Global properties of some nonlinear parabolic equations

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1. Introduction

In this paper we review some recent results concerning the class of nonlinear equations of evolution given by,

\[ u_t - \Delta u = \mu |\nabla u|^p, \quad \mu \in \mathbb{R}, \ p \geq 1, \]

(1.1)

\[ u(x, 0) = u_0(x), \ x \in \mathbb{R}^n. \]

(1.2)

We denote \( \nabla = \nabla_x = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right), \Delta = \sum_{i=1}^{n} \left( \frac{\partial}{\partial x_i} \right)^2. \) While equations of the type \( u_t - \Delta u = u^p \) have been extensively studied (see e.g. [14, 15, 29] and references there), the same is not true for (1.1). We note that most of the results mentioned in this paper apply to the more general case where the right-hand side of (1.1) is replaced by \( F(\nabla u), \) with suitable growth conditions on \( F. \) Thus, (1.1) can be viewed as a model for a “viscous Hamilton-Jacobi” equation. Indeed, this equation appears naturally in a variety of studies. Some examples include:

(a) The one-dimensional case \( n = 1. \) In this case the equation appears in the study of growth of surfaces and is labeled as the “generalized KPZ equation” [16, 17, 20, 21].

(b) Still in the one-dimensional case, we take \( \mu = -1 \) and \( p = 2, \) thus obtaining the equation \( u_t + u_x^2 = u_{xx}. \) Differentiating with respect to \( x \) and setting \( v = u_x \) we get for \( v(x, t) \) the equation \( v_t + (v^2)_x = v_{xx}, \) which is the well-known Burgers equation.

(c) Consider the Navier-Stokes equations in the plane \( (n = 2), \) which in vorticity form can be written as

\[ \xi_t + (u \cdot \nabla)\xi = \nu \Delta \xi \]
(\(\xi\) is the vorticity \(\partial_2 u^1 - \partial_1 u^2\) of the velocity field \(u = (u^1, u^2)\)). Suppose we know in advance that \(|u|\) is bounded. Then \(\xi\) satisfies the inequality \(\xi - \nu \Delta \xi \leq C |\nabla \xi|\). Thus, the methods used in the study of (1.1) are also applicable in the case of the inequality.

In the following sections we shall discuss the global well-posedness of (1.1) in various spaces and the decay properties of solutions as \(t \to +\infty\).

2. Existence of global solutions

Let \(C^2_b(\mathbb{R}^n) := C^2(\mathbb{R}^n) \cap W^{2,\infty}(\mathbb{R}^n)\), namely, the space of twice continuously differentiable functions with bounded derivatives. It was proved in [3] that \(C^2_b(\mathbb{R}^n)\) is a “persistence space” to classical solutions of (1.1). Namely, we have the theorem.

**Theorem 2.1 [3].** Let \(u_0 \in C^2_b(\mathbb{R}^n)\). Then for any \(\mu \in \mathbb{R}\), \(p \geq 1\), there exists a unique classical solution to (1.1) - (1.2), such that \(u(\cdot, t) \in C^2_b(\mathbb{R}^n)\) for all \(t \geq 0\) and the mapping

\[
u_0 \in C^2_b(\mathbb{R}^n) \to u \in C(\mathbb{R}_+, C^2_b(\mathbb{R}^n))
\]

is continuous.

Furthermore, the solution satisfies the following maximum-minimum principles.

(2.1) \[
\sup_{x \in \mathbb{R}^n} u(x, t) = \sup_{x \in \mathbb{R}^n} u_0(x), \quad \inf_{x \in \mathbb{R}^n} u(x, t) = \inf_{x \in \mathbb{R}^n} u_0(x), \forall T > 0,
\]

(2.2) \[
\|\nabla u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq \|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}, \forall t \geq 0.
\]

In the proof, one shows that the solution exists in a time interval \((0, T]\), where \(T\) depends only on \(\|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}\). The inequality (2.2) then allows the continuation of the solution to \([T, 2T], \ldots\). We remark that to prove (2.2) the equation (1.1) is differentiated with respect to \(x_j\). Denoting \(u_j = \frac{\partial}{\partial x_j} u\), we get

