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SCALAR CONSERVATION LAWS ON A HALF-LINE: A PARABOLIC APPROACH

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Abstract. The initial-boundary value problem for the (viscous) nonlinear scalar conservation law is considered,

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon, \quad x \in \mathbb{R}_+ = (0,\infty), \quad 0 \le t \le T, \quad \varepsilon > 0,$$
$$u^\varepsilon(x,0) = u_0(x),$$
$$u^\varepsilon(0,t) = g(t).$$

The flux $f(\xi) \in C^2(\mathbb{R})$ is assumed to be convex (but not strictly convex, i.e. $f''(\xi) \ge 0$). It is shown that a unique limit $u = \lim_{\varepsilon\to 0} u^\varepsilon$ exists. The classical duality method is used to prove uniqueness. To this end parabolic estimates for both the direct and dual solutions are obtained. In particular, no use is made of the Kružkov entropy considerations.

Keywords: Scalar conservation law; initial-boundary value problem; zero viscosity limit; uniqueness; duality method.

Mathematics Subject Classification 2010: Primary 35L65; Secondary 35K20

1. Introduction

Consider the (viscous) nonlinear conservation law

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon, \quad x \in \mathbb{R}_+ = (0,\infty), \quad 0 \le t \le T, \quad \varepsilon > 0, \tag{1.1}$$

subject to the initial condition

$$u^\varepsilon(x,0) = u_0(x), \tag{1.2}$$

and the boundary condition

$$u^\varepsilon(0,t) = g(t). \tag{1.3}$$

We assume that the flux $f(\xi) \in C^2(\mathbb{R})$ is convex, $f''(\xi) \geq 0$. In particular, we do not assume strict convexity.

The aim of this paper is to provide a *fully parabolic* proof of the uniqueness of the "zero viscosity" limit

$$u = \lim_{\varepsilon \to 0} u^\varepsilon,$$

which can then be viewed as the "physically correct" solution of the nonlinear conservation law

$$u_t + f(u)_x = 0.$$

In particular, no entropy considerations (in the sense of Kruzkov [13]) are employed. To the best of our knowledge, there has not been a previous attempt to prove the uniqueness of the vanishing viscosity limit without resorting to those entropy considerations.

This solution satisfies the initial condition (1.2), while the sense in which the boundary condition is satisfied (in the *inviscid case*) is more delicate.

Throughout the paper we employ the following standard notations: We denote by $\|\cdot\|_p$ the norm in $L^p(\mathbb{R}_+), 1 \leq p \leq \infty$.

For a normed (or semi-normed) space X, we denote by $C([0,T];X)$ (respectively, $B([0,T];X)$) the space of continuous (respectively, bounded) functions from $[0,T]$ into X.

We denote by $BV(\mathbb{R}_+)$ the space of functions of bounded variation on $\mathbb{R}_+$.

The basic theorem concerning the existence and uniqueness of solutions to (1.1) is the following.

Theorem 1.1. *Consider Eq.* (1.1) *subject to the initial condition* (1.2) *and the boundary condition* (1.3). *Assume that*

$$u_0 \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+) \cap BV(\mathbb{R}_+). \tag{1.4}$$

Fix any $T > 0$ and assume that $g \in C^1[0,T]$. Then, there exists a unique solution

$$\begin{aligned} u^\varepsilon \in C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}_+)) \cap B([0,T];L^1(\mathbb{R}_+)) \\ \cap B([0,T];L^\infty(\mathbb{R}_+)) \cap B([0,T];BV(\mathbb{R}_+)). \end{aligned}$$

The compactness properties of the family of solutions $(0 < \varepsilon < 1)$ are summarized in the following theorem.

Theorem 1.2. *Fix $T > 0$ and let $u^\varepsilon, 0 < \varepsilon < 1$, be the solution given in Theorem* 1.1, *where u_0 satisfies* (1.4) *and $g \in C^1[0,T]$.*

Then every subsequence $\{u^{\varepsilon_j}(x,t), \varepsilon_j \downarrow 0\}$, *of solutions to* (1.1), *subject to the initial condition* (1.2) *and the boundary condition* (1.3), *contains a subsequence* $\{u^{\varepsilon_{j_k}}(x,t)\}$ *converging in* $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}_+))$ *to a function*

$$u \in C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}_+)) \\ \cap B([0,T]; L^1(\mathbb{R}_+)) \cap B([0,T]; L^\infty(\mathbb{R}_+)) \cap B([0,T]; BV(\mathbb{R}_+)).$$

The function $u(x,t)$ *is a weak solution to the non-viscous equation*

$$u_t + f(u)_x = 0, \quad x \in \mathbb{R}_+, \quad 0 \le t \le T. \tag{1.5}$$

The proofs are standard, see e.g. [9, Chap. 2] or [11] (with suitable modifications due to the boundary).

As mentioned above, our purpose here is to provide a *fully parabolic* proof of the uniqueness of the "zero-viscosity" limit, as stated in the following theorem.

Theorem 1.3. *Let*

$$u_0 \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+) \cap BV(\mathbb{R}_+).$$

Fix any $T > 0$ *and let* $g \in C^1[0,T]$.

Then the weak solution u *obtained as a "zero viscosity" limit is unique. In other words, it is the limit*

$$u(x,t) = \lim_{\varepsilon \downarrow 0} u^\varepsilon(x,t),$$

in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}_+))$.

Remark 1.4. It is therefore justified to call this solution the "zero-viscosity" solution to (1.5). It is considered as the "physically meaningful" solution of the hyperbolic problem.

This solution satisfies the initial condition (1.2) but, as is well-known, the boundary condition (1.3) can only be satisfied on that part of the boundary where the characteristic lines "enter" the domain. However, such considerations are not used in our proof.

We emphasize that the uniqueness of the limit for the mixed initial-boundary value problem has already been established, even in more general settings, in the works of Bardos–Leroux–Nedelec [1], Dubois–LeFloch [5] and Otto [17]. We mention also the similar study of Benabdallah [2] concerning the "p system". However, these authors have used the classical entropy approach of Kruzkov [13]. In other words, certain "boundary entropy pairs" are introduced, and the uniqueness is obtained by invoking suitable entropy inequalities. These inequalities are applied to the "boundary trace" of the limit solution.

In contrast, our method of proof is "fully parabolic". In other words, we make no use of entropy functions but inspect more closely the dependence of the solutions u^ε on the parameter ε. We apply the classical "duality method", and in this sense we are closer to the method of Oleinik [16], [6, Chap. 3].

The basic estimates for the solution u^ε are derived in Sec. 2. In particular, these estimates include those needed in the proof of Theorems 1.1 and 1.2 (using "nice" initial and boundary data and an approximation argument).

In Sec. 3, we present our uniqueness proof, via the duality method. We first assume strict convexity of the flux function and obtain "Oleinik-type" upper bound estimates for the derivative of the solution u^ε up to the boundary (see Lemmas 3.3 and 3.7) as well as new surprising estimates for the dual solution (see Lemma 3.4 and Corollary 3.10). A stability argument then allows us to relax the strict convexity assumption.

We remark that in the simpler case of the pure initial-value problem, which we outline in Appendix A, the Oleinik estimate (for the hyperbolic case, $\varepsilon = 0$) is certainly known [6, Chap. 3], [4]. In fact, it was shown in [10] that it entails uniqueness if and only if the flux f is convex.

However, here we make use of it in a purely parabolic way. In particular, the estimate for the dual solution (Lemma A.5) serves as an essential ingredient in our proof.

2. Estimates for the Viscous Solution

Throughout this section we assume at least that

$$u_0 \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+) \cap BV(\mathbb{R}_+),$$

and that $g \in C^1[0, T]$.

If additional requirements are needed for certain estimates, they will be explicitly stated.

The solution u^ε is that given by Theorem 1.1.

Let

$$\begin{aligned} M_T &= \max\left(\sup_{0\le\tau\le T} |g(\tau)|, \|u_0\|_\infty\right), \\ N_T &= \max_{\xi\in[-M_T, M_T]} |f'(\xi)|. \end{aligned} \tag{2.1}$$

By the maximum principle, we have

$$|u^\varepsilon(x,t)| \le M_T, \quad (x,t) \in \mathbb{R}_+ \times [0,T].$$

We first estimate the L^1 norm of the solution as follows.

Lemma 2.1. *Let $0 < \varepsilon < 1$. We have*

$$\|u^\varepsilon(\cdot,t)\|_1 \le \|u_0\|_1 + C[|g(0)| + \int_0^t |g'(\tau)| d\tau], \quad 0 \le t \le T, \tag{2.2}$$

where $C > 0$ depends only on T, M_T.

It will be seen in the proof that the convexity of f is not used.

