SPECTRAL THEORY FOR DIVERGENCE-FORM OPERATORS

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1. Introduction and Statement of Results

Let $H = -\sum_{i,j=1}^{n} \partial_i a_{i,j}(x)\partial_j$, where $a_{i,j} = a_{j,i}$, be a formally self-adjoint operator in $L^2(\mathbb{R}^n)$, $n \geq 2$, where the notation $\partial_j = \frac{\partial}{\partial x_j}$ has been used.

We assume that the real measurable matrix function $a(x) = \{a_{i,j}(x)\}_{1\leq i,j\leq n}$ satisfies, with some positive constants $a_1 > a_0 > 0, \quad \Lambda_0 > 0$,

$$a_0 I \leq a(x) \leq a_1 I, \qquad x \in \mathbb{R}^n, \tag{1.1}$$

$$a(x) = I \qquad for \quad |x| > \Lambda_0. \tag{1.2}$$

In what follows we shall use the notation $H = -\nabla \cdot a(x)\nabla$.
We retain the notation H for the self-adjoint (Friedrichs) extension associated with the form $(a(x)\nabla\varphi, \nabla\psi)$, where (,) is the scalar product in $L^2(\mathbb{R}^n)$. When $a(x) \equiv I$ we get $H = H_0 = -\Delta$.

Let

$$R_0(z) = (H_0 - z)^{-1}, R(z) = (H - z)^{-1}, \quad z \in \mathcal{C}^{\pm} = \{z/ \quad \pm Imz > 0\},$$

be the associated resolvent operators.

The purpose of this paper is to study the continuity properties of $R(z)$ in certain operator topologies, as z approaches the real axis. To fix the ideas, we shall generally assume that $Imz > 0$, with obvious modifications for $Imz < 0$.

Definition 1.1. *Let $[\alpha, \beta] \subseteq \mathbb{R}$. We say that H satisfies the "Limiting Absorption Principle" (LAP) in $[\alpha, \beta]$ if $R(z), z \in \mathcal{C}^+$, can be extended continuously to $Imz = 0, Rez \in [\alpha, \beta]$, in a suitable operator topology. In this case we denote the limiting values by $R^+(\lambda), \quad \alpha \leq \lambda \leq \beta$.*

A similar definition applies for $z \in \mathcal{C}^-$, but the limiting values $R^-(\lambda)$ will be, generally speaking, different from $R^+(\lambda)$. Observe that the precise specification of the operator topology in the above definition is left open. Typically, it will be the uniform operator topology associated with weighted-L^2 or Sobolev spaces, which will be introduced later.

It is well-known that our assumptions (1.1), (1.2) imply that $\sigma(H)$, the spectrum of H, is the half-axis $[0, \infty)$, and is entirely absolutely continuous. The "threshold" $z = 0$ plays a special role in this setting, as we shall see later. Thus, consider first the case $[\alpha, \beta] \subseteq (0, \infty)$.

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Under assumptions close to ours here (but also assuming that $a(x)$ is continuously differentiable) a weaker version (roughly, "strong" instead of "uniform" convergence of the resolvents) was obtained by Eidus [14, Theorem 4 and Remark 1]. For $H = H_0$ the LAP has been established by Agmon [1]. Indeed, it was established for operators of the type $H_0 + V$, where V is a short-range perturbation. However, an inspection of Agmon's perturbation-theoretic proof shows that it cannot be extended to our operator H, in a straightforward way. Observe on the other hand that the short-range potential V can be replaced by a potential depending only on direction $(x/|x|)$ [15] or a perturbation of such a potential [23, 24]. In this case the condition $\alpha > 0$ is replaced by $\alpha > \limsup_{|x|\to\infty} V(x)$. The LAP for the periodic case (namely, $a(x)$ is symmetric and periodic) has recently been established in [22] . Note that in this case the spectrum is absolutely continuous and consists of a union of intervals ("bands").

We also refer to [16] where the existence and completeness of the wave operators $W_\pm(H, H_0)$ is established under suitable smoothness assumptions on $a(x)$ (however, $a(x) - I$ is not assumed to be compactly supported and H can include also magnetic and electric potentials). Note that by a well-known theorem of Kato and Kuroda [19] , if H, H_0 satisfy the LAP in $[\alpha, \beta]$ (with respect to the same operator topologies) then the wave operators over this interval exist and are complete.

