Global properties of some nonlinear parabolic equations

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1. Introduction

In this paper we review some recent results concerning the class of nonlinear equations of evolution given by,

$$u_t - \Delta u = \mu |\nabla u|^p, \ \mu \in \mathbb{R}, \ p \geq 1, \tag{1.1}$$

$$u(x,0) = u_0(x), \ x \in \mathbb{R}^n. \tag{1.2}$$

We denote $\nabla = \nabla_x = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right), \Delta = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^2$. While equations of the type $u_t - \Delta u = u^p$ have been extensively studied (see e.g. [14, 15, 29] and references there), the same is not true for (1.1). We note that most of the results mentioned in this paper apply to the more general case where the right-hand side of (1.1) is replaced by $F(\nabla u)$, with suitable growth conditions on F. Thus, (1.1) can be viewed as a model for a "viscous Hamilton-Jacobi" equation. Indeed, this equation appears naturally in a variety of studies. Some examples include:

(a) The one-dimensional case $n = 1$. In this case the equation appears in the study of growth of surfaces and is labeled as the "generalized KPZ equation" [16, 17, 20, 21].

(b) Still in the one-dimensional case, we take $\mu = -1$ and $p = 2$, thus obtaining the equation $u_t + u_x^2 = u_{xx}$. Differentiating with respect to x and setting $v = u_x$ we get for $v(x,t)$ the equation $v_t + (v^2)_x = v_{xx}$, which is the well-known Burgers equation.

(c) Consider the Navier-Stokes equations in the plane ($n = 2$), which in vorticity form can be written as

$$\xi_t + (\underline{u} \cdot \underline{\nabla})\xi = \nu \Delta \xi$$

(ξ is the vorticity $\partial_2 u^1 - \partial_1 u^2$ of the velocity field $\underline{u} = (u^1, u^2)$). Suppose we know in advance that $|\underline{u}|$ is bounded. Then ξ satisfies the inequality $\xi_t - \nu\Delta\xi \le C|\nabla\xi|$. Thus, the methods used in the study of (1.1) are also applicable in the case of the inequality.

In the following sections we shall discuss the global well-posedness of (1.1) in various spaces and the decay properties of solutions as $t \to +\infty$.

2. Existence of global solutions

Let $C_b^2(\mathbb{R}^n) := C^2(\mathbb{R}^n) \cap W^{2,\infty}(\mathbb{R}^n)$, namely, the space of twice continuously differentiable functions with bounded derivatives. It was proved in [3] that $C_b^2(\mathbb{R}^n)$ is a "persistence space" to classical solutions of (1.1). Namely, we have the theorem.

Theorem 2.1 [3]. *Let $u_0 \in C_b^2(\mathbb{R}^n)$. Then for any $\mu \in \mathbb{R}$, $p \ge 1$, there exists a unique classical solution to (1.1) - (1.2), such that $u(\cdot, t) \in C_b^2(\mathbb{R}^n)$ for all $t \ge 0$ and the mapping*

$$u_0 \in C_b^2(\mathbb{R}^n) \to u \in C(\overline{\mathbb{R}_+}, C_b^2(\mathbb{R}^n))$$

is continuous.

Furthermore, the solution satisfies the following maximum-minimum principles.

$$\sup_{\substack{x\in\mathbb{R}^n \\ t\in(0,T]}} u(x,t) = \sup_{x\in\mathbb{R}^n} u_0(x), \ \inf_{\substack{x\in\mathbb{R}^n \\ t\in(0,T]}} u(x,t) = \inf_{x\in\mathbb{R}^n} u_0(x), \forall T > 0, \tag{2.1}$$

$$\|\nabla u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le \|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}, \forall t \ge 0. \tag{2.2}$$