Proof. Let $\lambda(x) \in C_0^\infty[0,\infty)$ satisfy $\lambda(x) = 1$ for $0 \le x \le 1$ and $\lambda(x) = 0$ if $x > 3$. Assume in addition that $-1 \le \lambda'(x) \le 0$. Then (with $u^\varepsilon = u^\varepsilon(x,t)$ for simplicity)

$$\begin{aligned}(u^\varepsilon - \lambda(x)g(t))_t &+ (\nu^\varepsilon(x,t)(u^\varepsilon - \lambda(x)g(t)))_x - \varepsilon[u^\varepsilon - \lambda(x)g(t)]_{xx} \\ &= \varepsilon\lambda''(x)g(t) - \lambda(x)g'(t) - f'(\lambda(x)g(t))\lambda'(x)g(t),\end{aligned}$$

where

$$\nu^\varepsilon(x,t) = \frac{f(u^\varepsilon) - f(\lambda(x)g(t))}{u^\varepsilon - \lambda(x)g(t)} = \int_0^1 f'(\kappa u^\varepsilon + (1-\kappa)\lambda(x)g(t))d\kappa.$$

The linear semigroup associated with the equation $\frac{\partial}{\partial t}\psi(x,t) + \frac{\partial}{\partial x}(\nu^\varepsilon(x,t)\psi(x,t)) - \varepsilon\frac{\partial^2}{\partial x^2}\psi(x,t) = 0$, with zero boundary data, is an L^1-contraction. From Duhamel's principle, we infer

$$\begin{aligned}\|u^\varepsilon - \lambda(x)g(t)\|_1 &\le \|u_0 - \lambda(x)g(0)\|_1 \\ &\quad + \int_0^t [\|\lambda(x)g'(\tau) + \lambda'(x)g(\tau)f'(\lambda(x)g(\tau))\|_1 + \varepsilon\|\lambda''(x)g(\tau)\|_1]d\tau.\end{aligned}$$

Noting that $|f'(\lambda(x)g(t))| \le N_T$ and $|g(t)| \le |g(0)| + \int_0^t |g'(\tau)|d\tau$ we get (2.2). □

We next derive an estimate for the (spatial) total variation of the solution.

Remark. In what follows we perform formal integrations by parts, and take as zero boundary terms at infinity. These steps can be justified in a standard way by suitable smooth (and compactly supported) approximations.

We note that, as in the previous lemma, the proof of the following lemma does not use the convexity of f.

Lemma 2.2. *Assume that $u_0'(x) \in L^1(\mathbb{R}_+)$. Then, the total variation of u^ε can be estimated by the initial and boundary data as follows.*

$$\|u_x^\varepsilon(\cdot,t)\|_1 \le \|u_0'\|_1 + \int_0^t |g'(\tau)|d\tau, \quad 0 \le t \le T. \tag{2.3}$$

Proof. Differentiating (1.1) we get

$$(u_x^\varepsilon)_t + (f'(u^\varepsilon)u_x^\varepsilon)_x - \varepsilon(u_x^\varepsilon)_{xx} = 0. \tag{2.4}$$

Multiplying by $\mathrm{sgn}(u_x^\varepsilon)$ and using the inequality (in the sense of distributions)

$$|\theta|_{xx} \ge \mathrm{sgn}(\theta)\theta_{xx} \tag{2.5}$$

for any $\theta(x)$, we obtain

$$|u_x^\varepsilon|_t + (f'(u^\varepsilon)|u_x^\varepsilon|)_x - \varepsilon|u_x^\varepsilon|_{xx} \le 0. \tag{2.6}$$

Integrating with respect to x over $(0,\infty)$ we get

$$\frac{d}{dt}\|u_x^\varepsilon(\cdot,t)\|_1 \le [(f'(u^\varepsilon)|u_x^\varepsilon|) - \varepsilon|u_x^\varepsilon|_x]_{x=0} = -\mathrm{sgn}(u_x^\varepsilon(0,t))g'(t), \tag{2.7}$$

which yields (2.3) by integration. □

In order to study the continuity of the map $t \to u^\varepsilon(\cdot,t) \in L^1(\mathbb{R}_+)$ for smoother initial and boundary data, we need to estimate $\|u_t^\varepsilon(\cdot,t)\|_1$.

Lemma 2.3. *Assume that (in addition to the hypotheses of Theorem 1.1) $u_0', u_0'' \in L^1(\mathbb{R}_+), g'' \in L^1(0,T)$. Let $0 < \varepsilon < 1$. Then, there exists a constant $C > 0$, depending only on T, M_T, such that*

$$\|u_t^\varepsilon(\cdot,t)\|_1 \le 3\varepsilon\left(2|g'(0)| + \|u_0''\|_1 + 2\int_0^t |g''(\tau)|d\tau\right) + C\left\{\left(1 + \int_0^t |g'(\tau)|d\tau\right)(1 + \|u_0'\|_1) + \left(\int_0^t |g'(\tau)|d\tau\right)^2\right\}, \quad 0 \le t \le T. \tag{2.8}$$

Proof. Differentiating (1.1) with respect to t we get

$$(u_t^\varepsilon)_t + (f'(u^\varepsilon)u_t^\varepsilon)_x - \varepsilon(u_t^\varepsilon)_{xx} = 0.$$

Let $\lambda^\varepsilon(x) = \lambda(\frac{x}{\varepsilon})$, where $\lambda(x)$ is the function introduced above (see the proof of Lemma 2.1). The equation can be rewritten as

$$\begin{aligned}(u_t^\varepsilon - \lambda^\varepsilon(x)g'(t))_t &+ [f'(u^\varepsilon)(u_t^\varepsilon - \lambda^\varepsilon(x)g'(t))]_x - \varepsilon(u_t^\varepsilon - \lambda^\varepsilon(x)g'(t))_{xx} \\ &= \varepsilon(\lambda^\varepsilon)''(x)g'(t) - \lambda^\varepsilon(x)g''(t) - (f'(u^\varepsilon)\lambda^\varepsilon(x)g'(t))_x.\end{aligned} \tag{2.9}$$

The linear semigroup associated with the equation $\frac{\partial}{\partial t}\psi(x,t) + \frac{\partial}{\partial x}(f'(u^\varepsilon)\psi(x,t)) - \varepsilon\frac{\partial^2}{\partial x^2}\psi(x,t) = 0$, with zero boundary data, is an L^1-contraction. Thus, as in the proof of Lemma 2.1, we get

$$\begin{aligned}&\|u_t^\varepsilon - \lambda^\varepsilon(x)g'(t)\|_1 \\ &\quad\le \|u_t^\varepsilon(x,0) - \lambda^\varepsilon(x)g'(0)\|_1 + 3\varepsilon\int_0^t |g''(\tau)|d\tau + \int_0^3 |\lambda''(x)|dx\int_0^t |g'(\tau)|d\tau \\ &\qquad + \int_0^t |g'(\tau)|\|(\lambda^\varepsilon(\cdot)f'(u^\varepsilon)(\cdot,\tau))_x\|_1 d\tau.\end{aligned}$$

In view of Lemma 2.2, we have

$$\|(\lambda^\varepsilon(\cdot)f'(u^\varepsilon)(\cdot,\tau))_x\|_1 \le B(\|u_0'\|_1 + \int_0^\tau |g'(s)|ds) + 3N_T, \tag{2.10}$$

where $B = \max_{[-M_T,M_T]} f''(\xi)$.

We note that

$$2|g'(\tau)|\int_0^\tau |g'(s)|ds = \frac{d}{d\tau}\left(\int_0^\tau |g'(s)|ds\right)^2.$$

Combining this with $u_t^\varepsilon(x,0) = -f'(u_0(x))u_0'(x) + \varepsilon u_0''(x)$, we obtain (2.8). ☐

Corollary 2.4. *Let $0 < \varepsilon < 1$. Then, under the assumptions imposed in Lemma 2.3, there exists a constant $L > 0$, depending on T and on u_0, g, up to second-order derivatives, such that*

$$\|u^\varepsilon(\cdot,t_2) - u^\varepsilon(\cdot,t_1)\|_1 \le L(t_2 - t_1), \quad 0 \le t_1 < t_2 \le T.$$

An interesting and useful estimate that can be derived from the estimate (2.8) is the following.

Lemma 2.5. *Let u_0, g satisfy the assumptions imposed in Lemma 2.3. There exists a constant $C > 0$, depending on T and on u_0, g (up to second-order derivatives), but not on $\varepsilon > 0$, such that*

$$\varepsilon|u_x^\varepsilon(x,t)| \le C, \quad (x,t) \in \mathbb{R}_+ \times [0,T], \quad 0 < \varepsilon < 1. \tag{2.11}$$

Proof. Clearly it suffices to have such an estimate for $\|u_{xx}^\varepsilon\|_1$. However, using Eq. (1.1), this follows from the estimates in Lemmas 2.2 and 2.3. □

The method used in the proof of Lemma 2.2 can be used to obtain a "stability" result for the solution in terms of the initial and boundary data.

Lemma 2.6. *Assume $0 < \varepsilon < 1$. Let u^ε be a solution to (1.1), subject to the boundary condition (1.3). Let $\widetilde{u^\varepsilon}$ be a solution to the same equation but subject to the initial and boundary conditions*

$$\widetilde{u^\varepsilon}(0,x) = \tilde{u}_0(x), \quad \widetilde{u^\varepsilon}(0,t) = \tilde{g}(t).$$

Let $M_T' = \max(M_T, \widetilde{M_T})$, where

$$\widetilde{M_T} = \max\left(\sup_{0\le\tau\le T} |\tilde{g}(\tau)|, \|\tilde{u}_0\|_\infty\right).$$

Then, there exists a constant $C > 0$, depending only on T, M_T', such that

$$\begin{aligned}\|u^\varepsilon(\cdot,t) - \widetilde{u^\varepsilon}(\cdot,t)\|_1 &\le \|u_0 - \tilde{u}_0\|_1 \\ &\quad + CK(|g(0) - \tilde{g}(0)| + \int_0^t |g'(\tau) - \tilde{g}'(\tau)|d\tau), \quad 0 \le t \le T,\end{aligned} \tag{2.12}$$

where $K = 1 + \|u_0'\|_1 + \|\tilde{u}_0'\|_1 + \int_0^t (|g'(\tau)| + |\tilde{g}'(\tau)|)d\tau$.