(2.3) \[
\frac{\partial}{\partial t} u_j - \Delta u_j = \sum_{i=1}^n \Psi_i(x, t) \frac{\partial u_i}{\partial x_i},
\]

where \(\Psi_i(x, t) = \mu p |\nabla u|^{p-2} \frac{\partial}{\partial x_i} \in L^\infty(\mathbb{R}^n \times (0, T])\). However, the solution \(u_j\) to the linear parabolic equation (2.3) is not twice continuously differentiable hence some care must be
taken in deducing (2.2) from the standard (linear) maximum principle. See the Appendix in [3] for details.

Naturally, our next goal is to investigate the well-posedness of (1.1) in wider spaces of less regular functions, for instance, $L^q(\mathbb{R}^n)$ for suitable exponents $q$ (possibly depending on $p$). To allow such solutions, the equation (1.1) is first cast in the integral form,

\begin{equation}
(2.4) \quad u(x, t) = \int_{\mathbb{R}^n} G(x - y, t)u_0(y)dy + \mu \int_0^t \int_{\mathbb{R}^n} G(x - y, t - s)|\nabla u(y, s)|^pdyds,
\end{equation}

where $G(x, t) = (4\pi t)^{-n/2} \exp(-|x|^2/4t)$ is the heat kernel. Taking $\nabla_x$ of Eq. (2.4) and using norms of the type $\sup_{t \in (0, T]} t^\alpha \|\nabla u(\cdot, t)\|_{L^r(\mathbb{R}^n)}$ for suitable $r, \alpha$, as in [29] one obtains the local (in time) existence of solutions to (2.4) in $L^q(\mathbb{R}^n)$, for certain exponents. Then, by using the regularizing effect of the parabolic equation (2.4) (see also [9] for a direct argument) one shows that the solution $u(\cdot, t) \in C^2_b(\mathbb{R}^n)$ for $t > 0$, hence global existence follows from Theorem 2.1 above. As for uniqueness, we note that the solution was constructed by using “growth norms” of the type $\sup_{t \in (0, T]} t^\alpha \|\nabla u(\cdot, t)\|_{L^r(\mathbb{R}^n)}$. Thus, a contraction argument yields uniqueness using such norms (the “Kato-Fujita condition” [19]). However, an alternative approach as in [13] gives uniqueness for solutions in classes like $C([0, T]; L^q(\mathbb{R}^n)) \cap C((0, T]; C^2_b(\mathbb{R}^n))$. The exact exponents are summarized in the following theorem.

**Theorem 2.2** [11]. For $1 \leq p < 2$, let $q_c = n \frac{p + 1}{2 - p}$ and take any $q \geq \max(1, q_c)$, $q < \infty$ (but $q > 1$ if $q_c = 1$). Then, given any $u_0 \in L^q(\mathbb{R}^n)$ (and any $\mu \in \mathbb{R}$), the equation (2.4) has a unique, global (in time) solution $u \in C([0, \infty), L^q(\mathbb{R}^n))$.

In particular we note that if

\begin{equation}
(2.5) \quad p > p_c := \frac{n + 2}{n + 1}
\end{equation}

then $q_c > 1$ and the exponent $q = 1$ is outside the scope of Theorem 2.2. Indeed, as the following claim shows, one cannot expect, for $p > p_c$, to have solutions $u$ of (2.4) for any $u_0 \in L^1(\mathbb{R}^n)$, even under the mildest assumptions on $u$. In presenting the next claim, there is no attempt at achieving maximal generality.
Claim 2.3 [12]. Let $p > p_c = \frac{n+2}{n+1}$ and $\mu = 1$.

