**ORIGINAL PAPER** 



# **Regularity of Fluxes in Nonlinear Hyperbolic Balance Laws**

Matania Ben-Artzi<sup>1</sup> · Jiequan Li<sup>2,3</sup> 💿

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## Abstract

This paper addresses the issue of the formulation of weak solutions to systems of nonlinear hyperbolic conservation laws as integral balance laws. The basic idea is that the "meaningful objects" are the *fluxes*, evaluated across domain boundaries over time intervals. The fundamental result in this treatment is the regularity of the flux trace in the multidimensional setting. It implies that a weak solution indeed satisfies the balance law. In fact, it is shown that the flux is Lipschitz continuous with respect to suitable perturbations of the boundary. It should be emphasized that the weak solutions considered here need not be entropy solutions. Furthermore, the assumption imposed on the flux f(u) is quite minimal—just that it is locally bounded.

**Keywords** Balance laws · Hyperbolic conservation laws · Multi-dimensional · Discontinuous solutions · Finite-volume schemes · Flux · Trace on boundary

Mathematics Subject Classification 35L65 · 35L67 · 76N10

## 1 Introduction

This paper deals with the formulation of weak solutions of nonlinear hyperbolic conservation laws as solutions of integral "balance laws". Such laws are closely associated with the relevant physical laws. The basic idea is that the "meaningful objects" are the *fluxes*, evaluated across manifolds over time intervals. In contrast, the role played by the unknown u(x, t) is not its pointwise value but is limited to its integral over a given domain as a function of time. A fundamental issue is, therefore, the meaning (and regularity) of fluxes across domain boundaries.

 Jiequan Li jiequan@cnu.edu.cn
 Matania Ben-Artzi mbartzi@math.huji.ac.il

<sup>1</sup> Institute of Mathematics, Hebrew University of Jerusalem, 91904 Jerusalem, Israel

<sup>2</sup> Academy of Multidisciplinary Studies, Capital Normal University, Beijing 100048, China

<sup>&</sup>lt;sup>3</sup> State Key Laboratory for Turbulence Research and Complex System, Peking University, Beijing 100871, China

From the numerical point-of-view, finite-volume schemes rely on an appropriate approximation of these fluxes, so the present paper is a contribution to the validity of the finite-volume approach.

The case of a single space dimension has already been studied by the authors in [2], and the emphasis here is on systems of conservation laws in the multi-dimensional case. Let

$$f(u) = (f_1(u), \cdots, f_D(u)), f_i(u) \in \mathbb{R}^n, \quad i = 1, 2, \cdots, D$$

be a vector of "fluxes". We only assume that **these functions are locally bounded** as functions of  $u = (u_1, \dots, u_D) \in \mathbb{R}^D$ .

Consider a system of hyperbolic conservation laws in  $\mathbb{R}^n$  of the form

$$\begin{cases} u(x,t)_t + \nabla \cdot f(u(x,t)) = 0, & u = (u_1, \cdots, u_D) \in \mathbb{R}^D, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}_+, \\ \nabla \cdot f(u(x,t)) = (\nabla_x \cdot f_1(u), \cdots, \nabla_x \cdot f_D(u)) \end{cases}$$
(1)

subject to the initial data

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^n.$$
<sup>(2)</sup>

Let  $\Omega \subseteq \mathbb{R}^n$  be a bounded smooth domain with  $\Gamma = \partial \Omega$ , and let  $0 \leq t_1 < t_2$ . Formally, by integration of the equation in  $Q = \Omega \times [t_1, t_2] \subseteq \mathbb{R}^n \times \overline{\mathbb{R}_+}$  the following "balance" equality holds:

$$\int_{\Omega} u_i(x, t_2) dx - \int_{\Omega} u_i(x, t_1) dx = -\left[\int_{t_1}^{t_2} \int_{\Gamma} f_i(u(x, t)) \cdot v dS dt\right], \quad i = 1, 2, \cdots, D.$$
(3)

Here, v is the outward unit normal to  $\Gamma$  and dS is the surface Lebesgue measure.

**NOTATION** Let X be a space of *scalar* functions. Then, we denote by  $X \bigotimes \mathbb{R}^D$  the space of vector functions of D components, where each component is an element of X. Thus,  $C_0^{\infty}(\mathbb{R}^n) \bigotimes \mathbb{R}^D$  is the space of D-vectors whose components are test functions in  $\mathbb{R}^n$ .

