Highly Accurate Discretization of the Navier-Stokes Equations in Streamfunction Formulation

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Dedicated to the memory of Professor David Gottlieb for his Wisdom and Generosity

Abstract A discrete version of the pure streamfunction formulation of the Navier-Stokes equation is presented. The proposed scheme is fourth order in both two and three spatial dimensions.

1 Fourth order scheme for the Navier-Stokes equations in two dimensions

We consider the Navier-Stokes equations in pure streamfunction form, which in the two-dimensional case leads to the scalar equation

$$\begin{cases} \partial_t \Delta \psi + \nabla^{\perp} \psi \cdot \nabla \Delta \psi - \nu \Delta^2 \psi = f(x,y,t), \\ \psi(x,y,t) = \psi_0(x,y). \end{cases} \tag{1}$$

Recall that $\nabla^{\perp} \psi = (-\partial_y \psi, \partial_x \psi)$ is the velocity vector. The no-slip boundary condition associated with this formulation is

$$\psi = \frac{\partial \psi}{\partial n} = 0 \ , \ \ (x,y) \in \partial \Omega \ , \ \ t > 0 \tag{2}$$

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and the initial condition is

$$\psi(x,y,0) = \psi_0(x,y), \ \ (x,y) \in \Omega. \tag{3}$$

The spatial derivatives in Equation (1) are discretized as we describe next. The fourth order discrete Laplacian $\tilde{\Delta}_h \psi$ and biharmonic $\tilde{\Delta}_h^2 \psi$ operators introduced in [3] are perturbations of the second order operators $\Delta_h \psi = (\delta_x^2 + \delta_y^2)\psi$ and $\Delta_h^2 \psi = (\delta_x^4 + \delta_y^4 + 2\delta_x^2\delta_y^2)\psi$. They are designed as follows.

$$\tilde{\Delta}_h \psi_{i,j} = 2\Delta_h \psi_{i,j} - (\delta_x(\psi_x)_{i,j} + \delta_y(\psi_y)_{i,j}) = (\Delta \psi)_{i,j} + O(h^4). \tag{4}$$

Here, ψ_x, ψ_y are the fourth-order Hermitian approximations to $\partial_x \psi, \partial_y \psi$ described as

$$\begin{cases} \sigma_x \psi_x = \frac{1}{6}(\psi_x)_{i-1,j} + \frac{2}{3}(\psi_x)_{i,j} + \frac{1}{6}(\psi_x)_{i+1,j} = \delta_x \psi_{i,j} \quad , \quad 1 \le i,j \le N-1 \\ \sigma_y \psi_y = \frac{1}{6}(\psi_y)_{i,j-1} + \frac{2}{3}(\psi_y)_{i,j} + \frac{1}{6}(\psi_y)_{i,j+1} = \delta_y \psi_{i,j} \quad , \quad 1 \le i,j \le N-1. \end{cases} \tag{5}$$

We use the standard central difference operators δ_x, δ_y, δ_x^2, δ_y^2.

The fourth-order approximation to the biharmonic operator $\Delta^2 \psi$ is

$$\tilde{\Delta}_h^2 \psi = \delta_x^4 \psi + \delta_y^4 \psi + 2\delta_x^2 \delta_y^2 \psi - \frac{h^2}{6}(\delta_x^4 \delta_y^2 \psi + \delta_y^4 \delta_x^2 \psi) = \Delta^2 \psi + O(h^4), \tag{6}$$

where δ_x^4 and δ_y^4 are the compact approximations of ∂_x^4 and ∂_y^4, respectively.

$$\delta_x^4 \psi_{i,j} = \frac{12}{h^2}\left((\delta_x \psi_x)_{i,j} - \delta_x^2 \psi_{i,j}\right) \quad , \quad \delta_x^4 \psi = \partial_x^4 \psi - \frac{1}{720} h^4 \partial_x^8 \psi + O(h^6), \tag{7}$$

$$\delta_y^4 \psi_{i,j} = \frac{12}{h^2}\left((\delta_y \psi_y)_{i,j} - \delta_y^2 \psi_{i,j}\right) \quad , \quad \delta_y^4 \psi = \partial_y^4 \psi - \frac{1}{720} h^4 \partial_y^8 \psi + O(h^6). \tag{8}$$