Denote, for $0 < \delta < \frac{1}{2} \left( \frac{n+1}{n+2} - \frac{1}{p} \right)$,

$$v_\delta(x) = \left\{ \begin{array}{ll} |x|^{-n+\delta}, & |x| < 1, \\ 0 & |x| \geq 1. \end{array} \right.$$ 

Then, given any $T > 0$, there is no solution $u(x,t)$ of (2.4) in $(x,t) \in \mathbb{R}^n \times (0,T]$, where $u_0 = v_\delta$ and such that

$$u \in L^p((0,T); W^{1,p}(\mathbb{R}^n)).$$

Proof: Assume the existence of a solution $u(x,t)$ with the above properties. Since

$$\int_0^T \int_{\mathbb{R}^n} |\nabla u|^p dx \, dt < \infty,$$

given $\varepsilon > 0$ there exists a sequence $t_j \to 0$ such that

$$\int_{\mathbb{R}^n} |\nabla u(x,t_j)|^p dx < \varepsilon t_j^{-1}, \quad j = 1, 2, \ldots,$$

which implies, by the Sobolev inequality,

$$\int_{\mathbb{R}^n} u(x,t_j)^p dx \leq C(\varepsilon t_j^{-1})^{\frac{p^*}{p}}, \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n},$$

($u \geq 0$ in view of (2.4)).

Take $0 < \beta < \frac{1}{2}$ (to be specified later) and use Hölder’s inequality and (2.7) to get,

$$\int_{|x|<t_j^\beta} u(x,t_j) dx \leq (C\varepsilon t_j^{-1})^{\frac{1}{p}} \cdot (\omega_n t_j^{3n})^{1-\frac{1}{p^*}},$$

($\omega_n =$ volume of unit ball).

Now $p > \frac{n+2}{n+1}$ implies $n(1 - \frac{1}{p^*}) > \frac{2(n+1)}{n+2}$, so from (2.8),

$$\int_{|x|<t_j^\beta} u(x,t_j) dx \leq C\varepsilon^{\frac{1}{p}} t_j^{-\frac{1}{p} + \frac{2\beta(n+1)}{n+2}}, \quad j = 1, 2, \ldots$$

Since $p > \frac{n+2}{n+1}$ we can choose $\beta < \frac{1}{2}$ such that $\eta = -\frac{1}{p} + 2\beta \frac{n+1}{n+2} > 0$, hence

$$\int_{|x|<t_j^\beta} u(x,t_j) dx \leq C\varepsilon^{\frac{1}{p}} t_j^{-\eta} \to 0 \quad \text{as} \quad j \to \infty.$$
Let \( \tilde{u}(x,t) = G(x,t) * u_0 \) be the solution to the heat equation with the same initial data. Clearly \( u(x,t) \geq \tilde{u}(x,t) \). We have, for \( t > 0 \),

\[
\int_{|x| > t^\beta} \tilde{u}(x,t) \, dx = \int_{|x| > t^\beta} \int_{\mathbb{R}^n} G(x-y,t)u_0(y) \, dy \, dx
\]

\[
= \int_{|x| > t^\beta} \int_{|y| < \frac{1}{2}t^\beta} + \int_{|x| > t^\beta} \int_{|y| > \frac{1}{2}t^\beta} G(x-y,t)u_0(y) \, dy \, dx
\]

\[
\leq \int_{|\xi| > \frac{1}{2}t^\beta} G(\xi,t) d\xi \cdot \|u_0\|_{L^1(\mathbb{R}^n)} + \int_{|y| > \frac{1}{2}t^\beta} u_0(y) \, dy.
\]

Since \( \beta < \frac{1}{2} \) we have

\[
\int_{|\xi| > \frac{1}{2}t^\beta} G(\xi,t) d\xi = O(t^N) \text{ as } t \to 0, N = 1,2,\ldots
\]

and, for \( u_0 = v_\delta \)

\[
\int_{|y| > \frac{1}{2}t^\beta} u_0(y) \, dy = (1 - 2^{-\delta} t^{\beta\delta}) \|u_0\|_{L^1(\mathbb{R}^n)}, \, t < 1,
\]

so, since \( \|\tilde{u}(\cdot,t)\|_{L^1(\mathbb{R}^n)} = \|u_0\|_{L^1(\mathbb{R}^n)} \), we conclude that

\[
\int_{|x| < t^\beta} \tilde{u}(x,t) \, dx = 2^{-\delta} t^{\beta\delta} \|u_0\|_{L^1(\mathbb{R}^n)} + O(t^N) \text{ as } t \to 0.
\]