Proof. Let $z = u^\varepsilon - \widetilde{u^\varepsilon}$. It satisfies the equation

$$z_t + (r^\varepsilon z)_x = \varepsilon z_{xx},$$

where

$$r^\varepsilon(x,t) = \frac{f(u^\varepsilon) - f(\widetilde{u^\varepsilon})}{u^\varepsilon - \widetilde{u^\varepsilon}} = \int_0^1 f'(\kappa u^\varepsilon + (1-\kappa)\widetilde{u^\varepsilon})d\kappa.$$

This equation can be rewritten as

$$\begin{aligned}&(z - \lambda(x)\delta g(t))_t + [r^\varepsilon(x,t)(z - \lambda(x)\delta g(t))]_x - \varepsilon(z - \lambda(x)\delta g(t))_{xx} \\ &\quad = -\lambda(x)\delta g'(t) - \delta g(t)(\lambda(x) r^\varepsilon(x,t))_x + \varepsilon\lambda''(x)\delta g(t),\end{aligned}$$

where $\delta g(t) = g(t) - \tilde{g}(t)$ and $\lambda(x)$ is as above (see the proof of Lemma 2.1).

The linear semigroup associated with the equation $\frac{\partial}{\partial t}\psi(x,t)+\frac{\partial}{\partial x}(r^\varepsilon\psi(x,t))-\varepsilon\frac{\partial^2}{\partial x^2}\psi(x,t)=0$, with zero boundary data, is an L^1-contraction, so we have

$$\begin{aligned}
&\|z(\cdot,t)-\lambda(\cdot)\delta g(t)\|_1\\
&\quad\leq \|u_0-\tilde{u}_0-\lambda(x)\delta g(0)\|_1\\
&\qquad+\int_0^t [|\delta g(\tau)|\|(\lambda(\cdot)r^\varepsilon(\cdot,\tau))_x\|_1+\|\lambda(\cdot)\delta g'(\tau)\|_1+\varepsilon\|\lambda''(\cdot)\delta g(\tau)\|_1]d\tau. \qquad (2.13)
\end{aligned}$$

Noting that

$$r_x^\varepsilon(x,t)=\int_0^1 f''(\kappa u^\varepsilon+(1-\kappa)\widetilde{u^\varepsilon})(\kappa u_x^\varepsilon+(1-\kappa)\widetilde{u^\varepsilon}_x)d\kappa,$$

and using Lemmas 2.2 and 2.1, we have

$$\|(\lambda(\cdot)r^\varepsilon(\cdot,\tau))_x\|_1\leq CK.$$

Clearly

$$|\delta g(\tau)|\leq |g(0)-\tilde{g}(0)|+\int_0^\tau |g'(s)-\tilde{g}'(s)|ds.$$

Inserting these two estimates in (2.13) we get (2.12). □

3. Uniqueness of the Solution

The proof of Theorem 1.3 is given now, in several steps. It is based on the classical duality approach.

We first assume that the flux function is *strictly convex*,

$$A=\min_{\xi\in\mathbb{R}} f''(\xi)>0. \qquad (3.1)$$

At the very last step of the proof this assumption is relaxed (allowing convex fluxes which are not strictly convex).

Without loss of generality we assume that there exists a point $b\in\mathbb{R}$ such that $f'(b)=0$. Indeed, otherwise we can modify f outside the interval $[-M_T,M_T]$ (see Eq. (2.1)).

By a suitable shift we can therefore assume

$$f(0)=f'(0)=0. \qquad (3.2)$$

Some estimates obtained in the course of the proof are of interest in their own right, as they represent "parabolic versions" of the well-known hyperbolic "entropy inequalities".

3.1. *The case of nonnegative boundary data,* $g(t)\geq 0$

In addition to the hypotheses on u_0, g in Theorem 1.1, we assume here that $g\in C^2[0,T]$ and that $u_0', u_0''\in L^1(\mathbb{R}_+)$. Thus, all the estimates obtained in Sec. 2 can be used here. Observe that $u^\varepsilon, u_x^\varepsilon$ decay as $x\to\infty$ (to see this it suffices to note the integrability results of Lemmas 2.2 and 2.3).

For notational simplicity, we shall also use, in addition to u^ε (solution to (1.1)), a solution to the same equation, subject to the same initial and boundary data, but with a viscosity coefficient μ instead of ε.

So let $v^\mu(x,t)$ be a solution of

$$v_t^\mu + f(v^\mu)_x = \mu v_{xx}^\mu, \quad x \in \mathbb{R}_+, \quad 0 \le t \le T, \quad \mu > 0, \tag{3.3}$$

subject to the same initial and boundary conditions ((1.2) and (1.3)).

Let $w^{\mu,\varepsilon} = u^\varepsilon - v^\mu$. It satisfies the equation

$$w_t^{\mu,\varepsilon} + (b^{\mu,\varepsilon} w^{\mu,\varepsilon})_x - \varepsilon w_{xx}^{\mu,\varepsilon} = (\varepsilon - \mu) v_{xx}^\mu, \tag{3.4}$$

where

$$b^{\mu,\varepsilon}(x,t) = \frac{f(u^\varepsilon(x,t)) - f(v^\mu(x,t))}{u^\varepsilon(x,t) - v^\mu(x,t)}. \tag{3.5}$$

We have $w^{\mu,\varepsilon}(0,t) = 0, w^{\mu,\varepsilon}(x,0) = 0$. We need to show that $w^{\mu,\varepsilon} \to 0$ a.e. in $\mathbb{R}_+ \times [0,T]$ as $\mu, \varepsilon \to 0$ (along any two decreasing sequences).

3.1.1. *The dual equation*

Let $b^{\mu,\varepsilon}$ be given by (3.5) and consider the dual equation

$$\phi_t^{\mu,\varepsilon} + b^{\mu,\varepsilon}\phi_x^{\mu,\varepsilon} + \varepsilon \phi_{xx}^{\mu,\varepsilon} = 0, \quad x \in \mathbb{R}_+, \quad 0 \le t \le T, \tag{3.6}$$

with "terminal" condition

$$\phi^{\mu,\varepsilon}(x,T) = \phi_T(x) \in C_0^\infty(\mathbb{R}_+),$$

and boundary condition

$$\phi^{\mu,\varepsilon}(0,t) = 0, \quad 0 \le t \le T.$$

By standard estimates, $\phi^{\mu,\varepsilon}(x,t)$, as well as its derivatives, decay exponentially as $x \to \infty$, uniformly in $t \in [0,T]$, for every fixed $\varepsilon > 0$. In particular, the only boundary terms in the integrations below are those at $x = 0$.

The obvious maximum principle

$$\sup_{[0,\infty)\times(0,T)} |\phi^{\mu,\varepsilon}(x,t)| \le \|\phi_T\|_\infty,$$

can be strengthened as follows.

Lemma 3.1. *The total variation $TV(\phi^{\mu,\varepsilon}(\cdot,t)) = \int_0^\infty |\phi_x^{\mu,\varepsilon}(x,t)|dx$ is an increasing function of $t \in [0,T]$.*

Proof. Differentiating Eq. (3.6) and multiplying by $\operatorname{sgn}(\phi_x^{\mu,\varepsilon})$, we get

$$|\phi_x^{\mu,\varepsilon}|_t + (b^{\mu,\varepsilon}|\phi_x^{\mu,\varepsilon}|)_x = -\varepsilon \phi_{xxx}^{\mu,\varepsilon}\operatorname{sgn}(\phi_x^{\mu,\varepsilon}) \ge -\varepsilon|\phi_x^{\mu,\varepsilon}|_{xx}, \tag{3.7}$$

where in the last step we have used $|\theta|_{xx} \ge \operatorname{sgn}(\theta)\theta_{xx}$ for any $\theta(x)$.

Observe that at $x = 0$ we have

$$b^{\mu,\varepsilon}|\phi_x^{\mu,\varepsilon}| + \varepsilon|\phi_x^{\mu,\varepsilon}|_x = \mathrm{sgn}(\phi_x^{\mu,\varepsilon})[b^{\mu,\varepsilon}\phi_x^{\mu,\varepsilon} + \varepsilon\phi_{xx}^{\mu,\varepsilon}] = -\mathrm{sgn}(\phi_x^{\mu,\varepsilon})\phi_t^{\mu,\varepsilon} = 0.$$

Thus integrating (3.7) over $\mathbb{R}_+ \times [t_1, t_2]$ for any two values $0 \le t_1 < t_2 \le T$ concludes the proof. □

Remark 3.2. Note that Lemma 3.1 implies that the total variation of $\phi^{\mu,\varepsilon}(\cdot, t)$ is bounded by that of ϕ_T.