Equation (3) can be considered as an integrated (formal) form of (1) by using the Gauss-Green theorem. However, the application of this theorem is certainly not straightforward, since the function u(x, t) is not even continuous (see [9, Section 4.5]). We refer to [5, 13] and [6, Chapter I] for an abstract discussion of this topic. In particular, regarding the right-hand side of (3) it is not clear what the appropriate fluxes should be and one needs to keep in mind the following comment concerning them: "the drawback of this, functional analytic, demonstration is that it does not provide any clues on how the  $q_{\mathfrak{D}}$  may be computed from A" [6, Section 1.3].

In the context of theoretical continuum mechanics the quantity  $\int_A f(u(x,t)) \cdot v dS$ ,  $A \subseteq \Gamma$  is referred to as the **Cauchy flux across** A [11, 12]. The pointwise value  $f(u(x,t)) \cdot v$  is its density.

We now introduce the notion of a "solution to the balance law" as follows.

Definition 1 Let

$$u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \bigotimes \mathbb{R}^D$$

The function  $u(\cdot, t) \in C(\overline{\mathbb{R}_+}, L^1(\mathbb{R}^n)) \cap L^{\infty}(\mathbb{R}_+, L^{\infty}(\mathbb{R}^n)) \bigotimes \mathbb{R}^D$  is a solution to the balance law (3) corresponding to the partial differential equation (1) if the following conditions are satisfied.

- For every t≥ 0 and every smooth bounded domain Ω ⊆ ℝ<sup>n</sup>, the integral ∫<sub>Ω</sub> u(x, t)dx is well defined and is a continuous function of t.
- For every smooth bounded domain  $\Omega \subseteq \mathbb{R}^n$  and interval  $[t_1, t_2] \subseteq \overline{\mathbb{R}_+}$ , the trace

$$h_i(t_1, t_2) = \int_{t_1}^{t_2} \left[ \int_{\partial \Omega} f_i(u(x, t)) \cdot v \, \mathrm{d}S_x \right] \mathrm{d}t, \quad i = 1, 2, \cdots, D$$

is well defined and is continuous with respect to suitable perturbations of the boundary  $\partial \Omega$  (see Lemma 2 below for details). We denote by  $dS_x$  the Lebesgue surface measure on  $\partial \Omega$ .

• The balance equation (3) is satisfied.

**Definition 2** The quantities  $h_i(t_1, t_2)$ ,  $i = 1, 2, \dots, D$  are called the **fluxes** associated with the conservation law (1), across the boundary  $\partial \Omega$  over the time interval  $[t_1, t_2]$ .

**Remark 1** Our definition of a solution to the balance law conforms to that introduced in [6, Chapter I]. In fact, in Dafermos' book the balance equation is assumed to hold for any domain in spacetime. We note that other authors use various other terms, such as the "integral conservation law", and the term "balance law" is applied to a conservation law with a source term.

Definition 1 is closely related to the physical interpretation of fluid mechanics under the continuum hypothesis, that stipulates that the total quantities  $\int_{\Omega} u(x, t) dx$  in a fixed domain are well defined and continuous in time. The fluxes  $h_i(t_1, t_2)$  are defined over a time interval  $[t_1, t_2]$  rather than at any instant t, reflecting the dynamical process of fluid flows.

The important concept of **divergence-measure vector fields** was introduced in [4, Definition 1.2] and then generalized in [3]. In particular if u(x, t) is a weak solution to (1), then (u(x, t), f(u(x, t)) is divergence-measure in spacetime. For such vector fields the Gauss-Green equation can be justified [4, Theorem 2.2], provided the domain boundary is a deformable Lipschitz boundary. The resulting flux turns out to be a Radon measure on the boundary. In our treatment here we treat more specifically weak solutions to the conservation law (1). A distinction is made between the time coordinate and the spatial coordinates. Thus, the vector field f(u(x, t)) is shown (Sect. 2.1) to be divergence-measure in space, for a.e. time t. However, we do not invoke the Gauss-Green formula at *fixed time levels*, but show (Sect. 2.2) that the trace of the flux is well defined when integrated over time intervals. Imposing a geometric condition on the boundary (stronger than just deformable Lipschitz), as well as on the continuity in time of the total mass, it is shown (Theorem 1) that the resulting flux is *Lipschitz continuous* with respect to boundary deformations and the balance equation is satisfied.

Finally, while this paper is concerned with theoretical aspects of the balance law formulation, we emphasize its relevance to the numerical simulation of nonlinear hyperbolic conservation laws. More specifically, it serves as a theoretical basis of finite-volume schemes [8, 10]; in fact every cell of the discrete mesh is considered as a "control volume" in which the balance law is implemented between arbitrary time levels  $t_1 < t_2$ . The common points between our treatment here and finite volume schemes can be summarized as follows.

