_{t\to\infty} t^{1-\frac{1}{p}} \|\omega(\cdot, t)\|_p = 0, \qquad p \in [1, \infty] \tag{6.1}$$

Proof. We refer to [9, Theorem 4] for a proof of (a). The proof for (b) follows [15, Theorem 2.4].

Replacing (1.10) by

$$\mathbf{u}_0(\mathbf{x}) = \int_{\mathbb{R}^2} (\mathbf{K}(\mathbf{x}-\mathbf{y}) - \mathbf{K}(\mathbf{x}))\omega_0(\mathbf{y})d\mathbf{y}$$

we get $\mathbf{u}_0 \in L^2$. From the L^2 theory of the Navier-Stokes equations it follows that $\nabla \mathbf{u}(\mathbf{x},t) \in L^2(\mathbb{R}^2 \times \mathbb{R}_+)$, hence so is $\omega(\mathbf{x},t)$. In view of (3.1) the function $\|\omega(\cdot,t)\|_2$ is decreasing in t, so

$$t\|\omega(\cdot,t)\|_2^2 \leq 2\int_{t/2}^{t} \|\omega(\cdot,s)\|_2^2 ds$$

which proves (6.1) with $p=2$. To prove the case $p=1$ use the integral equation (3.7). Since the decay is known for the heat equation, we need only estimate the second term in the RHS of (3.7), in the $L^1(\mathbb{R}^2)$ norm. Denoting $\epsilon(s) = s^{\frac{1}{2}}\|\omega(\cdot,s)\|_2$ we have

$$\|\mathbf{u}(\cdot,s)\omega(\cdot,s)\|_1 \leq s^{-\frac{1}{2}}\epsilon(s)\|\mathbf{u}_0\|_2.$$

In view of (3.9) (with $r=1$) we conclude

$$\|\omega(\cdot,t) - G_\nu(\cdot,t) * \omega_0\|_1 \leq C\|\mathbf{u}_0\|_2 \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}}\epsilon(s)ds$$

and the RHS tends to 0 as $t \to \infty$ by the Lebesgue dominated convergence theorem. By interpolation we get (6.1) for $1 \leq p \leq 2$. The conclusion for $2 < p \leq \infty$ follows by duality, as in the proof of [1, Eq. (3.47)]. □

Remark 6.2. *The conclusion in part (b) of the theorem can be considerably improved. In fact, under the same assumptions (in fact, only exponential decay of ω_0 is required) we have*

$$\sup_{0\leq t<\infty} t^{\frac{1}{2}}\|\omega(\cdot,t)\|_1 < \infty$$

and combining this with (3.4) we obtain

$$\sup_{0<t<\infty} t^{\frac{3}{2}}\|\omega(\cdot,t)\|_\infty < \infty.$$

These estimates are identical to those obtained for the heat equation. We refer to [9, Theorem 4] *for details and sharp constants.*

The asymptotic behavior of solutions to the vorticity equation (1.9) can be studied in detail in terms of "scaling variables" ([10], [17], [15]). They are defined by

$$\boldsymbol{\xi} = (1+t)^{-\frac{1}{2}}\mathbf{x}, \qquad \tau = \ln(1+t).$$

Defining new functions $\mathbf{v}, \theta$ by

$$\mathbf{v}(\boldsymbol{\xi},\tau) = (1+t)^{\frac{1}{2}}\mathbf{u}(\mathbf{x},t), \qquad \theta(\boldsymbol{\xi},\tau) = (1+t)^{\frac{1}{2}}\omega(\mathbf{x},t)$$

and setting for simplicity $\nu = 1$, Eq. (1.9) is transformed into

$$\begin{aligned} &\partial_\tau \theta = \mathcal{L}\theta - (\mathbf{v}\cdot\nabla)\theta, \qquad\qquad \theta(\boldsymbol{\xi},0) = \theta_0(\boldsymbol{\xi}) \\ &\mathcal{L}\theta = \Delta\theta + \tfrac{1}{2}(\boldsymbol{\xi}\cdot\nabla)\theta + \theta \quad . \end{aligned} \tag{6.2}$$

(spatial derivatives are now with respect to $\boldsymbol{\xi}$). Clearly, the relation $\mathbf{v}(\cdot,\tau) = \mathbf{K} * \theta(\cdot,\tau)$ is still valid. The results of Sec. 4 yield readily the well-posedness of Eq. (6.2) in $L^1(\mathbb{R}^2)$, as well as decay estimates in τ. However, the interest in this transformed equation lies in its well-posedness in a scale of weighted-L^2 spaces defined as follows.

$$L^{2,s} = \Big\{ f, \qquad \|f\|^2_{L^{2,s}} := \int_{\mathbb{R}^2} (1+|\boldsymbol{\xi}|^2)^s |f(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \Big\}.$$