In this paper we focus on the study of the LAP for H in $[\alpha, \beta]$ where $\alpha < 0 < \beta$. This case has been studied for the Laplacian H_0 [6, Appendix A] and in the one-dimensional case ($n = 1$) in [3, 4, 10]. The present paper deals with the multi-dimensional case $n \geq 2$.

Throughout this paper we shall make use of the following weighted-L^2 and Sobolev spaces. First, for $s \in \mathbb{R}$ and m a nonnegative integer we define.

$$L^{2,s}(\mathbb{R}^n) := \{u(x) \quad / \quad \|u\|_{0,s}^2 = \int_{\mathbb{R}^n} (1 + |x|^2)^s |u(x)|^2 dx < \infty\} \tag{1.3}$$

$$H^{m,s}(\mathbb{R}^n) := \{u(x) \quad / D^\alpha u \in L^{2,s}, \quad |\alpha| \leq m, \quad \|u\|_{m,s}^2 = \sum_{|\alpha \leq m} \|D^\alpha u\|_{0,s}^2\} \tag{1.4}$$

(we write $\|u\|_0 = \|u\|_{0,0}$).

More generally, for any $\sigma \in \mathbb{R}$, let $H^\sigma \equiv H^{\sigma,0}$ be the Sobolev space of order σ, namely,

$$H^\sigma = \{\hat{u} \quad / u \in L^{2,\sigma}, \quad \|\hat{u}\|_{\sigma,0} = \|u\|_{0,\sigma}\} \tag{1.5}$$

where the Fourier transform is defined as usual by

$$\hat{u}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} u(x) \exp(-i\xi x) dx.$$

For negative indices we denote by $\left\{H^{-m,s}, \quad \|\cdot\|_{-m,s}\right\}$ the dual space of $H^{m,-s}$. In particular, observe that any function $f \in H^{-1,s}$ can be represented (not uniquely) as

$$f = f_0 + \sum_{k=1}^n i^{-1} \frac{\partial}{\partial x_k} f_k, \qquad f_k \in L^{2,s}, \quad 0 \leq k \leq n. \tag{1.6}$$

In the case $n = 2$ and $s > 1$, we define

$$L_0^{2,s}(\mathbb{R}^2) = \{u \in L^{2,s}(\mathbb{R}^2) \quad / \hat{u}(0) = 0\},$$

and set $H_0^{-1,s}(\mathbb{R}^2)$ to be the space of functions $f \in H^{-1,s}(\mathbb{R}^2)$ which have a representation (1.6) where $f_k \in L_0^{2,s}, \quad k = 0, 1, 2.$

For any two normed spaces X, Y, we denote by $B(X, Y)$ the space of bounded linear operators from X to Y, equipped with the operator-norm topology.

The fundamental result obtained in the present paper is given in the following theorem.

THEOREM A. *Suppose that $a(x)$ satisfies (1.1),(1.2). Then the operator H satisfies the LAP in $\mathbb{R}$. More precisely, using the density of $L^{2,s}$ in $H^{-1,s}$, consider the resolvent $R(z) = (H - z)^{-1}, \quad Im\, z \neq 0$, as two operator-valued functions, defined respectively in the lower and upper half-planes,*

$$z \to R(z) \in B(H^{-1,s}(\mathbb{R}^n), H^{1,-s}(\mathbb{R}^n)), \qquad s > 1, \quad \pm Im\, z > 0. \tag{1.7}$$

Then these functions can be extended continuously from $\mathcal{C}^\pm = \{z / \pm Im\, z > 0\}$ to $\overline{\mathcal{C}^\pm} = \mathcal{C}^\pm \bigcup \mathbb{R}$, with respect to the operator-norm topology. In the case $n = 2$ replace $H^{-1,s}$ by $H_0^{-1,s}$.