The convective term in (1) is $C(\psi) = -\partial_y \psi \Delta(\partial_x \psi) + \partial_x \psi \Delta(\partial_y \psi)$. Its fourth-order approximation needs special care. The mixed derivative $\partial_x \partial_y^2 \psi$ may be approximated to fourth-order accuracy by $\tilde{\psi}_{yyx}$ using a suitable combination of lower order approximations.

$$\tilde{\psi}_{yyx} = \delta_y^2 \psi_x + \delta_x \delta_y^2 \psi - \delta_x \delta_y \psi_y = \partial_x \partial_y^2 \psi + O(h^4). \tag{9}$$

For the pure third order derivative $\partial_x^3 \psi$ we note that if ψ is smooth then

$$\psi_{xxx} = \frac{3}{2h^2}\left(10\delta_x \psi - h^2 \delta_x^2 \partial_x \psi - 10\partial_x \psi\right)_{i,j} + O(h^4). \tag{10}$$

One needs to approximate $\partial_x \psi$ to sixth-order accuracy in order to obtain from (10) a fourth-order approximation for $\partial_x^3 \psi$. Denoting this approximation by $\tilde{\psi}_x$, we invoke the Pade formulation [2], having the following form.

$$\frac{1}{3}(\tilde{\psi}_x)_{i+1,j} + (\tilde{\psi}_x)_{i,j} + \frac{1}{3}(\tilde{\psi}_x)_{i-1,j} = \frac{14}{9}\frac{\psi_{i+1,j} - \psi_{i-1,j}}{2h} + \frac{1}{9}\frac{\psi_{i+2,j} - \psi_{i-2,j}}{4h}. \quad (11)$$

At near-boundary points we apply a special treatment as in [2]. Carrying out the same procedure for $\partial_y \psi$, which yields the approximate value $\tilde{\psi}_y$, and combining with all other mixed derivatives, a fourth order approximation of the convective term is

$$\begin{aligned}\tilde{C}_h(\psi) = & -\psi_y\big(\Delta_h \tilde{\psi}_x + \frac{5}{2}\big(6\frac{\delta_x \psi - \tilde{\psi}_x}{h^2} - \delta_x^2 \tilde{\psi}_x\big) + \delta_x \delta_y^2 \psi - \delta_x \delta_y \tilde{\psi}_y\big) \\ & + \psi_x\big(\Delta_h \tilde{\psi}_y + \frac{5}{2}\big(6\frac{\delta_y \psi - \tilde{\psi}_y}{h^2} - \delta_y^2 \tilde{\psi}_y\big) + \delta_y \delta_x^2 \psi - \delta_y \delta_x \tilde{\psi}_x\big) \\ & = C(\psi) + O(h^4).\end{aligned} \quad (12)$$

Our implicit-explicit time-stepping scheme is of the Crank-Nicholson type as follows.

$$\frac{(\tilde{\Delta}_h \psi_{i,j})^{n+1/2} - (\tilde{\Delta}_h \psi_{i,j})^n}{\Delta t/2} = -\tilde{C}_h \psi^{(n)} + \frac{\nu}{2}[\tilde{\Delta}_h^2 \psi_{i,j}^{n+1/2} + \tilde{\Delta}_h^2 \psi_{i,j}^n] \quad (13)$$

$$\frac{(\tilde{\Delta}_h \psi_{i,j})^{n+1} - (\tilde{\Delta}_h \psi_{i,j})^n}{\Delta t} = -\tilde{C}_h \psi^{(n+1/2)} + \frac{\nu}{2}[\tilde{\Delta}_h^2 \psi_{i,j}^{n+1} + \tilde{\Delta}_h^2 \psi_{i,j}^n]. \quad (14)$$

Due to stability reasons we have chosen an Explicit-Implicit time stepping scheme. It is possible however to use an explicit time-stepping scheme if one can afford a small time step in order to advance the solution in time. The set of linear equations is solved via a FFT solver using the Sherman-Morrison formula (see [4]). This solver is of $O(N^2 logN)$ operations, where N is the number of grid points in each spatial direction. For the application of the pure streamfunction formulation on an irregular domain see [5].