3.1.2. *The entropy inequality and its dual analog*

We show that the following "entropy" inequality (in the sense of Oleinik) holds, when a suitable positivity condition is imposed on the boundary data g.

Lemma 3.3. *Assume $g(t) \ge 0$ and that for some $K > 0$*

$$g'(t) + Kg(t) \ge 0. \tag{3.8}$$

Then

$$u_x^\varepsilon(x,t) \le Et^{-1}, \quad x \in \mathbb{R}_+, \quad 0 < t \le T, \tag{3.9}$$

where $E \ge A^{-1}$ depends on K but is independent of $\varepsilon > 0$, and of the initial data.

Proof. We first show that $u_x^\varepsilon(x,t) - (At)^{-1}$ cannot have an interior positive maximum at any $x > 0, 0 < t \le T$. Indeed, the function $u_x^\varepsilon(x,t) - (At)^{-1}$ satisfies the equation (compare (2.4))

$$\begin{aligned}(u_x^\varepsilon - (At)^{-1})_t + f'(u^\varepsilon)(u_x^\varepsilon - (At)^{-1})_x - \varepsilon(u_x^\varepsilon - (At)^{-1})_{xx} \\ = A^{-1}t^{-2} - f''(u^\varepsilon)(u_x^\varepsilon)^2. \end{aligned} \tag{3.10}$$

Let $\gamma = \max_{[0,\infty)\times[0,T]} |u_x^\varepsilon|$.

Assume to the contrary that,

$$u_x^\varepsilon(x_0, t_0) - (At_0)^{-1} = \max_{[0,\infty)\times[0,T]} (u_x^\varepsilon - (At)^{-1}) > 0,$$

for some

$$(x_0, t_0) \in \mathbb{R}_+ \times [\delta, T], \quad 0 < \delta < (\gamma A)^{-1}.$$

Clearly $t_0 > \delta$ and at (x_0, t_0) the left-hand side of (3.10) is nonnegative while

$$[A^{-1}t^{-2} - f''(u^\varepsilon)(u_x^\varepsilon)^2]_{(x_0,t_0)} < A^{-1}t_0^{-2} - A(At_0)^{-2} = 0.$$

Thus, it remains to check the function $u_x^\varepsilon(x,t) - (At)^{-1}$ at the boundary $x = 0$.

Define the function

$$\psi(x,t) = u^\varepsilon(x,t) - g(t) - \frac{Ex}{t}, \quad t > 0,$$

where $E > 0$ will be determined in the process of the proof.

In view of Eq. (1.1) it satisfies

$$\psi_t + f'(u^\varepsilon)\psi_x - \varepsilon\psi_{xx} = -g'(t) + \frac{Ex}{t^2} - \frac{E}{t}f'(u^\varepsilon). \tag{3.11}$$

Suppose that ψ has a positive maximum at $(x_0, t_0) \in (0,\infty) \times (0,T]$. Then the left-hand side of Eq. (3.11) is nonnegative at this point.

On the other hand, since $f'' \geq A$, and $g(t_0) \geq 0$ by assumption,

$$f'(u^\varepsilon(x_0,t_0)) > f'\left(g(t_0) + \frac{Ex_0}{t_0}\right) \geq A\left(g(t_0) + \frac{Ex_0}{t_0}\right),$$

and for right-hand side of Eq. (3.11) we have at this point,

$$-g'(t_0) + \frac{Ex_0}{t_0^2} - \frac{E}{t_0}f'(u^\varepsilon(x_0,t_0)) < -g'(t_0) + \frac{Ex_0}{t_0^2} - \frac{EA}{t_0}\left(g(t_0) + \frac{Ex_0}{t_0}\right).$$

We now take $E > 0$ so large that

$$EA \geq 1, \quad \frac{EA}{T} \geq K,$$

where K is as in the above assumption on g.

It follows that for this choice of E the right-hand side of Eq. (3.11) is negative at (x_0, t_0), which is a contradiction.

Since $\psi(0,t) = 0$, and $\lim_{t\to 0+}\psi(x,t) = -\infty$, we conclude that $\psi(x,t) \leq 0$.

In particular, we obtain on the boundary

$$u_x^\varepsilon(0,t) \leq \frac{E}{t}, \quad t > 0.$$

Recalling the fact that $u_x^\varepsilon - \frac{1}{At}$ does not have an interior positive maximum, the proof is complete. □

Let $v^\mu(x,t)$ be the solution to (3.3) subject to the same initial and boundary conditions ((1.2) and (1.3)). In particular, it satisfies the same entropy inequality (3.9).

Let $w^{\mu,\varepsilon} = u^\varepsilon - v^\mu$. It satisfies Eq. (3.4), with $b^{\mu,\varepsilon}(x,t)$ as in Eq. (3.5), and $w^{\mu,\varepsilon}(0,t) \equiv 0, w^{\mu,\varepsilon}(x,0) \equiv 0$. We have

$$b^{\mu,\varepsilon} = \int_0^1 f'(\kappa u^\varepsilon + (1-\kappa)v^\mu)d\kappa.$$

It follows from the entropy inequality that also

$$b_x^{\mu,\varepsilon}(x,t) < st^{-1}, \quad x \in \mathbb{R}_+, \quad 0 < t \leq T, \tag{3.12}$$

for $s > BE > 1, \quad B = \max_{[-M_T,M_T]} f''(\xi)$. In particular, s is independent of ε, μ.

Lemma 3.4. *Assume that g satisfies the condition in Lemma 3.3. Assume further that $\phi_T \geq 0$. Then, the x-derivative of $\phi^{\mu,\varepsilon}$ satisfies the estimate*

$$t^s \phi_x^{\mu,\varepsilon}(x,t) \geq -T^s \|\phi_T'\|_\infty, \quad x \in \mathbb{R}_+, \quad 0 < t \leq T. \tag{3.13}$$

Proof. Differentiating (3.6) with respect to x and multiplying by t^s, we get

$$(\phi_x^{\mu,\varepsilon} t^s)_t + \left(b_x^{\mu,\varepsilon} - \frac{s}{t}\right) \phi_x^{\mu,\varepsilon} t^s + b^{\mu,\varepsilon} (\phi_x^{\mu,\varepsilon} t^s)_x + \varepsilon (\phi_x^{\mu,\varepsilon} t^s)_{xx} = 0,$$
$$x \in \mathbb{R}_+, \quad 0 < t \leq T. \tag{3.14}$$

We check that $\phi_x^{\mu,\varepsilon} t^s$ has no negative minimum in $\mathbb{R}_+ \times [\tau, T)$ for any $\tau \in (0,T)$. Indeed, suppose to the contrary that the function has a negative minimum at $(x_0, \tau_0) \in \mathbb{R}_+ \times [\tau, T)$. However, at this point the left-hand side of (3.14) is strictly positive, in view of (3.12).

On the other hand, the maximum principle, the zero boundary condition for $\phi^{\mu,\varepsilon}$ and the assumption that $\phi_T \geq 0$ imply that $\phi^{\mu,\varepsilon} \geq 0$ and in particular

$$\phi_x^{\mu,\varepsilon}(0,t) \geq 0,$$

which concludes the proof. □

3.1.3. *Concluding the uniqueness argument, $g \geq 0$*

Multiplying (3.4) by $\phi^{\mu,\varepsilon}$, integrating over $\mathbb{R}_+ \times [0,T]$ and noting (3.6), we obtain

$$\int_0^\infty w^{\mu,\varepsilon}(x,T)\phi_T(x)dx = (\mu - \varepsilon) \int_0^T \int_0^\infty \phi_x^{\mu,\varepsilon}(x,t) v_x^\mu(x,t) dx dt. \tag{3.15}$$

We now estimate the right-hand side of (3.15).

Lemma 3.5. *Assume that g satisfies the condition in Lemma 3.3. Assume further that $\phi_T \geq 0$. Then for any two sequences $\mu_i > \varepsilon_i > 0, \quad \mu_i \to 0$,*

$$\limsup_{\mu_i \to 0} (\mu_i - \varepsilon_i) \int_0^T \int_0^\infty \phi_x^{\mu_i,\varepsilon_i}(x,t) v_x^{\mu_i}(x,t) dx dt \leq 0. \tag{3.16}$$

Proof. In view of Remark 3.2 and the estimate (2.11) (applied to v^μ) we have, for $0 < \delta < T$,

$$|(\mu_i - \varepsilon_i) \int_0^\delta \int_0^\infty \phi_x^{\mu_i,\varepsilon_i}(x,t) v_x^{\mu_i}(x,t) dx dt| \leq C \|\phi_T'\|_1 \delta, \tag{3.17}$$

where $C > 0$ is independent of μ, ε, δ. Denote

$$D_1^{\mu,\varepsilon} = \{(x,t) \in \mathbb{R}_+ \times [\delta, T], \quad \phi_x^{\mu,\varepsilon}(x,t) > 0, v_x^\mu(x,t) > 0\},$$
$$D_2^{\mu,\varepsilon} = \{(x,t) \in \mathbb{R}_+ \times [\delta, T], \quad \phi_x^{\mu,\varepsilon}(x,t) < 0, v_x^\mu(x,t) < 0\}.$$