- The fact that the integral  $\int_{\Omega} u(x, t) dx$  is assumed to be a continuous function of *t* is very natural when referring to the conserved quantities, such as the mass, the momentum, and the energy.
- The construction of approximate fluxes is a primary building block of the finite volume schemes. The fact that the fluxes (evaluated over time intervals) are Lipschitz continuous places them at the position of the "most regular elements" in this context. This is in contrast to the complex discontinuities experienced by the flow variables. It, therefore, makes good sense, from the numerical point-of-view, to aim at approximating these regular fluxes, and then incorporate the approximate fluxes into the balance law.

This is indeed reflected in the GRP methodology [1], the MUSCL-Hancock [15] scheme, as well as the full plethora of "Godunov-type" schemes.

## 2 The Fundamental Principle of the Hyperbolic Balance Law

As is well known, the meaning of the x and t derivatives in the conservation equation (1) must be clarified since the solutions generate discontinuities, such as shocks or interfaces. The concept of a **weak solution** is introduced precisely in order to handle this difficulty [7, Chapter 11] as follows.

**Definition 3** The function u(x, t) is a weak solution of (1) if the following condition is satisfied: for every cylinder  $Q = \Omega \times [t_1, t_2] \subseteq \mathbb{R}^n \times \overline{\mathbb{R}_+}$ , if

$$\phi(x,t) = \left(\phi_1(x,t), \cdots, \phi_D(x,t)\right) \in C_0^{\infty}(Q) \bigotimes \mathbb{R}^D,$$

then

$$\sum_{i=1}^{D} \int_{t_1}^{t_2} \int_{\Omega} \left[ u_i(x,t) \frac{\partial}{\partial t} \phi_i + f_i(u(x,t)) \cdot \nabla_x \phi_i \right] \mathrm{d}x \, \mathrm{d}t = 0.$$
(4)

Recall that our only assumption on the fluxes  $\{f_i(u)\}_{i=1}^D$  is that they are *locally bounded* (as functions of *u*).

#### 2.1 Boundedness of the Flux Divergence

Definition 3 is a mathematical artifact and does not yield (in a straightforward fashion) the desired balance equality (3). The following lemma pretty much summarizes what can be said about the pointwise regularity of the flux function. Observe that in the one-dimensional (spatial) case the lemma already implies the Lipschitz regularity of the flux [2]. Nevertheless, this is not true in the higher dimensional case.

**Lemma 1** Let u(x, t) be a weak solution to the system (1) with the initial function  $u_0 \in (L^1(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)) \bigotimes \mathbb{R}^D$ .

Assume that u(x, t) satisfies the following properties:

- u(x, t) is locally bounded in  $\mathbb{R}^n \times \overline{\mathbb{R}_+}$ ;
- for every fixed bounded  $\Omega \subseteq \mathbb{R}^n$  the mass

$$m(t) = \int_{\Omega} u(x, t) dx \text{ is a well defined and continuous function of } t \in \overline{\mathbb{R}_+}$$
(5)

(more precisely, the function m(t) that is defined a.e.t is a restriction of a continuous function).

Then, for every fixed  $[t_1, t_2] \subseteq \mathbb{R}$ , the function  $g(x; t_1, t_2) = \int_{t_1}^{t_2} f(u(x, t)) dt$  satisfies  $\nabla_x \cdot g(x; t_1, t_2) \in L^{\infty}_{loc}(\mathbb{R}^n) \bigotimes \mathbb{R}^D$ .

**Proof** For every cylinder  $Q = \Omega \times [t_1, t_2] \subseteq \mathbb{R}^n \times \overline{\mathbb{R}_+}$ , we define

$$C_Q = \sup\{|u(x,t)|, (x,t) \in Q\}.$$
(6)

Note that in (3), the "fixed time" integrals in the left-hand side exist by the assumed continuity (in time) of m(t). Pick  $\phi(x, t) = \theta(t)\psi(x)$  in (4), where  $\theta \in C_0^{\infty}(t_1, t_2)$  and  $\psi \in C_0^{\infty}(\Omega) \bigotimes \mathbb{R}^D$ . Take  $0 \le \theta \le 1$  and such that

$$\theta(t) = \begin{cases} 1, & t_1 + \epsilon \leq t \leq t_2 - \epsilon, \\ 0, & t < t_1 - \epsilon \text{ or } t > t_2 + \epsilon. \end{cases}$$