2 The pure streamfunction formulation in three dimensions

Let Ω be a bounded domain in R^3. The three-dimensional Navier-Stokes equations in vorticity-velocity formulation is

$$\begin{aligned}&\omega_t + \nabla \times (\omega \times \mathbf{u}) - \nu \Delta \omega = \nabla \times \mathbf{f}, \quad \text{in} \quad \Omega \\ &\omega = \nabla \times \mathbf{u}, \ \nabla \cdot \mathbf{u} = 0, \quad \text{in} \quad \Omega \\ &\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega \\ &\omega(\mathbf{x}, 0) = \omega_0(\mathbf{x}) := \nabla \times \mathbf{u}_0, \quad \text{in} \quad \Omega.\end{aligned} \quad (15)$$

where $\omega = \nabla \times \mathbf{u}$ and the no-slip boundary condition has been imposed. The pure streamfunction formulation for this system is obtained by introducing a streamfunction $\psi(\mathbf{x},t) \in R^3$, such that

$$\mathbf{u} = -\nabla \times \psi. \quad (16)$$

This is always possible since $\nabla \cdot \mathbf{u} = \mathbf{0}$. Thus,

$$\omega = \nabla \times \mathbf{u} = \Delta \psi - \nabla(\nabla \cdot \psi). \tag{17}$$

Imposing a gauge condition

$$\nabla \cdot \psi = 0, \tag{18}$$

yields

$$\omega = \Delta \psi. \tag{19}$$

The system (15) can now be rewritten as

$$\frac{\partial \Delta \psi}{\partial t} - \nabla \times (\Delta \psi \times (\nabla \times \psi)) = \nu \Delta^2 \psi + \nabla \times \mathbf{f}, \quad \text{in} \quad \Omega. \tag{20}$$

The boundary conditions $\mathbf{u} = 0$ translates to $\nabla \times \psi = 0$ on $\partial\Omega$. We require that

$$\mathbf{n} \times \psi = \mathbf{0}, \quad \mathbf{n} \times (\nabla \times \psi) = \mathbf{0}, \quad \text{on} \quad \partial\Omega. \tag{21}$$

The condition $\mathbf{n} \times \psi = \mathbf{0}$ means that ψ is parallel to $\mathbf{n}$, hence the normal component of the velocity vector is zero on the boundary. Adding the condition $\mathbf{n} \times (\nabla \times \psi) = \mathbf{0}$ ensures that the full velocity vector vanishes on the boundary. The requirements in (21) are equivalent to four scalar conditions, namely the vanishing of the two tangential components of ψ and $\nabla \times \psi$.

Turning now to the gauge condition $\nabla \cdot \psi = 0$, we add the condition

$$\frac{\partial(\psi \cdot \mathbf{n})}{\partial n} = 0, \quad \text{on} \quad \partial\Omega. \tag{22}$$

Together with the vanishing of the tangential components of ψ, it implies that $\nabla \cdot \psi = 0$ on $\partial\Omega$.

Equations (21)-(22) consist of five scalar conditions for ψ on the boundary. We can still add one more scalar boundary condition, as the equations for the 3- component streamfunction ψ contain the fourth order biharmonic operator. The sixth scalar boundary condition that we choose to add is

$$\Delta(\nabla \cdot \psi) = 0, \quad \text{on} \quad \partial\Omega. \tag{23}$$

We thus obtain

$$\nabla \cdot \psi = 0, \quad \Delta(\nabla \cdot \psi) = 0, \quad \text{on} \quad \partial\Omega. \tag{24}$$

We assume that the initial value $\psi(\mathbf{x},0)$ satisfies $(\nabla \cdot \psi)(\mathbf{x},0) = 0$. Taking the divergence of (20) we obtain an evolution equation for $\nabla \cdot \psi$.

$$\frac{\partial \Delta(\nabla \cdot \psi)}{\partial t} = \nu \Delta^2(\nabla \cdot \psi), \quad \text{in} \quad \Omega. \tag{25}$$

Equations (24)-(25) together with the assumption that $\nabla \cdot \psi = 0$ initially ensure that $\nabla \cdot \psi = 0$ for all $t > 0$. See also [1], [6] and [7]. Finally, we have the following three-dimensional pure streamfunction formulation

$$\begin{cases} \frac{\partial \Delta \psi}{\partial t} - \nabla \times (\Delta \psi \times (\nabla \times \psi)) = \nu \Delta^2 \psi + \nabla \times \mathbf{f}, & \text{in} \quad \Omega \\ \mathbf{n} \times \psi = \mathbf{0}, \frac{\partial (\psi \cdot \mathbf{n})}{\partial n} = 0, & \text{on} \quad \partial \Omega \\ \mathbf{n} \times (\nabla \times \psi) = \mathbf{0}, \quad \Delta (\nabla \cdot \psi) = 0, & \text{on} \quad \partial \Omega. \end{cases} \tag{26}$$