Clearly

$$\int_\delta^T\int_0^\infty \phi_x^{\mu,\varepsilon}(x,t)v_x^\mu(x,t)dxdt$$
$$\leq \iint\limits_{D_1^{\mu,\varepsilon}} \phi_x^{\mu,\varepsilon}(x,t)v_x^\mu(x,t)dxdt + \iint\limits_{D_2^{\mu,\varepsilon}} \phi_x^{\mu,\varepsilon}(x,t)v_x^\mu(x,t)dxdt. \qquad (3.18)$$

Now, for $(x,t) \in D_1^{\mu,\varepsilon}$, we have, due to Lemma 3.3,

$$0 < \phi_x^{\mu,\varepsilon}(x,t)v_x^\mu(x,t) \leq |\phi_x^{\mu,\varepsilon}(x,t)|\frac{E}{t} \leq |\phi_x^{\mu,\varepsilon}(x,t)|\frac{E}{\delta}.$$

For $(x,t) \in D_2^{\mu,\varepsilon}$, we have, by Lemma 3.4,

$$0 < \phi_x^{\mu,\varepsilon}(x,t)v_x^\mu(x,t) \leq \left(\frac{T}{t}\right)^s \|\phi_T'\|_\infty |v_x^\mu(x,t)| \leq \left(\frac{T}{\delta}\right)^s \|\phi_T'\|_\infty |v_x^\mu(x,t)|.$$

Thus, by (3.18) and the above estimates,

$$(\mu_i - \varepsilon_i)\int_\delta^T\int_0^\infty \phi_x^{\mu_i,\varepsilon_i}(x,t)v_x^{\mu_i}(x,t)dxdt$$
$$\leq (\mu_i - \varepsilon_i)\left(\frac{E}{\delta}\int_\delta^T \|\phi_x^{\mu_i,\varepsilon_i}(\cdot,t)\|_1 dt + \left(\frac{T}{\delta}\right)^s \|\phi_T'\|_\infty \int_\delta^T \|v_x^{\mu_i}(\cdot,t)\|_1 dt\right)$$

In view of Remark 3.2 and Lemma 2.2 (applied to v^μ), we have

$$\limsup_{\mu_i\to 0}(\mu_i - \varepsilon_i)\int_\delta^T\int_0^\infty \phi_x^{\mu_i,\varepsilon_i}(x,t)v_x^{\mu_i}(x,t)dxdt \leq 0,$$

and combining this with (3.17) we obtain (3.16). □

We can now complete the proof of the uniqueness Theorem 1.3 in this case (subject to the additional restrictions imposed on the initial and boundary conditions) as follows.

Let u^ε be a solution to (1.1) and let v^μ be a solution to (3.3), both subject to the same initial and boundary conditions ((1.2) and (1.3)).

Assume that $u^{\varepsilon_i} \underset{\varepsilon_i\to 0}{\to} \tilde{u}$, $\quad v^{\mu_i} \underset{\mu\to 0}{\to} \tilde{v}$. We want to show that $\tilde{u} \equiv \tilde{v}$.

Assume first that the boundary function g satisfies the condition in Lemma 3.3.

Let $\tilde{w} = \tilde{u} - \tilde{v}$. Let $0 \leq \phi_T \in C_0^\infty(\mathbb{R}_+)$. In view of Eq. (3.15) and Lemma 3.5, we have

$$\int_0^\infty \tilde{w}(x,T)\phi_T(x)dx \leq 0,$$

which implies $(\tilde{w})^+(x,T) = \max(\tilde{w}(x,T),0) = 0$. Since the roles of $\tilde{u}$ and $\tilde{v}$ can be interchanged, it follows that $\tilde{w} = 0$.

Now take a general function $g \geq 0$, which we assume to be (at least) continuously differentiable on $[0,T]$. We denote by $g^\theta = g + \theta$, for any nonnegative constant θ. Clearly, if $\theta > 0$, the condition of Lemma 3.3 is satisfied, hence, by the above argument, the solution $u^{\varepsilon,\theta}$ of (1.1), subject to the boundary condition g^θ, has a unique limit as $\varepsilon \to 0$. In view of the stability Lemma 2.6, the set of values of θ for which the solution $u^{\varepsilon,\theta}$ of (1.1) has a unique "zero viscosity" limit is closed. Thus, we can take $\theta = 0$. This completes the proof for $g \geq 0$.

3.2. *The case of nonpositive boundary data, $g(t) \leq 0$*

In addition to the hypotheses imposed in Theorem 1.1, we assume that the initial function

$$u_0 \in C_0^2(c,d), \tag{3.19}$$

where $[c,d] \subseteq (0,\infty)$, and that $g \in C^2[0,T]$. These assumptions are made in order to guarantee a rapid decay of the solutions (as $|x| \to \infty$) and will eventually be relaxed. Recall that we are still assuming the strict convexity (3.1) of the flux function.

Lemma 3.6. *For a suitable constant $C > 0$, the function $\theta(x,t) = \frac{Cx}{t}$ is a super-solution to* (1.1) *in $\mathbb{R}_+ \times (0,T]$, for all $0 < \varepsilon < 1$.*

Proof. We note that

$$f(\theta(x,t))_x = \frac{C}{t} f'(\theta(x,t)) \geq A\frac{C^2 x}{t^2},$$

where we have used the strict convexity assumption (3.1).

It follows that

$$\theta_t + f(\theta)_x - \varepsilon\theta_{xx} \geq \left(-\frac{C}{t^2} + A\frac{C^2}{t^2}\right) x \geq 0, \quad (x,t) \in \mathbb{R}_+ \times (0,T],$$

provided we take $C > A^{-1}$. □

Lemma 3.7. *Let u^ε be a solution to* (1.1), *where the initial function satisfies* (3.19) *and the boundary function g is nonpositive. Then, for $0 < \varepsilon < 1$, and $C > 0$ as in Claim* 3.6,

$$u^\varepsilon \leq \frac{Cx}{t}, \quad (x,t) \in \mathbb{R}_+ \times (0,T].$$

Proof. For any fixed $0 < \varepsilon < 1$ there exists $\tau > 0$ such that

$$u^\varepsilon(x,t) \leq \frac{Cx}{t}, \quad x \in \mathbb{R}_+, \quad t \in (0,\tau), \tag{3.20}$$

where we take into account the assumption $g(t) \leq 0$ and Lemma 2.5.

The assertion of the lemma now follows from the previous claim and the comparison principle. □

Our approach here is to show that the effect of the boundary data "disappears" as $\varepsilon \to 0$, namely, that the limiting solution will be identical to that obtained with $g \equiv 0$. This is understandable from the "hyperbolic point-of-view", since the characteristic lines from the boundary "escape to the left" (as t increases).

To that end, we take another boundary function $0 \geq \tilde{g}(t) \geq g(t), t \in [0, T]$. Let $\widetilde{u^\varepsilon}(x,t)$ be the solution to (1.1), subject to the initial data (1.2) (which satisfies (3.19)), but with boundary condition $\tilde{g}$.

The difference $w^\varepsilon = \widetilde{u^\varepsilon} - u^\varepsilon$ satisfies the equation

$$w_t^\varepsilon + (r^\varepsilon w^\varepsilon)_x = \varepsilon w_{xx}^\varepsilon, \quad (x,t) \in \mathbb{R}_+ \times (0,T), \tag{3.21}$$

where

$$r^\varepsilon(x,t) = \frac{f(\widetilde{u^\varepsilon}) - f(u^\varepsilon)}{\widetilde{u^\varepsilon} - u^\varepsilon} = \int_0^1 f'(\kappa \widetilde{u^\varepsilon}(x,t) + (1-\kappa)u^\varepsilon(x,t))d\kappa,$$
$$(x,t) \in \mathbb{R}_+ \times (0,T). \tag{3.22}$$

By the comparison principle, $w^\varepsilon \geq 0$. Furthermore, in view of Lemma 3.7 we have

$$r^\varepsilon(x,t) \leq \int_0^1 f'(\kappa \widetilde{u^\varepsilon}(x,t) + (1-\kappa)u^\varepsilon(x,t))_+ d\kappa \leq BC\frac{x}{t}, \quad (x,t) \in \mathbb{R}_+ \times (0,T), \tag{3.23}$$

where $B = \max\{f''(\xi),\ |\xi| \leq M_T\}$.

3.2.1. *The dual equation*

Let r^ε be given by (3.22) and consider the dual equation

$$\phi_t^\varepsilon + r^\varepsilon \phi_x^\varepsilon + \varepsilon \phi_{xx}^\varepsilon = 0, \quad x \in \mathbb{R}_+, \quad 0 \leq t \leq T, \tag{3.24}$$

with "terminal" condition

$$\phi^\varepsilon(x,T) = \phi_T(x) \in C_0^\infty(\mathbb{R}_+),$$

and boundary condition

$$\phi^\varepsilon(0,t) = 0, \quad 0 \leq t \leq T.$$

The obvious maximum principle

$$\sup_{[0,\infty)\times(0,T)} |\phi^\varepsilon(x,t)| \leq \|\phi_T\|_\infty,$$

can be strengthened exactly as in the case of Lemma 3.1 (and with identical proof) as follows.