Proposition 6.3. *Equation (6.2) is well-posed in $L^{2,s}$ for any $s > 1$. More explicitly, for any $\theta_0 \in L^{2,s}$ there exists a unique global solution*

$$\theta(\cdot,\tau) \in C(\overline{\mathbb{R}}_+, L^{2,s}).$$

Furthermore, for any $M > 0$ there exists a constant $C = C(M,s)$ such that

$$\|\theta_0\|_{L^{2,s}} \le M \qquad \Rightarrow \|\theta(\cdot,\tau)\|_{L^{2,s}} \le C, \quad \tau \in [0,\infty)$$

and $C \to 0$ as $M \to 0$.

We refer to [15, Section 3] for a proof. Observe that $L^{2,s} \subseteq L^1$ if $s > 1$. By Proposition 6.3 it is a "persistence" space for the vorticity, in analogy with the space Y in Lemma 4.1.

The spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$ in $L^{2,s}$, for any $s > 0$, is given by (see [15, Appendix A])

$$\sigma(\mathcal{L}) = \Big\{ \lambda \in \mathbb{C}, \quad Re(\lambda) \le -\frac{s-1}{2} \Big\} \bigcup \Big\{ -\frac{k}{2}, \quad k = 0,1,2,\dots \Big\}.$$

In particular, for a fixed $s > 1$, the finite set of real nonpositive numbers

$$\Lambda(k) = \Big\{ \lambda_j = -\frac{j}{2},\ j = 0,1,\dots,k \Big\} \tag{6.3}$$

consists of isolated eigenvalues of $\mathcal{L}$ if $k < s-1$ (in the space $L^{2,s}$).

Gallay and Wayne [15] construct finite-dimensional invariant manifolds for the semiflow of Eq. (6.2) (which can easily be translated to the solutions of (1.9)), for sufficiently small initial data. It is based on this spectral structure, and on methods used in the study of dynamical systems. The construction can be described as follows.

Fix $k \in \mathbb{N}$ and $s \ge k+2$. Let $\mathcal{H}_k \subseteq L^{2,s}$ be the finite-dimensional subspace spanned by the eigenvectors associated with $\Lambda(k)$, and let $\mathcal{J}_k = L^{2,s} \ominus \mathcal{H}_k$ be its orthogonal complement. For $r > 0$ we denote by $\mathcal{B}_r$ the ball of radius r in $L^{2,s}$ (centered at 0).

Theorem 6.4. *Fix $r > 0$ sufficiently small and s, k as above. Let $\mu \in (\frac{k}{2}, \frac{k+1}{2})$.*

(a)There exists a globally Lipschitz C^1 map $g : \mathcal{H}_k \to \mathcal{J}_k$ such that $g(0) = 0,\ Dg(0) = 0$ and such that the manifold

$$\mathcal{T}_g = \{w + g(w), \quad w \in \mathcal{H}_k\}$$

is locally invariant in the following sense.
There exists $0 < r_1 < r$ such that the semiflow associated with (6.2), commencing at any point $\theta_0 \in \mathcal{T}_g \cap \mathcal{B}_{r_1}$ stays in $\mathcal{T}_g \cap \mathcal{B}_r$ for all $\tau \geq 0$.

(b)This invariant manifold "attracts" all trajectories having small initial data. More explicitly, for every $\theta_0 \in \mathcal{T}_g \cap \mathcal{B}_r$ there exists a manifold $\mathcal{S}_{\theta_0}$, such that all trajectories beginning at points of $\mathcal{S}_{\theta_0} \cap \mathcal{B}_{r_1}$ (with θ_0 restricted also to $\mathcal{B}_{r_1}$) approach the trajectory $\theta(\cdot, \tau)$ starting at θ_0. We have, if $\phi(\cdot, \tau)$ is a solution to (6.2), with $\phi(\cdot, 0) = \phi_0 \in \mathcal{S}_{\theta_0} \cap \mathcal{B}_{r_1}$,

$$\limsup_{\tau \to \infty} \tau^{-1} \ln \|\phi(\cdot, \tau) - \theta(\cdot, \tau)\|_{L^{2,s}} \leq -\mu. \tag{6.4}$$

(c) The manifold $\mathcal{S}_{\theta_0}$ is a continuous map of $\mathcal{J}_k$. It intersects $\mathcal{T}_g \cap \mathcal{B}_r$ only at θ_0 and the family

$$\{\mathcal{S}_{\theta_0}, \qquad \theta_0 \in \mathcal{T}_g \cap \mathcal{B}_r\}$$

is a foliation of $\mathcal{B}_{r_1}$.