Letting  $\epsilon \to 0$ , (4) yields by the assumed continuity of m(t),

$$\int_{\Omega} [u(x,t_2) - u(x,t_1)] \cdot \psi(x) \mathrm{d}x = \int_{\Omega} \int_{t_1}^{t_2} f(u(x,t)) \mathrm{d}t \cdot \nabla \psi(x) \mathrm{d}x.$$
(7)

Equation (7) can be rewritten as

$$\int_{\Omega} [u(x,t_2) - u(x,t_1)] \cdot \psi(x) \mathrm{d}x = \int_{\Omega} g(x;t_1,t_2) \cdot \nabla \psi(x) \mathrm{d}x = \sum_{i=1}^{D} \int_{\Omega} g_i(x;t_1,t_2) \cdot \nabla \psi_i(x) \mathrm{d}x.$$

Note that the scalar product in the integral in the left-hand side is in  $\mathbb{R}^D$  while the one in the right-hand side is in  $\mathbb{R}^n$ .

Since  $|u(x,t)| \leq C_0$ , it follows that

$$\left| \int_{\Omega} g(x; t_1, t_2) \cdot \nabla \psi(x) \mathrm{d}x \right| \leq 2C_Q \|\psi\|_1.$$
(8)

Define the linear functional for  $\psi \in C_0^{\infty}(\Omega) \bigotimes \mathbb{R}^D$ 

$$\mathcal{G}\psi = \int_{\Omega} g(x; t_1, t_2) \cdot \nabla \psi(x) dx = \sum_{i=1}^{D} \int_{\Omega} g_i(x; t_1, t_2) \cdot \nabla \psi_i(x) dx$$

The estimate (8) shows that  $\mathcal{G}$  is continuous with respect to the  $L^1$  norm. The density of  $C_0^{\infty}(\Omega)$  in  $L^1(\Omega)$  and the  $L^1$ ,  $L^{\infty}$  duality entail that there exists a function  $r(x) \in L^{\infty}(\Omega) \bigotimes \mathbb{R}^D$  such that

$$\int_{\Omega} g(x;t_1,t_2) \cdot \nabla \psi(x) dx = \int_{\Omega} r(x) \cdot \psi(x) dx, \quad \psi \in C_0^{\infty}(\Omega) \bigotimes \mathbb{R}^D.$$
(9)

We conclude that the distributional divergence of  $g(x; t_1, t_2)$  satisfies  $\nabla_x \cdot g(x; t_1, t_2) = -r(x)$  in  $\Omega$ .

This concludes the proof of the lemma.

**Remark 2** We could replace the continuity assumption (5) by the stronger assumption that the map  $t \rightarrow u(\cdot, t) \in L^{\infty}(\mathbb{R})$  weak<sup>\*</sup> is continuous. This latter assumption is universally imposed when dealing with entropy solutions to nonlinear conservation laws [6, Section 4.5]. However, the continuity condition (5) is valid for weak solutions that are not necessarily entropy solutions. In fact, it holds for weak solutions that have bounded (locally in time) total variation. This is expressed by Dafermos as "mechanism of regularity transfer from the spatial to the temporal variables" [6, Theorem 4.3.1].

## 2.2 Traces of Fluxes—Geometric Approach

In order to replace "weak solutions" by "solutions to balance laws" and make good sense of (3) we need to establish the meaning of *fluxes across domain boundaries*. The regularity result of Lemma 1 falls short of this goal. We, therefore, need to address directly such traces.

Let  $\Omega = \Omega_0 \subseteq \mathbb{R}^n$  be a bounded domain with the smooth boundary  $\Gamma = \Gamma_0 = \partial \Omega$ .

Starting with  $\Gamma_0$ , we can construct a tubular neighborhood [14, Chapter 9, Addendum] with the following properties. For some small  $0 < \delta < 1$ , there exists a family of "expanding" smooth bounded domains  $\{\Omega_y \subseteq \mathbb{R}^n, y \in (-\delta, 1 - \delta)\}$  so that their respective boundaries  $\{\Gamma_y, y \in (-\delta, 1 - \delta)\}$  form a foliation of a tubular neighborhood of  $\Gamma_0$ . The coordinate  $y \in (-\delta, 1 - \delta)$  is normal to  $\Gamma_y$  so that  $\frac{\partial}{\partial y} = v$  is the unit normal. We designate by  $dS_y$  the Lebesgue surface measure on  $\Gamma_y$ ,  $y \in (-\delta, 1 - \delta)$ .