Lemma 3.8. *The total variation $TV(\phi^\varepsilon(\cdot,t)) = \int_0^\infty |\phi_x^\varepsilon(x,t)|dx$ is an increasing function of $t \in [0,T]$.*

In analogy with Lemma 3.6, we have

Lemma 3.9. *For suitable constants $C_1, \beta > 0$, the function $\theta(x,t) = \frac{C_1 x}{t^\beta}$ is a subsolution to (3.24) in $\mathbb{R}_+ \times (0,T]$, for all $0 < \varepsilon < 1$.*

Proof. By the estimate (3.23) we have

$$r^\varepsilon \theta_x \le BC\frac{x}{t} \cdot \frac{C_1}{t^\beta} = B\frac{CC_1 x}{t^{\beta+1}}.$$

Thus

$$\theta_t + r^\varepsilon \theta_x + \varepsilon \theta_{xx} \le -\frac{\beta C_1 x}{t^{\beta+1}} + B\frac{CC_1 x}{t^{\beta+1}},$$

and the right-hand side is nonpositive if $\beta > BC$.

In addition, since ϕ_T is compactly supported we can determine $C_1 > 0$ so that

$$\phi_T(x) \le \frac{C_1 x}{T^\beta}, \quad x \in \mathbb{R}_+. \qquad \square$$

The comparison principle now yields (since we consider a *terminal value* problem),

Corollary 3.10. *Let ϕ^ε be a solution to (3.24), where the terminal function is $\phi_T(x) \in C_0^\infty(\mathbb{R}_+)$ and $\phi^\varepsilon(0,t) \equiv 0$. Then, for $\beta, C_1 > 0$ as in Lemma 3.9,*

$$\phi^\varepsilon \le \frac{C_1 x}{t^\beta}, \quad (x,t) \in \mathbb{R}_+ \times (0,T].$$

3.2.2. *The duality argument, concluding uniqueness for $g \le 0$*

Lemma 3.11. *The difference $w^\varepsilon = \widetilde{u^\varepsilon} - u^\varepsilon$ of the two solutions associated with the boundary data $g \le \tilde{g} \le 0$, respectively, satisfies*

$$\lim_{\varepsilon \to 0} w^\varepsilon(x,t) = 0, \quad \textit{in } L^1_{\mathrm{loc}}(\mathbb{R}_+ \times (0,T)).$$

Proof. Let ϕ^ε be the solution to (3.24) and assume that the terminal value $\phi_T \ge 0$. Fix $0 < \eta < T$. Multiplying (3.21) by ϕ^ε and integrating over $\mathbb{R}_+ \times [\eta, T]$ we get, in view of (3.24),

$$\int_0^\infty [w^\varepsilon(x,T)\phi_T(x) - w^\varepsilon(x,\eta)\phi^\varepsilon(x,\eta)]dx = \varepsilon \int_\eta^T w^\varepsilon(0,t)\phi_x^\varepsilon(0,t)dt. \tag{3.25}$$

We note that by Corollary 2.4

$$\left| \int_0^\infty w^\varepsilon(x,\eta)\phi^\varepsilon(x,\eta)dx \right| \le L\eta \|\phi_T\|_\infty.$$

By Corollary 3.10 and the assumption $\phi_T \ge 0$, we have

$$0 \le \phi_x^\varepsilon(0,t) \le \frac{C_1}{\eta^\beta}, \quad t \in (\eta, T].$$

Using the last two estimates in (3.25),

$$\limsup_{\varepsilon \to 0} \int_0^\infty w^\varepsilon(x,T)\phi_T(x)dx \le L\eta \|\phi_T\|_\infty,$$

and letting now $\eta \to 0$, we conclude

$$\limsup_{\varepsilon\to 0} \int_0^\infty w^\varepsilon(x,T)\phi_T(x)dx \le 0.$$

Since $w^\varepsilon(x,T) \ge 0$ and T can be replaced by any value $t \in [0,T]$ the lemma is proved. $\square$

Take now $\tilde{g} \equiv 0$. We know from the treatment above ($g \ge 0$) that $\widetilde{u^\varepsilon}$ converges to a unique limit as $\varepsilon \to 0$. Hence by Lemma 3.11 the same is true for u^ε. This concludes the proof of Theorem 1.3 in this case.

3.3. *The general case for the boundary function $g(t)$*

3.3.1. *Conclusion of the proof of Theorem 1.3*

Fix $T > 0$. We still assume the strict convexity (3.1). In addition to the hypotheses imposed on u_0, g in Theorem 1.1, we assume also that:

$$\begin{cases} \text{(i)} & u_0 \in C_0^2(\mathbb{R}_+), \\ \text{(ii)} & g \in C^2[0,T], \\ \text{(iii)} & \text{For every } \tau \in [0,T) \text{ there exists a } \delta > 0 \text{ such that} \\ & g(t_1)g(t_2) \ge 0 \text{ for any } \tau \le t_1 < t_2 < \tau + \delta. \end{cases} \tag{3.26}$$

Let $u^\varepsilon, \varepsilon > 0$, be the unique solution given by Theorem 1.1.

Let $\bar{t}$ be defined by:

$$\begin{aligned} \bar{t} = \sup\{\tau \in [0,T],\ u^\varepsilon(x,t) &\text{ converges as } \varepsilon \downarrow 0 \text{ to a unique limit} \\ u &\in C([0,\tau]; L^1_{\mathrm{loc}}(\mathbb{R}_+)) \cap B([0,\tau]; L^1(\mathbb{R}_+)) \\ &\cap B([0,\tau]; L^\infty(\mathbb{R}_+)) \cap B([0,\tau]; BV(\mathbb{R}_+))\} \end{aligned}$$

By our assumption on g and the above treatments (where g has unique sign) we have $\bar{t} > 0$. Suppose that $\bar{t} < T$.

Let $\eta > 0$ and let $R > 0$ be sufficiently large so that

$$\int_R^\infty (|u(x,\bar{t})| + |u^\varepsilon(x,\bar{t})|)dx < \eta, \quad 0 < \varepsilon < 1.$$

The existence of such a number R follows from the (uniform) rapid decay of $\{u^\varepsilon\}_{0<\varepsilon<1}$ as $x \to \infty$ [11].

Let $0 < \varepsilon_0 < 1$ be such that

$$\int_0^R |u(x,\bar{t}) - u^\varepsilon(x,\bar{t})|dx < \eta, \quad 0 < \varepsilon < \varepsilon_0.$$

We can choose $v_0 \in C_0^\infty(0,R)$ such that

$$\int_0^R |u(x,\bar{t}) - v_0(x)|dx < \eta.$$

By the results above and the assumption on g there exists a $\delta > 0$ such that the solution v^ε to (1.1), subject to the initial condition $v^\varepsilon(x,\bar{t}) = v_0(x)$ and the boundary condition g, exists in $[\bar{t}, \bar{t}+\delta]$ and converges to a unique limit $v(x,t)$ (as $\varepsilon \downarrow 0$).

In view of the stability Lemma 2.6, we have

$$\int_0^\infty |v^\varepsilon(x,t) - u^\varepsilon(x,t)|dx < 3\eta, \quad 0 < \varepsilon < \varepsilon_0,\ t \in [\bar{t}, \bar{t}+\delta].$$

It follows that if $u, \bar{u}$ are two limits of subsequences of $\{u^\varepsilon\}_{0<\varepsilon<\varepsilon_0}$, then

$$\int_0^\infty |u(x,t) - \bar{u}(x,t)|dx < 6\eta, \quad t \in [\bar{t}, \bar{t}+\delta].$$

Since $\eta > 0$ is arbitrary, it follows that in fact the limit is unique also in $[\bar{t}, \bar{t}+\delta]$. Thus, $\bar{t} = T$.

Finally, let g, u_0 satisfy the hypotheses of Theorem 1.3, and let u^ε be the solution given in Theorem 1.1. Let $\{g_n(t)\}_{n=1}^\infty \subseteq C^2[0,T]$ be a sequence of polynomials such that:

$$\begin{cases} \text{(a)} \quad \|g'(\cdot) - g_n'(\cdot)\|_1 \xrightarrow[n\to\infty]{} 0. \\ \text{(b)} \quad g_n(0) = g(0). \end{cases}$$

Let $\{u_{0,n}(x)\}_{n=1}^\infty \subseteq C_0^\infty(\mathbb{R}_+)$ be a sequence converging to u_0 in $L^1(\mathbb{R}_+)$. For every $n = 1, 2, \ldots$, let $u^{\varepsilon,n}$ be the solution given by Theorem 1.1, subject to the initial condition $u_{0,n}(x)$ and the boundary condition $g_n(t)$.

In view of Lemma 2.6, we have

$$\sup_{0<\varepsilon<1} \sup_{0\le t\le T} \|u^\varepsilon(\cdot,t) - u^{\varepsilon,n}(\cdot,t)\|_1 \xrightarrow[n\to\infty]{} 0.$$

On the other hand, since g_n is a polynomial, it certainly satisfies (3.26)(iii), so that by the above considerations, there exists a unique limit (as $\varepsilon \to 0$) to $u^{\varepsilon,n}$, for every fixed n. Hence, there is a unique limit to u^ε.

3.3.2. *Removing the strict convexity assumption*

We now assume only $f''(\xi) \ge 0$.

Fix $T > 0$. We consider the solution u^ε to (1.1), subject to the initial condition (1.2) and the boundary condition (1.3).