3 The Numerical Scheme

Our numerical scheme is based on the approximation of the following equation

$$\frac{\partial \Delta \psi}{\partial t} - ((\nabla \times \psi) \cdot \nabla) \Delta \psi + (\Delta \psi \cdot \nabla)(\nabla \times \psi) - \nu \Delta^2 \psi = \nabla \times \mathbf{f}, \quad \text{in} \quad \Omega, \tag{27}$$

assuming that $\psi \in H_0^2(\Omega)$. For the vector function ψ we construct a fourth-order approximation to the the biharmonic operator as follows. The pure fourth-order derivatives are approximated by $\delta_x^4, \delta_y^4, \delta_z^4$ as in (7)-(8).

The mixed terms ψ_{xxyy}, ψ_{yyzz} and ψ_{zzxx} are approximated by

$$\begin{cases} \tilde{\delta}_{xy}^2 \psi_{i,j,k} = 3\delta_x^2 \delta_y^2 \psi_{i,j,k} - \delta_x^2 \delta_y \psi_{y,i,j,k} - \delta_y^2 \delta_x \psi_{x,i,j,k} = \partial_x^2 \partial_y^2 \psi_{i,j,k} + O(h^4) \\ \tilde{\delta}_{yz}^2 \psi_{i,j,k} = 3\delta_y^2 \delta_z^2 \psi_{i,j,k} - \delta_y^2 \delta_z \psi_{z,i,j,k} - \delta_z^2 \delta_y \psi_{y,i,j,k} = \partial_y^2 \partial_z^2 \psi_{i,j,k} + O(h^4) \\ \tilde{\delta}_{zx}^2 \psi_{i,j,k} = 3\delta_z^2 \delta_x^2 \psi_{i,j,k} - \delta_z^2 \delta_x \psi_{z,i,j,k} - \delta_x^2 \delta_z \psi_{x,i,j,k} = \partial_z^2 \partial_x^2 \psi_{i,j,k} + O(h^4). \end{cases} \tag{28}$$

A fourth order approximation of the biharmonic operator is then obtained as

$$\tilde{\Delta}_h^2 \psi = \delta_x^4 \psi + \delta_y^4 \psi + \delta_z^4 \psi + 2\tilde{\delta}_{xy}^2 \psi + 2\tilde{\delta}_{yz}^2 \psi + 2\tilde{\delta}_{zx}^2 \psi. \tag{29}$$

The approximate derivatives ψ_x, ψ_y and ψ_z are related to ψ via the Hermitian derivatives as in (5).

Equation (29) provides a fourth order compact operator for $\Delta^2 \psi$, which involves values of ψ, ψ_x, ψ_y and ψ_z at (i, j, k) and at its twenty six nearest neighbors. The Laplacian operator is approximated by a fourth order operator via

$$\tilde{\Delta}_h \psi = 2\Delta_h \psi - (\delta_x \psi_x + \delta_y \psi_y + \delta_z \psi_z). \tag{30}$$

The nonlinear part in (27) consists of two terms, the convective term and the stretching term. We design a fourth-order scheme which approximates the convective term. The convective term in the three-dimensional case is

$$C(\psi) = -((\nabla \times \psi) \cdot \nabla) \Delta \psi = u \Delta \partial_x \psi + v \Delta \partial_z \psi + w \Delta \partial_z \psi. \tag{31}$$

Here $(u, v, w) = \mathbf{u} = -\nabla \times \psi$ is the velocity vector, whose components contain first order derivatives of the streamfunction, and thus may be approximated to fourth-order accuracy. The terms $\Delta \partial_x \psi, \Delta \partial_z \psi, \Delta \partial_z \psi$ may be approximated as in the two-dimensional case. The term $\Delta \partial_x \psi$, for example, may be written as

$$\Delta \partial_x \psi = \partial_x^3 \psi + \partial_x \partial_y^2 \psi + \partial_x \partial_z^2 \psi. \tag{32}$$

Here, the pure and mixed type derivatives may be approximated as in the two-dimensional Navier-Stokes equations (see (10), (9)). We denote the approximation to the convective term by $\tilde{C}_h(\psi)$.