In direct continuation to Lemma 1, we now have.

**Lemma 2** Let u(x, t) be a weak solution to the system (1) with the initial function  $u_0 \in (L^1(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)) \bigotimes \mathbb{R}^D$ .

Assume that u(x, t) satisfies the following properties:

- u(x, t) is locally bounded in  $\mathbb{R}^n \times \overline{\mathbb{R}_+}$ ;
- for every fixed bounded  $\Omega \subseteq \mathbb{R}^n$ , the mass

$$m(t) = \int_{\Omega} u(x, t) dx \text{ is a well defined and continuous function of } t \in \overline{\mathbb{R}_+}.$$
 (10)

For every smooth domain  $\Omega$  and the geometric construction above, and for every fixed  $[t_1, t_2] \subseteq \mathbb{R}$ , define the trace function  $h(y; t_1, t_2) = (h_1(y; t_1, t_2), \dots, h_D(y; t_1, t_2))$  by

$$h_i(y; t_1, t_2) = \int_{t_1}^{t_2} \left[ \int_{\Gamma_y} f_i(u(x, t)) \cdot v \, \mathrm{d}S_y(x) \right] \mathrm{d}t, \quad i = 1, 2, \cdots, D, \ y \in (-\delta, 1 - \delta).$$

Then, h is Lipschitz continuous with respect to  $y \in (-\delta, 1 - \delta)$ .

**Proof** As in the proof of Lemma 1, we obtain (see (7)) for every smooth domain  $\widetilde{\Omega}$ 

$$\int_{\widetilde{\Omega}} [u(x,t_2) - u(x,t_1)] \cdot \psi(x) \mathrm{d}x = \int_{\widetilde{\Omega}} \int_{t_1}^{t_2} f(u(x,t)) \mathrm{d}t \cdot \nabla \psi(x) \mathrm{d}x.$$
(11)

We construct  $\widetilde{\Omega}$  as the tubular domain

$$\widetilde{\Omega} = \cup \Big\{ \Gamma_y, \ y \in (-\delta, 1 - \delta) \Big\}.$$

Let  $\psi \in C_0^{\infty}(\widetilde{\Omega}) \bigotimes \mathbb{R}^D$  such that

$$\psi(x) = \theta(y), \quad x \in \Gamma_y,$$
 (12)

where  $\theta(y) \in C_0^{\infty}(-\delta, 1-\delta) \bigotimes \mathbb{R}^D$ . Equation (11) can now be rewritten as (where  $\theta = (\theta_1, \dots, \theta_D)$ )

$$\int_{\widetilde{\Omega}} [u(x,t_2) - u(x,t_1)] \cdot \psi(x) dx = \sum_{i=1}^{D} \int_{t_1}^{t_2} \left[ \int_{-\delta}^{1-\delta} \int_{\Gamma_y} f_i(u(x,t)) \frac{\partial}{\partial y} \theta_i(y) \cdot v \, dS_y(x) dy \right] dt,$$

namely,

$$\int_{\widetilde{\Omega}} [u(x,t_2) - u(x,t_1)] \cdot \psi(x) dx = \sum_{i=1}^{D} \int_{-\delta}^{1-\delta} h_i(y;t_1,t_2) \frac{\partial}{\partial y} \theta_i(y) dy.$$
(13)

Define the linear functional

$$\mathcal{G}\theta = \sum_{i=1}^{D} \int_{-\delta}^{1-\delta} h_i(y; t_1, t_2) \frac{\partial}{\partial y} \theta_i(y) dy, \quad \theta(y) \in C_0^{\infty}(-\delta, 1-\delta) \bigotimes \mathbb{R}^D.$$

From (13) and the boundedness assumption on u, we infer that  $\mathcal{G}$  is continuous with respect to the  $L^1(-\delta, 1-\delta)$  norm. The density of  $C_0^{\infty}(-\delta, 1-\delta)$  in  $L^1(-\delta, 1-\delta)$  and the  $L^1, L^{\infty}$  duality entail that there exists a function  $r(y) \in L^{\infty}(-\delta, 1-\delta) \bigotimes \mathbb{R}^D$  such that

$$\int_{-\delta}^{1-\delta} h(y;t_1,t_2) \cdot \frac{\partial}{\partial y} \theta(y) dy = \int_{-\delta}^{1-\delta} r(y) \cdot \theta(y)(y) dy, \quad \theta \in C_0^{\infty}(-\delta, 1-\delta) \bigotimes \mathbb{R}^D.$$
(14)

It follows that the distributional derivative  $\frac{\partial}{\partial y}h(y;t_1,t_2) = -r(y)$  is bounded, which concludes the proof of the lemma.