In particular, it follows that the limiting values $R^\pm(\lambda)$ are continuous at $\lambda = 0$ and H has no resonance there. The study of the resolvent near the threshold $\lambda = 0$ is sometimes referred to as "low energy estimates". As mentioned earlier, this result has been established in the case $H = H_0$ [6, Appendix A]. The paper [25] deals with the two-dimensional ($n = 2$) case, but the resolvent $R(z)$ is restricted to continuous compactly supported functions f , thus enabling the use of pointwise decay estimates of $R(z)f$ at infinity. The case of the closely related "acoustic propagator" , where the matrix $a(x) = b(x_1)I$ is scalar and dependent on a single coordinate , has been extensively studied [4, 9, 12, 17, 18, 20], as well as the "anisotropic" case where $b(x_1)$ is a general positive matrix [5]. The proof of the theorem will be given in Section 3. It is based on an extended version of the LAP for H_0, with the resolvent $R_0(z)$ acting on elements of $H^{-1,s}$, for suitable positive values of s (see Section 2).

An important application of the LAP in the case of perturbations of the Laplacian is the derivation of an "eigenfunction expansion theorem", where the eigenfunctions are perturbations of plane waves $\exp(i\xi x)$ [1, 29]. We can use the LAP result of Theorem A in order to derive a similar expansion for the operator H. In fact, our generalized eigenfunctions are given by the following definition.

Definition 1.2. *For every $\xi \in \mathbb{R}^n$ let*

$$\begin{aligned} \psi_\pm(x, \xi) &= -R^\mp(|\xi|^2)((H - |\xi|^2)\exp(i\xi x)) = \\ &R^\mp(|\xi|^2)(\textstyle\sum_{l,j=1}^n \partial_l(a_{l,j}(x) - \delta_{l,j})\partial_j)\exp(i\xi x). \end{aligned} \tag{1.8}$$

The generalized eigenfunctions *of H are defined by*

$$\varphi_\pm(x, \xi) = \exp(i\xi x) + \psi_\pm(x, \xi). \tag{1.9}$$

Remark 1.3. *We label the eigenfunctions as "generalized" because they do not belong to the Hilbert space $L^2(\mathbb{R}^n)$.*

In analogy with the eigenfunction expansion theorem for short or long range perturbations of the Laplacian [1, 29] we can now state an eigenfunction expansion theorem for the operator H. We assume $n \geq 3$ in order to simplify the statement of the theorem. As we show below (see Proposition **??**) the generalized eigenfunctions are (at least) continuous in x, so that the integral in the statement makes sense.

THEOREM B
Suppose that $n \geq 3$ and that $a(x)$ satisfies (1.1),(1.2). For any compactly supported $f \in L^2(\mathbb{R}^n)$ define

$$(\mathbb{F}_\pm f)(\xi) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} f(x)\overline{\varphi_\pm(x,\xi)}dx, \quad \xi \in \mathbb{R}^n. \tag{1.10}$$

Then the transformations $\mathbb{F}_\pm$ can be extended as unitary transformations (for which we retain the same notation) of $L^2(\mathbb{R}^n)$ onto itself. Furthermore, these transformations "diagonalize" H in the following sense.
$f \in L^2(\mathbb{R}^n)$ is in the domain $D(H)$ if and only if $|\xi|^2(\mathbb{F}_\pm f)(\xi) \in L^2(\mathbb{R}^n)$ and

$$H = \mathbb{F}_\pm^* M_{|\xi|^2} \mathbb{F}_\pm, \tag{1.11}$$

where $M_{|\xi|^2}$ is the multiplication operator by $|\xi|^2$.

As is well-known from the theory of Schrödinger operators, the LAP and the eigenfunction expansion theorem provide powerful tools for the treatment of a wide array of related problems. Here we give one such application, dealing with global space-time estimates for a generalized wave equation.

We consider the equation

$$\frac{\partial^2 u}{\partial t^2} = Hu = -\sum_{i,j=1}^n \partial_i a_{i,j}(x)\partial_j u, \tag{1.12}$$

subject to the initial data

$$u(x,0) = u_0(x), \quad \partial_t u(x,0) = v_0(x), \quad x \in \mathbb{R}^n. \tag{1.13}$$

We next replace the assumptions (1.1),(1.2) by stronger ones as follows.