In the proof, one shows that the solution exists in a time interval $(0, T]$, where T depends only on $\|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}$. The inequality (2.2) then allows the continuation of the solution to $[T, 2T], \ldots$. We remark that to prove (2.2) the equation (1.1) is differentiated with respect to x_j. Denoting $u_j = \frac{\partial}{\partial x_j} u$, we get

$$\frac{\partial}{\partial t} u_j - \Delta u_j = \sum_{i=1}^n \Psi_i(x,t) \frac{\partial u_j}{\partial x_i}, \tag{2.3}$$

where $\Psi_i(x,t) = \mu p |\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \in L^\infty(\mathbb{R}^n \times (0, T]$. However, the solution u_j to the linear parabolic equation (2.3) is not twice continuously differentiable hence some care must be taken in deducing (2.2) from the standard (linear) maximum principle. See the Appendix in [3] for details.

Naturally, our next goal is to investigate the well-posedness of (1.1) in wider spaces of less regular functions, for instance, $L^q(\mathbb{R}^n)$ for suitable exponents q (possibly depending on p). To allow such solutions, the equation (1.1) is first cast in the integral form,

$$u(x,t) = \int_{\mathbb{R}^n} G(x-y,t)u_0(y)dy + \mu \int_0^t \int_{\mathbb{R}^n} G(x-y,t-s)|\nabla u(y,s)|^p dyds, \tag{2.4}$$

where $G(x,t) = (4\pi t)^{-n/2}\exp(-|x|^2/4t)$ is the heat kernel. Taking ∇_x of Eq. (2.4) and using norms of the type $\sup\limits_{t\in(0,T]} t^\alpha \|\nabla u(\cdot,t)\|_{L^r(\mathbb{R}^n)}$ for suitable r, α, as in [29] one obtains the local (in time) existence of solutions to (2.4) in $L^q(\mathbb{R}^n)$, for certain exponents. Then, by using the regularizing effect of the parabolic equation (2.4) (see also [9] for a direct argument) one shows that the solution $u(\cdot,t) \in C_b^2(\mathbb{R}^n)$ for $t > 0$, hence global existence follows from Theorem 2.1 above. As for uniqueness, we note that the solution was constructed by using "growth norms" of the type $\sup\limits_{t\in(0,T]} t^\alpha \|\nabla u(\cdot,t)\|_{L^r(\mathbb{R}^n)}$. Thus, a contraction argument yields uniqueness using such norms (the "Kato-Fujita condition" [19]). However, an alternative approach as in [13] gives uniqueness for solutions in classes like $C([0,T];L^q(\mathbb{R}^n)) \cap C((0,T];C_b(\mathbb{R}^n))$. The exact exponents are summarized in the following theorem.

THEOREM 2.2 [11]. *For $1 \le p < 2$, let $q_c = n\frac{p-1}{2-p}$ and take any $q \ge \max(1,q_c)$, $q < \infty$ (but $q > 1$ if $q_c = 1$). Then, given any $u_0 \in L^q(\mathbb{R}^n)$ (and any $\mu \in \mathbb{R}$), the equation (2.4) has a unique, global (in time) solution $u \in C([0,\infty), L^q(\mathbb{R}^n))$.*

In particular we note that if

$$p > p_c := \frac{n+2}{n+1} \tag{2.5}$$

then $q_c > 1$ and the exponent $q = 1$ is outside the scope of Theorem 2.2. Indeed, as the following claim shows, one cannot expect, for $p > p_c$, to have solutions u of (2.4) for any $u_0 \in L^1(\mathbb{R}^n)$, even under the mildest assumptions on u. In presenting the next claim, there is no attempt at achieving maximal generality.