Setting \( t = t_j \) and comparing with (2.10) we get, for \( j = 1,2,\ldots \),

\[
C e^{1/p} f_j \geq 2^{-\delta} t_j^{\beta\delta} \|u_0\|_{L^1(\mathbb{R}^n)} + O(t_j^N)
\]

which is a contradiction by the choice of \( \delta \), since \( \beta < \frac{1}{2} \) can be chosen such that \( \beta \delta < \eta \). \( \square \)

**Remark 2.4:** In view of the last claim, one may ask, in the case \( p > p_c, \mu = 1 \), what is the set of initial data \( u_0(x) \in L^1(\mathbb{R}^n) \) for which a solution to (2.4) does exist. Theorem 2.2 implies that this set contains all \( u_0 \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \), \( q \geq q_c \), and in particular, all \( u_0(x) \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \). However, Claim 2.3 says this set is not all of \( L^1(\mathbb{R}^n) \). The situation is still not clear for \( \mu = -1 \). On the other hand, if \( \mu = 1 \) and \( p \geq 2 \), Claim 2.3 can be strengthened as follows.
Proposition 2.5. [11]. Let $u(x,t)$ be a classical solution of (1.1), with $\mu = 1, p \geq 2$, in a strip $\mathbb{R}^n \times (0,T)$. Assume that $\lim_{t \to 0} u(\cdot,t) = u_0$ in $L^1_{loc}(\mathbb{R}^n)$. Then $\exp(u_0) \in L^1_{loc}(\mathbb{R}^n)$.

Finally, while (for $p > p_\ast, \mu = 1$) existence is not guaranteed for all $u_0 \in L^q(\mathbb{R}^n), 1 \leq q < q_\ast$, uniqueness can also fail, as the following theorem shows.

Theorem 2.6. [11]. Assume $2 > p > p_\ast$ and let $1 \leq q < q_\ast$ and $\mu = 1$. Then, for $u_0 = 0$, there exists a positive solution $u$ to (2.4). In fact, $u$ is self-similar,

$$u(x,t) = t^{-k} U(|x|t^{-\frac{1}{2}}), k = \frac{2 - p}{2(p - 1)},$$

where $U = U(r) \in C^2([0,\infty))$.

Remark 2.7: The case of a coupled system of equations of the type (1.1) was treated in [4].

3. Further extensions. The case $\mu = -1$.

We consider here some further results for solutions of Eq. (1.1) (or (2.4)) under the assumptions that $\mu = -1$ and $u_0 \geq 0$. The maximum-minimum principle guarantees that the solution $u$ is nonnegative and is majorized by the corresponding solution of the heat equation.

In this case, the subcritical part of Theorem 2.2 has been extended by Benachour and Laurençot [5] to include positive bounded measures, as follows.

Theorem 3.1 [5]. Let $1 < p < p_\ast = \frac{n+2}{n+1}, \mu = -1$ and $u_0 \in M^+_b(\mathbb{R}^n)$ (= the space of positive bounded Borel measures). Then there exists a unique weak solution (in the sense of (2.4)) $u$ such that, $u \in C((0,\infty); L^1(\mathbb{R}^n)) \cap L^p_{loc}((0,\infty); W^{1,p}(\mathbb{R}^n))$.

Remark 3.2: (a) We refer to [5] for a precise definition of a “weak solution”. Also, for the uniqueness a “growth condition” (as $t \to 0$) of the “Kato-Fujita” type is required, as in the discussion preceding Theorem 2.2 above.

(b) The case $p = 1$ (and $\mu = -1$) was treated in [7], by probabilistic methods, producing a spherically symmetric solution for any initial data $u_0(x)$ which is a “profiled” spherically symmetric bounded positive measure.
(c) The more general equation \( u_t - \Delta u = -a(x)u^p(\nabla u)^q, u_0 \geq 0 \), was treated in [23].