Let $g(x) = (g_{i,j}(x))_{1 \leq i,j \leq n}$ be a smooth Riemannian metric on $\mathbb{R}^n$ such that

$$g(x) = I \qquad for \quad |x| > \Lambda_0. \tag{1.14}$$

and assume that

$$a(x) = g^{-1}(x) = (g^{i,j}(x))_{1 \leq i,j \leq n}. \tag{1.15}$$

We have the following theorem.

THEOREM C
Suppose that $n \geq 3$ and that $a(x)$ satisfies (1.14),(1.15). *Assume further that the geometry defined by the metric g has no "trapped geodesics"* [27]. *Then for any $s > 1$ there exists a constant $C = C(s,n) > 0$ such that the solution to (1.12),(1.13) satisfies*

$$\int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^n} (1+|x|^2)^{-s}|u(x,t)|^2 dxdt \leq C[\|u_0\|_0^2 + \||D|^{-1}v_0\|_0^2], \tag{1.16}$$

where as usual $|D|^{-1}$ *denotes multiplication by the symbol* $|\xi|^{-1}$.

This estimate generalizes similar estimates obtained for the classical ($g = I$) wave equation [2, 21].

We do not provide proofs of the theorems, but we include below the treatment of the *unperturbed* operator H_0. This treatment is already new in the sense that it extends the treatment of the LAP beyond the L^2 setting (see the statement of Theorem A).

2. **The Operator** $H_0 = -\Delta$

Let $\{E_0(\lambda)\}$ be the spectral family associated with H_0, so that

$$(E_0(\lambda)h, h) = \int\limits_{|\xi|^2 \le \lambda} |\hat{h}|^2 d\xi, \qquad \lambda \ge 0, \quad h \in L^2(\mathbb{R}^n). \tag{2.1}$$

Following the methodology of [7, 13] we see that the weak derivative $A_0(\lambda) = \frac{d}{d\lambda} E_0(\lambda)$ exists in $B(L^{2,s}, L^{2,-s})$ for any $s > \frac{1}{2}$ and $\lambda > 0$. (Here and below we write $L^{2,s}$ for $L^{2,s}(\mathbb{R}^n)$). Furthermore,

$$< A_0(\lambda)h, h >= (2\sqrt{\lambda})^{-1} \int\limits_{|\xi|^2 = \lambda} |\hat{h}|^2 d\tau, \tag{2.2}$$

where $<,>$ is the $(L^{2,-s}, L^{2,s})$ pairing and $d\tau$ is the Lebesgue surface measure. Recall that by the standard trace lemma we have

$$\int\limits_{|\xi|^2 = \lambda} |\hat{h}|^2 d\tau \le C\|\hat{h}\|^2_{H^s}, \quad s > \frac{1}{2}. \tag{2.3}$$

However, we can refine this estimate near $\lambda = 0$ as follows.

Proposition 2.1. *Let* $\frac{1}{2} < s < \frac{3}{2}$, $\quad h \in L^{2,s}$. *For* $n = 2$ *assume further that* $s > 1$ *and* $h \in L^{2,s}_0$. *Then*

$$\int\limits_{|\xi|^2 = \lambda} |\hat{h}|^2 d\tau \le C \min(\lambda^\gamma, 1)\|\hat{h}\|^2_{H^s}, \tag{2.4}$$

where

$$0 < \gamma < s - \frac{1}{2}, \tag{2.5}$$

and $C = C(s, \gamma, n)$. *(Actually we can take* $\gamma = s - \frac{1}{2}$ *if* $s \le 1$ *and* $n \ge 3$*).*

Proof. If $n \ge 3$, the proof follows from [8, Appendix]. If $n = 2$ and $1 < s < \frac{3}{2}$ we have, for $h \in L^{2,s}_0$,

$$|\hat{h}(\xi)| = |\hat{h}(\xi) - \hat{h}(0)| \le C_{s,\delta}|\xi|^\delta \|\hat{h}\|_{H^s},$$

for any $0 < \delta < \min(1, s-1)$. Using this estimate in the integral in the right-hand side of (2.4) the claim follows also in this case. $\square$