Claim 2.3 [12]. *Let $p > p_c = \frac{n+2}{n+1}$ and $\mu = 1$.*

Denote, for $0 < \delta < \frac{1}{2}(\frac{n+1}{n+2} - \frac{1}{p})$,

$$v_\delta(x) = \begin{cases} |x|^{-n+\delta}, & |x| < 1, \\ 0 & |x| \geq 1. \end{cases}$$

Then, given any $T > 0$, there is no solution $u(x,t)$ of (2.4) in $(x,t) \in \mathbb{R}^n \times (0,T]$, where $u_0 = v_\delta$ and such that

$$u \in L^p((0,T); W^{1,p}(\mathbb{R}^n)).$$

Proof: Assume the existence of a solution $u(x,t)$ with the above properties. Since $\int_0^T \int_{\mathbb{R}^n} |\nabla u|^p dx dt < \infty$, given $\varepsilon > 0$ there exists a sequence $t_j \to 0$ such that

$$\int_{\mathbb{R}^n} |\nabla u(x,t_j)|^p dx < \varepsilon t_j^{-1}, j = 1, 2, \ldots, \tag{2.6}$$

which implies, by the Sobolev inequality,

$$\int_{\mathbb{R}^n} u(x,t_j)^{p^*} dx \leq C(\varepsilon t_j^{-1})^{\frac{p^*}{p}}, \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}. \tag{2.7}$$

($u \geq 0$ in view of (2.4)).

Take $0 < \beta < \frac{1}{2}$ (to be specified later) and use Hölder's inequality and (2.7) to get,

$$\int\limits_{|x|<t_j^\beta} u(x,t_j) dx \leq (C\varepsilon t_j^{-1})^{\frac{1}{p}} \cdot (\omega_n t_j^{\beta n})^{1-\frac{1}{p^*}}, \tag{2.8}$$

(ω_n= volume of unit ball).

Now $p > \frac{n+2}{n+1}$ implies $n(1 - \frac{1}{p^*}) > \frac{2(n+1)}{n+2}$, so from (2.8),

$$\int\limits_{|x|<t_j^\beta} u(x,t_j) dx \leq C\varepsilon^{1/p} t_j^{-\frac{1}{p} + \frac{2\beta(n+1)}{n+2}}, j = 1, 2, \ldots \tag{2.9}$$

Since $p > \frac{n+2}{n+1}$ we can choose $\beta < \frac{1}{2}$ such that $\eta = -\frac{1}{p} + 2\beta\frac{n+1}{n+2} > 0$, hence

$$\int\limits_{|x|<t_j^\beta} u(x,t_j) dx \leq C\varepsilon^{1/p} t_j^\eta \to 0 \text{ as } \quad j \to \infty. \tag{2.10}$$

Let $\tilde{u}(x,t) = G(x,t) * u_0$ be the solution to the heat equation with the same initial data. Clearly $u(x,t) \geq \tilde{u}(x,t)$. We have, for $t > 0$,

$$\begin{aligned}
\int\limits_{|x|>t^\beta} \tilde{u}(x,t)dx &= \int\limits_{|x|>t^\beta} \int\limits_{\mathbb{R}^n} G(x-y,t)u_0(y)dydx \\
&= \int\limits_{|x|>t^\beta} \int\limits_{|y|<\frac{1}{2}t^\beta} + \int\limits_{|x|>t^\beta} \int\limits_{|y|>\frac{1}{2}t^\beta} G(x-y,t)u_0(y)dydx \qquad (2.11)\\
&\leq \int\limits_{|\xi|>\frac{1}{2}t^\beta} G(\xi,t)d\xi \cdot \|u_0\|_{L^1(\mathbb{R}^n)} + \int\limits_{|y|>\frac{1}{2}t^\beta} u_0(y)dy.
\end{aligned}$$

Since $\beta < \frac{1}{2}$ we have

$$\int\limits_{|\xi|>\frac{1}{2}t^\beta} G(\xi,t)d\xi = O(t^N) \text{ as } \ t \to 0, N = 1, 2, \ldots$$

and, for $u_0 = v_\delta$

$$\int\limits_{|y|>\frac{1}{2}t^\beta} u_0(y)dy = (1 - 2^{-\delta}t^{\beta\delta})\|u_0\|_{L^1(\mathbb{R}^n)}, t < 1, \qquad (2.13)$$

so, since $\|\tilde{u}(\cdot,t)\|_{L^1(\mathbb{R}^n)} = \|u_0\|_{L^1(\mathbb{R}^n)}$, we conclude that

$$\int\limits_{|x|<t^\beta} \tilde{u}(x,t)dx = 2^{-\delta}t^{\beta\delta}\|u_0\|_{L^1(\mathbb{R}^n)} + O(t^N) \text{ as } t \to 0.$$