The supercritical case \((p \geq p_c)\) is more difficult. Clearly, the method of proof of Claim 2.3 does not work here and the question whether or not the equation is well-posed in \( L^1(\mathbb{R}^n) \) remains an open problem. However, Benachour and Laurencot [5] have managed to prove the non-existence of “source-type” solutions, namely, solutions that converge (in the sense of distributions) to a multiple of the Delta-function. The exact formulation of the theorem is as follows.

**Theorem 3.3.** [5]. Let \( M, T > 0, p \geq p_c \). There is no \( u \in L^\infty((0,T); L^1(\mathbb{R}^n)) \cap L^p((0,T); W^{1,p}(\mathbb{R}^n)) \) such that \( u_t - \Delta u = -|\nabla u|^p \) in \( \mathcal{D}'(\mathbb{R}^n \times (0,T)) \) and \( \lim_{t \to 0^-} \int_{\mathbb{R}^n} u(x,t) \Psi dx = M \Psi(o), \forall \Psi \in C_0^\infty(\mathbb{R}^n) \).

4. **Decay as \( t \to +\infty \).**

Let us go back to classical (say, as in Theorem 2.1) solutions to (1.1), where we assume now that \( u_0 \geq 0 \) and \( \mu = -1 \). Then the solution \( u(x,t) \) is nonnegative and an integration of (1.1) shows that if, in addition, \( u_0 \in L^1(\mathbb{R}^n) \) then \( u(\cdot,t) \in L^1 \) for all \( t \geq 0 \) and the nonnegative function \( I(t) = \int_{\mathbb{R}^n} u(x,t) dx \) is nonincreasing. Thus, the limit \( I_\infty = \lim_{t \to -\infty} I(t) \geq 0 \) always exists. It is interesting that the question whether or not \( I_\infty = 0 \) is determined uniquely by \( p_c = \frac{n+2}{n+1} \), the same critical value as in the previous sections. We have the following theorem.

**Theorem 4.1** [10]. Let \( 0 \leq u_0 \in C_0^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), u_0 \neq 0 \). Let \( u(x,t) \) be the solution to (1.1), with \( \mu = -1 \). Then

\[
I_\infty > 0 \iff p > p_c = \frac{n+2}{n+1}.
\]

**Remark 4.2:** As was seen in Theorem 2.2, the well-posedness of (1.1) in \( L^1(\mathbb{R}^n) \) was also linked to the same critical index \( p_c \). However, there is yet no direct argument connecting this well-posedness (essentially a short-time feature) with the long-time decay as expressed in Theorem 4.1.

**Remark 4.3:** In the case \( p < p_c \) the equation is well-posed in \( L^1(\mathbb{R}^n) \). Then, as in the discussion preceding Theorem 2.2, if \( 0 \leq u_0 \in L^1(\mathbb{R}^n) \) (and \( \mu = -1 \)), it follows that
$u(\cdot, t) \in C^2_0(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ for $t > 0$. Hence, Theorem 4.1 is applicable also, in the subcritical case, to all $0 \leq u_0 \in L^1(\mathbb{R}^n)$.

**Remark 4.4:** In the case $p \leq p_c$, the rate of decay of $I(t)$ to zero becomes slower as $p$ approaches $p_c$. More precisely, let $1 < p \leq p_c$ and $\alpha > \frac{2-p}{2(p-1)} - \frac{n}{2}$. Then $[10] I(t) \leq C t^{-\alpha}$ (for all sufficiently large $t$) implies $u_0 = 0$. In particular, if $p = p_c$ then $I(t)$ cannot decay like $t^{-\alpha}$ for any $\alpha > 0$. On the other hand, if $p = 1$ and $u_0$ is compactly supported then, for some $A, \theta > 0$ we have

$$\sup_{0 \leq t < \infty} \exp(At^\theta) I(t) < \infty.$$ 

(see [3]).

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