For $\delta > 0$ we define $f_\delta(\xi) = f(\xi) + \delta\xi^2$ and let $u^{\varepsilon,\delta}$ be the solution to

$$u_t^{\varepsilon,\delta} + f_\delta(u^{\varepsilon,\delta})_x = \varepsilon u_{xx}^{\varepsilon,\delta},$$

subject to the same initial and boundary conditions.

The function $w^{\varepsilon,\delta} = u^\varepsilon - u^{\varepsilon,\delta}$ satisfies the equation

$$w_t^{\varepsilon,\delta} + (b^{\varepsilon,\delta} w^{\varepsilon,\delta})_x = \varepsilon w_{xx}^{\varepsilon,\delta} + 2\delta u^{\varepsilon,\delta} u_x^{\varepsilon,\delta},$$

where

$$b^{\varepsilon,\delta} = \frac{f(u^\varepsilon) - f(u^{\varepsilon,\delta})}{u^\varepsilon - u^{\varepsilon,\delta}}.$$

The linear semigroup associated with the equation $\frac{\partial}{\partial t}\psi(x,t) + \frac{\partial}{\partial x}(b^{\varepsilon,\delta}\psi(x,t)) - \varepsilon\frac{\partial^2}{\partial x^2}\psi(x,t) = 0$, with zero boundary data, is an L^1-contraction. Taking into account the maximum principle and the total variation estimate for $u^{\varepsilon,\delta}$ (Lemma 2.2), we obtain from the Duhamel principle

$$\|w^{\varepsilon,\delta}(\cdot,t)\|_1 \leq 2\delta T M_T(\|u_0'\|_1 + \int_0^T |g'(\tau)|d\tau), \quad 0 \leq t \leq T,$$

where M_T is as in (2.1).

Since f_δ is strictly convex, there exists a unique limit $u^{0,\delta} = \lim_{\varepsilon\to 0} u^{\varepsilon,\delta}$. It is sufficient to take this convergence in the sense of $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}_+))$. The function $z^{\varepsilon,\delta} = u^\varepsilon - u^{0,\delta}$ satisfies therefore the same estimate

$$\|z^{\varepsilon,\delta}(\cdot,t)\|_1 \leq 2\delta T M_T(\|u_0'\|_1 + \int_0^T |g'(\tau)|d\tau), \quad 0 \leq t \leq T. \tag{3.27}$$

Invoking Theorem 1.2, let $\{u^{\varepsilon_j}(x,t),\ \varepsilon_j \downarrow 0\}$, $\{u^{\mu_j}(x,t),\ \mu_j \downarrow 0\}$, be two subsequences of solutions converging in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}_+))$ to limit functions u, v, respectively.

From (3.27), it follows that

$$\|u(\cdot,t) - v(\cdot,t)\|_1 \leq 4\delta T M_T\left(\|u_0'\|_1 + \int_0^T |g'(\tau)|d\tau\right), \quad 0 \leq t \leq T,$$

and since $\delta > 0$ is arbitrary, we conclude that $u = v$. □

Appendix A. The Conservation Law on the Whole Line

The method used in the paper to prove the uniqueness of the zero-viscosity limit to solutions of (1.1) is of interest even in the simpler case of the pure Cauchy problem. It avoids completely the classical entropy argument [13] and instead uses a "duality" approach. In this sense, it is close to the duality argument used by Oleinik [16], [6, Chap. 3]. However, the latter treated directly the hyperbolic limit ($\varepsilon = 0$), while our method deals solely with the parabolic equation and yields new interesting estimates for the solution of the dual equation.

We consider

$$u_t^\varepsilon(x,t) + f(u^\varepsilon(x,t))_x = \varepsilon u_{xx}^\varepsilon(x,t), \quad (x,t) \in \mathbb{R}\times[0,T], \quad \varepsilon > 0. \tag{A.1}$$

subject to the initial condition

$$u^\varepsilon(x,0) = u_0(x). \tag{A.2}$$

We assume that the flux function $f(u)$ is convex, and that without loss of generality, it satisfies (3.2). Furthermore, using an argument as in the end of the preceding section, we may in fact assume that f is strictly convex and satisfies (3.1).

By a density argument we may assume that the initial data $u_0(x)$ is smooth and rapidly decaying at infinity, so that the initial value problem (A.1), (A.2) has a unique smooth solution $u^\varepsilon(x,t)$ for any fixed $\varepsilon > 0$. Such a solution $u^\varepsilon(x,t)$

can be constructed by the fixed point argument (see [9, Chap. 2]). By extension, using the notation introduced at the end of the Introduction, with $\mathbb{R}_+$ replaced by $\mathbb{R}$, if

$$u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap BV(\mathbb{R}), \tag{A.3}$$

then for every sequence $\{u^{\varepsilon_j}, \varepsilon_j \to 0\}$, there exists a converging subsequence $\{u^{\varepsilon_{j_k}}(x,t)\}$,

$$\lim_{\varepsilon_{j_k} \to 0+} u^{\varepsilon_{j_k}}(\cdot,t) = u(\cdot,t) \quad \text{in } C([0,T]; L^1_{\text{loc}}(\mathbb{R})).$$

We need to show that this limit is unique.

A.1. *Estimates for u^ε*

In analogy with (2.1) we denote

$$\begin{aligned} M &= \|u_0\|_\infty, \\ N &= \max_{|p| \le M} |f'(p)| . \end{aligned} \tag{A.4}$$

The flux f is assumed to satisfy (3.1) and we define

$$B = \max_{|p| \le M} f''(p). \tag{A.5}$$

We have the basic estimates [9], which certainly do not depend on the convexity of f,

Lemma A.1. *The solution to* (A.1) *satisfies the estimates*

(i) $\|u^\varepsilon(\cdot,t)\|_\infty \le M$.
(ii) $\|u^\varepsilon(\cdot,t)\|_1 \le \|u_0\|_1$.
(iii) *The total variation of u^ε is a non-increasing function of time,*

$$TV(u^\varepsilon(\cdot,t)) = \|u^\varepsilon_x(\cdot,t)\|_1 \le \|u_0'\|_1, \quad \textit{for any } t \in [0,T]. \tag{A.6}$$

We note that the total variation estimate has been generalized [18, Theorem 1.4] to any monotone continuous function of the solution. We refer also to [20, Theorem 2.1] for precise total variation estimates in the (non-convex hyperbolic, $\varepsilon = 0$) case $f(u) = u^k$ (and compactly supported initial data).

We have here also the analog of the estimate (2.11), but with weaker conditions (and a very different proof):

Lemma A.2. *The space derivative of the direct equation solution can be estimated by the initial data,*

$$\|u^\varepsilon_x(\cdot,t)\|_\infty \le \|u_0'\|_\infty + \frac{2N\|u_0\|_\infty}{\varepsilon}. \tag{A.7}$$

Proof. We check directly that

$$(\varepsilon u_x^\varepsilon - f(u^\varepsilon))_t + f'(u^\varepsilon)(\varepsilon u_x^\varepsilon - f(u^\varepsilon))_x - \varepsilon(\varepsilon u_x^\varepsilon - f(u^\varepsilon))_{xx} = 0, \tag{A.8}$$

so that the maximum principle yields

$$\|(\varepsilon u_x^\varepsilon - f(u^\varepsilon))(\cdot,t)\|_\infty \le \|(\varepsilon u_x^\varepsilon - f(u^\varepsilon))(\cdot,0)\|_\infty. \tag{A.9}$$

Since $f(0) = 0$, it follows from (A.4) that $\|f(u^\varepsilon(\cdot,t))\|_\infty \le N\|u_0\|_\infty$. Thus (A.9) leads to

$$\|(\varepsilon u_x^\varepsilon - f(u^\varepsilon))(\cdot,t)\|_\infty \le N\|u_0\|_\infty + \varepsilon\|u_0'\|_\infty,$$

which proves (A.7). □

In Lemma 3.3 we derived an upper bound estimate for the x-derivative of u^ε, for a certain class of boundary conditions. In the case of the whole line, such a condition is known, even for more general (degenerate) parabolic equations [7, Sec. 3]. It is independent of ε (and the initial data u_0), and constitutes a viscous version of Oleinik's (hyperbolic) entropy condition for the case of a convex flux ([6, Chap. 3], [4]).

Lemma A.3. *Let u^ε be a solution to* (A.1), *then*

$$u_x^\varepsilon(x,t) \le \frac{1}{At}, \tag{A.10}$$

(A is as in (3.1)*).*

A.2. *The duality approach*

We proceed as in Sec. 3.1. Let

$$\varepsilon_j > 0, \quad \mu_j > 0, \quad \lim_{j\to\infty} \varepsilon_j = 0, \quad \lim_{j\to\infty} \mu_j = 0$$

be such that the corresponding sequences of solutions of the parabolic equations

$$u_t^{\varepsilon_j} + f(u^{\varepsilon_j})_x = \varepsilon_j u_{xx}^{\varepsilon_j} \tag{A.11}$$

and

$$v_t^{\mu_j} + f(v^{\mu_j})_x = \mu_j v_{xx}^{\mu_j} \tag{A.12}$$

subject to the same initial data

$$u^{\varepsilon_j}(x,0) = v^{\mu_j}(x,0) = u_0(x), \tag{A.13}$$

converge in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}))$-space.