We refer to [15, Section 3] for a proof of the theorem.

Remark 6.5. *Observe that the decay rate in Eq. (6.4) corresponds to a decay rate of $t^{-\mu}$ for solutions of the vorticity equation (1.9). Thus, for sufficiently small initial data in weighted-L^2 spaces, the asymptotic behavior of the vorticity is determined, to any order, by "finite-dimensional dynamics".*

Remark 6.6. *In analogy with Theorem 6.1, if $\int_{\mathbb{R}^2} \theta_0(\boldsymbol{\xi}) d\boldsymbol{\xi} = 0$ then $\|\theta(\cdot, \tau)\|_s \to 0$ as $\tau \to 0$ (see* [15, Theorem 3.2]*) and Theorem 6.4 can be applied to determine its asymptotic behavior. Note that in this case the velocity field is square-integrable (assuming $s > 1$).*

The Gaussian

$$G(\boldsymbol{\xi}) = (4\pi)^{-1} \exp(-\frac{|\boldsymbol{\xi}|^2}{4}), \qquad \boldsymbol{\xi} \in \mathbb{R}^2$$

is a stationary solution of (6.2) and an eigenfunction of $\mathcal{L}$ (with zero eigenvalue). In terms of the original vorticity, it corresponds to the solution of (1.9) obtained by the heat kernel with singularity at $t = -1$. It is called the "Oseen Vortex". Taking $k = 0$ and $s = 2$ in Theorem 6.4, it is easily seen that $\mathcal{H}_0$ is the one-dimensional subspace spanned by G and coincides with the invariant manifold $\mathcal{T}$ (i.e., $g \equiv 0$). Thus,combining Theorem 6.4 and the conservation of $\int_{\mathbb{R}^2} \theta(\boldsymbol{\xi}, \tau) d\boldsymbol{\xi}$ we get

Corollary 6.7 (stability of Oseen's vortex)**.** *Fix $0 < \mu < \frac{1}{2}$. There exists $r > 0$ such that if $\theta(\boldsymbol{\xi}, \tau)$ is a solution to (6.2) with $\|\theta_0\|_{L^{2,2}} < r$ and $\int_{\mathbb{R}^2} \theta_0(\boldsymbol{\xi}) d\boldsymbol{\xi} = a$ then*

$$\|\theta(\cdot, \tau) - aG(\cdot)\|_{L^{2,2}} \leq Ce^{-\mu\tau}, \qquad \tau \geq 0. \tag{6.5}$$

We refer to [15, Section 4] for a detailed analysis of this convergence.

7. Concluding remarks and open problems

It is common to say that the case of the Navier-Stokes equations in two-dimensional is "resolved". Admittedly, the situation here is much better than that of the 3-D case. Furthermore, the L^2 theory of existence and uniqueness is complete. However, as we have seen, there are important problems, related to "rough" initial data, that remain yet unresolved. Rather than "purely mathematical", they touch upon very relevant issues of fluid dynamics and numerical simulations of singular flows. Even in the (weighted)-L^2 context, the asymptotic results discussed in Section 6 show that the two-dimensional case still carries much interest. Another aspect of this interest is the (relatively) recent interplay between the methods used here and those used in the study of various classes of nonlinear parabolic equations.

In what follows we list a number of yet unresolved problems.

(1) **Uniqueness for measure-valued initial data with large atomic part**
As was mentioned in Section 5, the uniqueness of the solution to (1.9)-(1.10) when ω_0 is a measure with large atomic part is unknown. It seems that tools developed in this context could prove useful for other classes of nonlinear parabolic equations.

(2) **Uniform estimates with respect to ν and Euler's equations**
It is known that for smooth initial data one can obtain estimates which are uniform in $\nu \in (0, 1]$, where ν is the coefficient of viscosity (see [18], [31]). The solutions converge, as $\nu \to 0$, to the unique solution ("zero viscosity limit") to Euler's equations with the same initial data. When the initial data is not sufficiently smooth (say, in $L^1 \cap L^p,\ p > 2$) we can still obtain the convergence of a *subsequence* to a solution of Euler's equations. However, the uniqueness of such a solution is not known. Thus,

one might try to establish at least the uniqueness of the "zero viscosity limit".

(3) **The case of bounded domains**
In this case there is no existence theory for solutions of the Navier-Stokes equations in vorticity form (with "no-slip" boundary conditions), if the initial vorticity is only known to be in L^1. We refer to [32] for the case of measures as initial data, but with homogeneous boundary condition on the vorticity. As already mentioned in the Introduction, this is a case of prime importance in applications. Indeed, if this problem is ill-posed, then the numerical procedure of approximating singular vorticities by smooth ones needs to be justified.

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