We summarize the above result as the fundamental theorem of fluxes.

**Theorem 1** Let u(x, t) be a weak solution to the system (1) with the initial function  $u_0 \in (L^1(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)) \bigotimes \mathbb{R}^D.$ 

Assume that u(x, t) satisfies the following properties:

- u(x, t) is locally bounded in ℝ<sup>n</sup> × ℝ<sub>+</sub>;
  for every fixed bounded Ω ⊆ ℝ<sup>n</sup> the mass

$$m(t) = \int_{\Omega} u(x, t) dx \text{ is a well defined and continuous function of } t \in \overline{\mathbb{R}_+}.$$
 (15)

Then, for any smooth bounded domain  $\Omega \subseteq \mathbb{R}^n$  and for every time interval  $[t_1, t_2]$  the flux

$$h_i(t_1, t_2) = \int_{t_1}^{t_2} \int_{\partial \Omega} f_i(u(x, t)) \cdot v dS(x) dt, \quad i = 1, 2, \cdots, D$$

is well defined and (3) holds.

**Proof** In light of Lemma 2, it only remains to establish the validity of the balance equation (3). Starting from (7) and using the geometric construction above, we select the test function  $\psi(x) = (\psi_1(x), \dots, \psi_D(x))$  as follows:

$$\psi_i(x) \in C_0^{\infty}(\Omega_0) \text{ and } \psi_i(x) \equiv 1, \ x \in \Omega_{-\delta}.$$

Letting  $\delta \to 0$  and using the continuity of the traces obtained in Lemma 2, we obtain (3).

The statement of Theorem 1 is closely related to the more fluid dynamical viewpoint: the "conservation law", which is a **partial differential equation**, is replaced by a <u>"balance law"</u>.

Note that as in the case of weak solutions, no uniqueness assumption is imposed on the solution.

Theorem 1 implies that a weak solution satisfying certain hypotheses (in particular an entropy solution) is a solution to the balance law in the sense of Definition 1. It is easy to see that conversely, a solution to the balance law is a weak solution of the conservation law (1).

An important observation is that the flux  $h(t_1, t_2)$  is defined *over a time interval*. In other words, there is no meaning attached to the instantaneous value  $\int_{\partial\Omega} f_i(u(x, t)) \cdot v dS$ . However, the flux is continuous with respect to the time interval, as in the following proposition.

**Proposition 1** Under the conditions of Theorem 1, the flux  $h(t_1, t_2)$  is continuous with respect to  $t_1, t_2$ .

**Proof** This follows from the balance equation (3) and the assumption about the continuity of m(t).

It is easy to see how to generalize the theorem to bounded domains with piecewisesmooth boundaries. From the point-of-view of applications, the most important instance is that of polygonal domains. For finite-volume schemes on regular meshes, every cell is a rectangular box, and we state the result explicitly for this case.

**Corollary 1** Let u(x, t) be a weak solution to the system (1), satisfying the conditions of Theorem 1. Let

$$\Omega = \prod_{i=1}^{n} [a_i, x_i],$$

and let

$$S_j = \{ y = (y_1, \cdots, y_{j-1}, x_j, y_{j+1}, \cdots, y_n), y_i \in [a_i, x_i], i \neq j \}$$

be the section of  $\partial \Omega$  at  $x_i$ .

*For any*  $1 \le j \le n$  *and any*  $0 \le t_1 < t_2$ *, define the flux* 

$$F^{j}(x_{j};t_{1},t_{2}) = \int_{t_{1}}^{t_{2}} \int_{S_{j}} f(u(y,t)) \cdot e_{j} \mathrm{d}S_{y} \, \mathrm{d}t \in \mathbb{R}^{D},$$

where  $e_i$  is the unit vector in the  $x_i$  direction.

Then,  $F^{j}(x_{j}; t_{1}, t_{2})$ , is well defined and indeed is a locally Lipschitz function of  $x_{j}$ . Furthermore, the following balance equation holds:

$$\int_{\Omega} u(x,t_2) \mathrm{d}x - \int_{\Omega} u(x,t_1) \mathrm{d}x = -\sum_{j=1}^n \left[ F^j(x_j;t_1,t_2) - F^j(a_j;t_1,t_2) \right].$$
(16)

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