Without loss of generality we can assume

$$\mu_j > \varepsilon_j > 0 \quad \text{for all } j = 0,1,2,\ldots \tag{A.14}$$

Let $u^{\varepsilon_j}(\cdot,t) \to u(\cdot,t)$ and $v^{\mu_j}(\cdot,t) \to v(\cdot,t)$. We want to prove

$$u = v \text{ a.e.} \quad \text{in } \mathbb{R} \times [0,T]. \tag{A.15}$$

Denote

$$w^j(x,t) = u^{\varepsilon_j}(x,t) - v^{\mu_j}(x,t),$$

and let $w(\cdot,t) = u(\cdot,t) - v(\cdot,t)$ be the limit of $w^j(\cdot,t)$ in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}))$. Thus, we have to show that

$$w = 0 \text{ a.e.} \quad \text{in } \mathbb{R} \times [0,T]. \tag{A.16}$$

Let

$$b^j(x,t) = \frac{f(u^{\varepsilon_j}) - f(v^{\mu_j})}{u^{\varepsilon_j} - v^{\mu_j}} = \frac{f(u^{\varepsilon_j}) - f(v^{\mu_j})}{w^j}, \tag{A.17}$$

so $b^j(x,t)$ is smooth and satisfies $\left|b^j(x,t)\right| \le N$.

By (A.10) we have

$$b^j_x \le \frac{B}{At} \quad \text{for all } t > 0, \tag{A.18}$$

where A and B are as in (3.1) and (A.5). Fix $\tau \in (0,T]$. For $t \in [0,\tau]$, $w^j(x,t)$ satisfies the parabolic non-homogeneous equation.

$$w^j_t + (b^j w^j)_x - \varepsilon_j w^j_{xx} + (\mu_j - \varepsilon_j) v^{\mu_j}_{xx} = 0, \tag{A.19}$$

$$w^j(x,0) = 0. \tag{A.20}$$

The dual equation is given by

$$\varphi^j_t + b^j \varphi^j_x + \varepsilon_j \varphi^j_{xx} = 0, \quad 0 < t < \tau, \tag{A.21}$$

with the terminal condition

$$\varphi^j(x,\tau) = \varphi_\tau(x) \in C_0^\infty(\mathbb{R}). \tag{A.22}$$

Multiplying (A.19) by $\varphi^j(x,t)$, and integrating over $\mathbb{R} \times [0,\tau]$, we obtain

$$\int_{\mathbb{R}} w^j(x,\tau)\varphi_\tau(x)dx = (\mu_j - \varepsilon_j) \int_0^\tau \int_{\mathbb{R}} \varphi^j_x v^{\mu_j}_x \, dxdt. \tag{A.23}$$

It suffices to show that

$$(\mu_j - \varepsilon_j) \int_0^\tau \int_{\mathbb{R}} \varphi^j_x(x,t) v^{\mu_j}_x(x,t) dxdt \to 0 \quad \text{as } j \to \infty. \tag{A.24}$$

To this end, we require various estimates of $\varphi^j_x(\cdot,t)$.

A.2.1. *Estimates for the dual solution*

We list estimates of the dual solution φ^j, which are analogous to the estimates in Lemma A.1.

Lemma A.4.

$$\|\varphi^j(\cdot,t)\|_\infty \le \|\varphi_\tau\|_\infty, \quad t\in[0,\tau], \tag{A.25}$$

$$\|\varphi^j_x(\cdot,t)\|_1 \le \|\varphi'_\tau\|_1, \quad t\in[0,\tau]. \tag{A.26}$$

(For the estimate (A.26), see the proof of Lemma 3.1.)

Our key estimate on the dual solution is an L^∞-estimate for $\varphi^j_x(\cdot,t)$, independent of ε_j and μ_j (compare the one-sided estimate (3.13)).

Lemma A.5. *Let $s > \frac{B}{A} \ge 1$, where A, B are as in Eqs.* (3.1) and (A.5). *Then*

$$\|\varphi^j_x(\cdot,t)\|_\infty \le \left(\frac{\tau}{t}\right)^s \|\varphi'_\tau\|_\infty, \quad t\in(0,\tau]. \tag{A.27}$$

Proof. Differentiating (A.21) with respect to x leads to

$$(\varphi^j_x)_t + b^j(\varphi^j_x)_x + b^j_x\varphi^j_x + \varepsilon_j(\varphi^j_x)_{xx} = 0, \tag{A.28}$$

thus

$$(t^s\varphi^j_x)_t + b^j(t^s\varphi^j_x)_x + \left(b^j_x - \frac{s}{t}\right)t^s\varphi^j_x + \varepsilon_j(t^s\varphi^j_x)_{xx} = 0, \tag{A.29}$$

with the terminal condition

$$\tau^s\varphi^j_x(x,\tau) = \tau^s\varphi'_\tau(x). \tag{A.30}$$

In view of (A.18), since $s > \frac{B}{A}$,

$$b^j_x - \frac{s}{t} < 0. \tag{A.31}$$

By the maximum principle, $t^s\varphi^j_x(x,t)$ does not attain a positive maximum or a negative minimum in $\mathbb{R}\times(0,\tau)$, therefore $t^s\|\varphi^j_x(\cdot,t)\|_\infty \le \tau^s\|\varphi'_\tau\|_\infty$ for all $t\in(0,\tau]$, which proves the lemma. ◻

A.3. *The main result*

Theorem A.6. *Let u, v be the limit functions of subsequences $u^{\varepsilon_j}, v^{\mu_j}$ as in Sec.* A.2. *Then $u = v$ a.e. in $\mathbb{R}\times[0,T]$.*

Proof. As observed in Sec. A.2, the proof that $w = u - v$ vanishes a.e. in $\mathbb{R}\times[0,T]$ reduces to the proof of (A.24).

We assume $\varphi_\tau(x)\in C_0^\infty$. In fact, we show that

$$(\mu_j - \varepsilon_j)\int_0^\tau \|\varphi^j_x(\cdot,t)v^{\mu_j}_x(\cdot,t)\|_1 dt \to 0 \quad \text{as } j\to\infty. \tag{A.32}$$

Interpolating the estimates

$$\|\varphi_x^j(\cdot,t)v_x^{\mu_j}(\cdot,t)\|_1 \le \|\varphi_x^j(\cdot,t)\|_\infty \|v_x^{\mu_j}(\cdot,t)\|_1,$$

$$\|\varphi_x^j(\cdot,t)v_x^{\mu_j}(\cdot,t)\|_1 \le \|\varphi_x^j(\cdot,t)\|_1 \|v_x^{\mu_j}(\cdot,t)\|_\infty,$$

we get, for any $0 \le \gamma \le 1$

$$\|\varphi_x^j(\cdot,t)v_x^{\mu_j}(\cdot,t)\|_1 \le (\|\varphi_x^j(\cdot,t)\|_\infty \|v_x^{\mu_j}(\cdot,t)\|_1)^\gamma (\|\varphi_x^j(\cdot,t)\|_1 \|v_x^{\mu_j}(\cdot,t)\|_\infty)^{1-\gamma}.$$

Using the four estimates (A.6), (A.7), (A.26) and (A.27) (assuming j sufficiently large), we can write for $0 < t \le \tau$

$$\|\varphi_x^j(\cdot,t)v_x^{\mu_j}(\cdot,t)\|_1 \le \left(\left(\frac{\tau}{t}\right)^s \|\varphi_\tau'\|_\infty \|u_0'\|_1\right)^\gamma \left(\|\varphi_\tau'\|_1 \frac{2N\|u_0\|_\infty + 1}{\mu_j}\right)^{1-\gamma}.$$

We choose $\gamma > 0$ such that $\gamma s < 1$. Then

$$\int_0^\tau \|\varphi_x^j(\cdot,t)v_x^{\mu_j}(\cdot,t)\|_1 dt$$
$$\le \tau^{\gamma s}\|\varphi_\tau'\|_\infty^\gamma \|u_0'\|_1{}^\gamma \|\varphi_\tau'\|_1{}^{1-\gamma} \frac{(2N\|u_0\|_\infty + 1)^{1-\gamma}}{\mu_j{}^{1-\gamma}} \int_0^\tau t^{-\gamma s} dt,$$

so that

$$(\mu_j - \varepsilon_j)\int_0^\tau \|\varphi_x^j(\cdot,t)v_x^{\mu_j}(\cdot,t)\|_1 dt \le \frac{\tau(\mu_j - \varepsilon_j)}{\mu_j{}^{1-\gamma}(1-\gamma s)} K, \tag{A.33}$$

where

$$K = \|\varphi_\tau'\|_\infty^\gamma \|u_0'\|_1{}^\gamma \|\varphi_\tau'\|_1{}^{1-\gamma}(2N\|u_0\|_\infty + 1)^{1-\gamma}.$$

Then since $\mu_j > \varepsilon_j > 0$ (see (A.14)),

$$(\mu_j - \varepsilon_j)\int_0^\tau \|\varphi_x^j(\cdot,t)v_x^{\mu_j}(\cdot,t)\|_1 dt \le \frac{\tau\mu_j{}^\gamma}{(1-\gamma s)} K. \tag{A.34}$$

The right-hand side of (A.34) tends to 0 as $j \to \infty$, which proves (A.32) and thus concludes the proof of the theorem. □

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