Setting $t = t_j$ and comparing with (2.10) we get, for $j = 1, 2, \ldots$,

$$C\varepsilon^{1/p}t_j^\eta \geq 2^{-\delta}t_j^{\beta\delta}\|u_0\||_{L^1(\mathbb{R}^n)} + O(t_j^N) \qquad (2.14)$$

which is a contradiction by the choice of δ, since $\beta < \frac{1}{2}$ can be chosen such that $\beta\delta < \eta$. □

Remark 2.4: In view of the last claim, one may ask, in the case $p > p_c, \mu = 1$, what is the set of initial data $u_0(x) \in L^1(\mathbb{R}^n)$ for which a solution to (2.4) does exist. Theorem 2.2 implies that this set contains all $u_0 \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, $q \geq q_c$, and in particular, all $u_0(x) \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. However, Claim 2.3 says this set is not all of $L^1(\mathbb{R}^n)$. The situation is still not clear for $\mu = -1$. On the other hand, if $\mu = 1$ and $p \geq 2$, Claim 2.3 can be strengthened as follows.

PROPOSITION 2.5. [11]. *Let* $u(x,t)$ *be a classical solution of (1.1), with* $\mu = 1, p \geq 2$, *in a strip* $\mathbb{R}^n \times (0,T)$. *Assume that* $\lim_{t \to 0} u(\cdot,t) = u_0$ *in* $L^1_{loc}(\mathbb{R}^n)$. *Then* $\exp(u_0) \in L^1_{loc}(\mathbb{R}^n)$.

Finally, while (for $p > p_c, \mu = 1$) existence is not guaranteed for all $u_0 \in L^q(\mathbb{R}^n), 1 \leq q < q_c$, uniqueness can also fail, as the following theorem shows.

THEOREM 2.6. [11]. *Assume* $2 > p > p_c$ *and let* $1 \leq q < q_c$ *and* $\mu = 1$. *Then, for* $u_0 = 0$, *there exists a positive solution* u *to (2.4). In fact,* u *is self-similar,*

$$u(x,t) = t^{-k} U(|x| t^{-\frac{1}{2}}), k = \frac{2-p}{2(p-1)},$$

where $U = U(r) \in C^2([0,\infty))$.

REMARK 2.7: The case of a coupled system of equations of the type (1.1) was treated in [4].

3. FURTHER EXTENSIONS. THE CASE $\mu = -1$.

We consider here some further results for solutions of Eq. (1.1) (or (2.4)) under the assumptions that $\mu = -1$ and $u_0 \geq 0$. The maximum-minimum principle guarantees that the solution u is nonnegative and is majorized by the corresponding solution of the heat equation.

In this case, the subcritical part of Theorem 2.2 has been extended by Benachour and Laurencot [5] to include positive bounded measures, as follows.

THEOREM 3.1 [5]. *Let* $1 < p < p_c = \frac{n+2}{n+1}, \mu = -1$ *and* $u_0 \in M_b^+(\mathbb{R}^n)$ *(= the space of positive bounded Borel measures). Then there exists a unique weak solution (in the sense of (2.4))* u *such that,* $u \in C((0,\infty); L^1(\mathbb{R}^n)) \cap L^p_{loc}((0,\infty); W^{1,p}(\mathbb{R}^n))$.

REMARK 3.2: (a) We refer to [5] for a precise definition of a "weak solution". Also, for the uniqueness a "growth condition" (as $t \to 0$) of the "Kato-Fujita" type is required, as in the discussion preceding Theorem 2.2 above.

(b) The case $p = 1$ (and $\mu = -1$) was treated in [7], by probabilistic methods, producing a spherically symmetric solution for any initial data $u_0(x)$ which is a "profiled" spherically symmetric bounded positive measure.