Combining Equations (2.2),(2.3) and (2.4) we conclude that,

$$
\begin{aligned}
&| < A_0(\lambda)f, g > | \leq < A_0(\lambda)f, f >^{\frac{1}{2}} < A_0(\lambda)g, g >^{\frac{1}{2}} \\
&\leq C \min(\lambda^{-\frac{1}{2}}, \lambda^{\eta}) \|f\|_{0,s} \|g\|_{0,\sigma}, \quad f \in L^{2,s}, \quad g \in L^{2,\sigma},
\end{aligned} \tag{2.6}
$$

where either

$$
\begin{aligned}
&(i) \qquad n \geq 3, \qquad \frac{1}{2} < s, \sigma < \frac{3}{2}, \quad s + \sigma > 2 \quad \text{and} \quad 0 < 2\eta < s + \sigma - 2, \\
&(2.7) \qquad \text{or} \\
&(ii) \qquad n = 2, \qquad 1 < s < \frac{3}{2}, \quad \frac{1}{2} < \sigma < \frac{3}{2}, \quad s + \sigma > 2, \quad 0 < 2\eta < s + \sigma - 2 \\
&\qquad \text{and} \quad \hat{f}(0) = 0.
\end{aligned}
$$

In both cases, $A_0(\lambda)$ is Hölder continuous and vanishes at $0, \infty$, so as in [7] we obtain

Proposition 2.2. *The operator-valued function*

$$
z \to R_0(z) \in \begin{cases} B(L^{2,s}, L^{2,-\sigma}), & n \geq 3, \\ B(L_0^{2,s}, L^{2,-\sigma}), & n = 2, \end{cases} \tag{2.8}
$$

where s, σ satisfy (2.7), can be extended continuously from $\mathcal{C}^{\pm}$ to $\overline{\mathcal{C}^{\pm}}$, in the respective uniform operator topologies.

We shall now extend this proposition to more general function spaces. Let $g \in H^{1,\sigma}$, where s, σ satisfy (2.7).Let $f \in H^{-1,s}$ have a representation of the form (1.6). Equation (2.2) can be extended in an obvious way to yield

$$
i^{-1} < A_0(\lambda) \frac{\partial}{\partial x_k} f_k, g >= (2\sqrt{\lambda})^{-1} \int\limits_{|\xi|^2 = \lambda} \xi_k \hat{f}_k(\xi) \overline{\hat{g}(\xi)} d\tau, \quad k = 1, ..., n. \tag{2.9}
$$

We therefore obtain

Proposition 2.3. *The operator-valued function of Proposition 2.2 is well-defined (and analytic) for nonreal z in the following functional setting.*

$$
z \to R_0(z) \in \begin{cases} B(H^{-1,s}, H^{1,-\sigma}), & n \geq 3, \\ B(H_0^{-1,s}, H^{1,-\sigma}), & n = 2, \end{cases} \tag{2.10}
$$

where s, σ satisfy (2.7). Furthermore, it can be extended continuously from $\mathcal{C}^{\pm}$ to $\overline{\mathcal{C}^{\pm}}$, in the respective uniform operator topologies.

Proof. In view of (2.9) and the considerations preceding Proposition 2.2 , since $g \in H^{1,\sigma}$, we have instead of (2.6),

$$
\begin{aligned}
&| < A_0(\lambda) \frac{\partial}{\partial x_k} f_k, g > | \\
&\leq C \min(\lambda^{-\frac{1}{2}}, \lambda^{\eta}) \|f\|_{-1,s} \|g\|_{1,\sigma} \quad , \quad f \in H^{-1,s}, \quad g \in H^{1,\sigma},
\end{aligned} \tag{2.11}
$$

so that the claim holds true if $H^{1,-\sigma}$ is replaced by $H^{-1,-\sigma}$. However, using that $H_0 R_0(z) = I + z R_0(z)$ we see that also $H_0 R_0(z)$ can be extended continuously (as z approaches the real line from either half-plane) with values in $H^{-1,-\sigma}$. The conclusion of the proposition follows since the norm of $H^{1,-\sigma}$ is equivalent to the graph-norm of H_0 in $H^{-1,-\sigma}$. $\square$

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