Now, we construct a fourth-order approximation to the stretching term $S = (\omega\cdot\nabla)\mathbf{u} = -(\Delta\psi\cdot\nabla)(\nabla\times\psi)$. Note that the stretching term contains $\Delta\psi$ and mixed second order derivatives of the streamfunction. The Laplacian of ψ may be approximated to fourth-order accuracy, as in (30). The second order mixed terms, such as $\partial_x\partial_y\psi$, may be approximated using a Hermitian approximation of the type

$$(\sigma_x\sigma_y)(\psi_{xy})_{i,j,k} = \delta_x\delta_y\psi_{i,j,k}. \tag{33}$$

Hence,

$$(I+\frac{h^2}{6}\delta_x^2)(I+\frac{h^2}{6}\delta_y^2)(\psi_{xy})_{i,j,k} = \delta_x\delta_y\psi_{i,j,k} \quad ,1\le i,j,k\le N-1 \tag{34}$$

is an implicit equation for ψ_{xy}. We denote the approximation of the stretching term by $\tilde{S}_h(\psi)$. For the approximation in time, we apply a Crank-Nicholson scheme (see the comment after (13)-(14)).

We obtain the following scheme

$$\frac{(\tilde{\Delta}_h\psi_{i,j,k})^{n+1/2}-(\tilde{\Delta}_h\psi_{i,j,k})^n}{\Delta t/2} = -\tilde{C}_h\psi_{i,j,k}^{(n)} + \tilde{S}_h\psi_{i,j,k}^{(n)} + \frac{\nu}{2}[\tilde{\Delta}_h^2\psi_{i,j,k}^{n+1/2} + \tilde{\Delta}_h^2\psi_{i,j,k}^n] \tag{35}$$

$$\frac{(\tilde{\Delta}_h\psi_{i,j,k})^{n+1}-(\tilde{\Delta}_h\psi_{i,j,k})^n}{\Delta t} = -\tilde{C}_h\psi_{i,j,k}^{(n+1/2)} + \tilde{S}_h\psi_{i,j,k}^{(n+1/2)} + \frac{\nu}{2}[\tilde{\Delta}_h^2\psi_{i,j}^{n+1} + \tilde{\Delta}_h^2\psi_{i,j,k}^n]. \tag{36}$$

At present, a direct solver is invoked to solve the linear set of equations (35)-(36).

Some preliminary MATLAB computations with coarse grids confirm the fourth order accuracy of the scheme. We first show numerical results for the time-dependent Stokes equations

$$\frac{\partial\Delta\psi}{\partial t} = \nu\Delta^2\psi + \mathbf{f}, \quad \text{in} \quad \Omega. \tag{37}$$

We have picked the exact solution ψ

$$\psi^T(\mathbf{x},t) = -\frac{1}{4}e^{-t}\left(z^4, x^4, y^4\right) \tag{38}$$

in the cube $\Omega = (0,1)^3$. Here, $\mathbf{f}$ is chosen such that ψ in (38) satisfied (37) exactly. In the numerical results shown here we have chosen the time step Δt of order h^2 in order to retain the overall fourth-order accuracy of the scheme. In practice, if we are interested mainly in the steady state solution, a larger time step, which is independent of h, may be used. In Table 1 we show results for the Stokes problem with $\Delta t = 0.1h^2$ and $t = 0.00625$. Here e is the error in the l_h^2 norm, i.e.

$$e^2 = \sum_i\sum_j\sum_k(\psi_3(x_i,y_j,z_k) - \tilde{\psi}_3(x_i,y_j,z_k))^2h^3,$$

	grid $5\times5\times5$	rate	grid $9\times9\times9$	rate	grid $17\times17\times17$
e	2.5460(-9)	3.82	1.8017(-10)	3.98	1.1443(-11)
e_y	7.7417(-9)	3.73	5.8037(-10)	3.96	3.7391(-11)
div (ψ)	1.3409(-8)	3.74	1.0052(-9)	3.96	6.4621(-11)

Table 1 Stokes equations for $t = 0.00625$ using $\Delta t = 0.1h^2$.

where ψ_3 is the z component of the exact solution and $\tilde{\psi}_3$ is the z component of the approximate solution. e_y is the l_h^2 in the y derivative of ψ_3. In Table 2 we display the results for $t = 0.0625$ using $\Delta t = h^2$.

	grid $5\times5\times5$	rate	grid $9\times9\times9$	rate	grid $17\times17\times17$
e	9.6461(-7)	4.41	4.5309(-8)	4.00	2.8291(-9)
e_y	3.0293(-6)	4.33	1.5049(-7)	3.99	9.4269(-9)
div (ψ)	5.2470(-6)	4.33	2.6066(-7)	4.00	1.6328(-8)

Table 2 Stokes equations with $\Delta t = h^2$ for $t = 0.0625$.