(c) The more general equation $u_t - \Delta u = -a(x)u^q(\nabla u)^p, u_0 \geq 0$, was treated in [23].

The supercritical case ($p \geq p_c$) is more difficult. Clearly, the method of proof of Claim 2.3 does not work here and the question whether or not the equation is well-posed in $L^1(\mathbb{R}^n)$ remains an open problem. However, Benachour and Laurencot [5] have managed to prove the non-existence of "source-type" solutions, namely, solutions that converge (in the sense of distributions) to a multiple of the Delta-function. The exact formulation of the theorem is as follows.

THEOREM 3.3. [5]. *Let $M, T > 0, p \geq p_c$. There is no $u \in L^\infty((0,T); L^1(\mathbb{R}^n)) \cap L^p((0,T); W^{1,p}(\mathbb{R}^n))$ such that $u_t - \Delta u = -|\nabla u|^p$ in $\mathcal{D}'(\mathbb{R}^n \times (0,T))$ and $\lim\limits_{t\to 0} \int\limits_{\mathbb{R}^n} u(x,t)\Psi dx = M\Psi(o), \forall \Psi \in C_0^\infty(\mathbb{R}^n)$.*

4. DECAY AS $t \to +\infty$.

Let us go back to classical (say, as in Theorem 2.1) solutions to (1.1), where we assume now that $u_0 \geq 0$ and $\mu = -1$. Then the solution $u(x,t)$ is nonnegative and an integration of (1.1) shows that if, in addition, $u_0 \in L^1(\mathbb{R}^n)$ then $u(\cdot,t) \in L^1$ for all $t \geq 0$ and the nonnegative function $I(t) = \int\limits_{\mathbb{R}^n} u(x,t)dx$ is nonincreasing. Thus, the limit $I_\infty = \lim\limits_{t\to\infty} I(t) \geq 0$ always exists. It is interesting that the question whether or not $I_\infty = 0$ is determined uniquely by $p_c = \frac{n+2}{n+1}$, the same critical value as in the previous sections. We have the following theorem.

THEOREM 4.1 [10]. *Let $0 \leq u_0 \in C_b^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), u_0 \not\equiv 0$. Let $u(x,t)$ be the solution to (1.1), with $\mu = -1$. Then*

$$I_\infty > 0 \Leftrightarrow p > p_c = \frac{n+2}{n+1}.$$

REMARK 4.2: As was seen in Theorem 2.2, the well-posedness of (1.1) in $L^1(\mathbb{R}^n)$ was also linked to the same critical index p_c. However, there is yet no direct argument connecting this well-posedness (essentially a short-time feature) with the long-time decay as expressed in Theorem 4.1.

REMARK 4.3: In the case $p < p_c$ the equation is well-posed in $L^1(\mathbb{R}^n)$. Then, as in the discussion preceding Theorem 2.2, if $0 \leq u_0 \in L^1(\mathbb{R}^n)$ (and $\mu = -1$), it follows that

$u(\cdot,t) \in C_b^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ for $t > 0$. Hence, Theorem 4.1 is applicable also, in the subcritical case, to all $0 \le u_0 \in L^1(\mathbb{R}^n)$.

Remark 4.4: In the case $p \le p_c$, the rate of decay of $I(t)$ to zero becomes slower as p approaches p_c. More precisely, let $1 < p \le p_c$ and $\alpha > \frac{2-p}{2(p-1)} - \frac{n}{2}$. Then [10] $I(t) \le Ct^{-\alpha}$ (for all sufficiently large t) implies $u_0 = 0$. In particular, if $p = p_c$ then $I(t)$ cannot decay like $t^{-\alpha}$ for any $\alpha > 0$. On the other hand, if $p = 1$ and u_0 is compactly supported then, for some $A, \theta > 0$ we have

$$\sup_{0 \le t < \infty} \exp(At^{\theta}) I(t) < \infty.$$

(see [3]).

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