Next we show results for the Navier-Stokes Equations

$$\frac{\partial \Delta \psi}{\partial t} - ((\nabla\times\psi)\cdot\nabla)\Delta\psi + (\Delta\psi\cdot\nabla)(\nabla\times\psi) - \nu\Delta^2\psi = \nabla\times\mathbf{f}, \quad \text{in} \quad \Omega \qquad (39)$$

in the cube $\Omega = (0,1)^3$. Here, the source term $\mathbf{g} = \nabla\times\mathbf{f}$ is chosen such that $\psi^T(\mathbf{x},t) = -\frac{1}{4}e^{-t}\left(z^4, x^4, y^4\right)$ is an exact solution of (39). In Table 3 we present results for $t = 0.00625$ using $\Delta t = 0.1h^2$.

	grid $5\times5\times5$	rate	grid $9\times9\times9$	rate	grid $17\times17\times17$
e	2.4497(-9)	3.86	1.6924(-10)	4.01	1.0473(-11)
e_y	7.6486(-9)	3.75	5.6845(-10)	3.98	3.5917(-11)
div (ψ)	1.2294(-8)	3.71	9.3619(-10)	3.92	6.1700(-11)

Table 3 Navier-Stokes equations for $t = 0.00625$ using $\Delta t = 0.1h^2$.

In Table 4 we show results for the Navier-Stokes Equations with $\Delta t = h^2$ for $t = 0.0625$. In Figures 1(a) and 1(b) we display the errors for Navier-Stokes equations in ψ_3 and $(\psi_3)_y$ at $t = 0.0625$ with $dt = h^2$ and a 17^3 grid.

	grid $5 \times 5 \times 5$	rate	grid $9 \times 9 \times 9$	rate	grid $17 \times 17 \times 17$
e	9.4418(-7)	4.46	4.2709(-8)	4.04	2.5934(-9)
e_y	2.9836(-6)	4.38	1.4334(-7)	4.03	8.7800(-9)
div (ψ)	5.0471(-6)	4.40	2.3944(-7)	4.02	1.4778(-8)

Table 4 Navier-Stokes equations for $t = 0.0625$ using $\Delta t = h^2$.

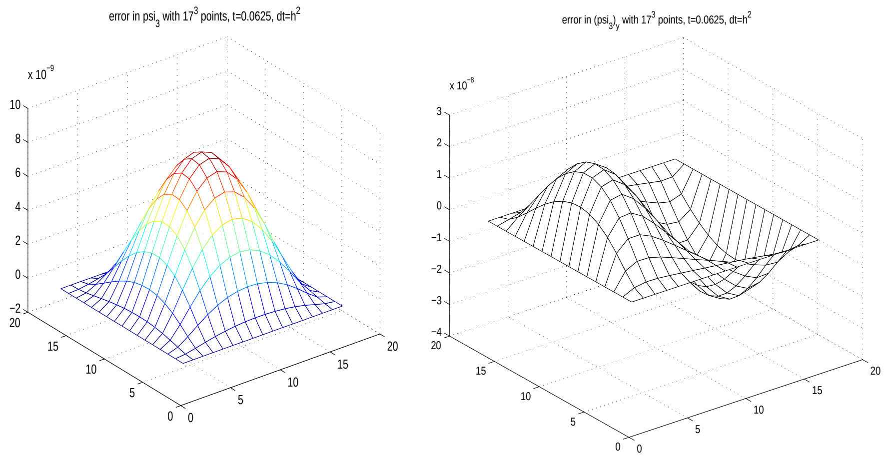

Fig. 1 Navier-Stokes : Errors in (a) ψ_3 and (b) $(\psi_3)_y$ for $N = 17$, $t = 0.0625$, $dt = h^2$.

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