Reviews in Mathematical Physics
Vol. 26, No. 7 (2014) 1492001 (3 pages)

DOI: 10.1142/S0129055X14920019

Erratum

Eigenfunction expansions and spacetime estimates for generators in divergence-form

[Reviews in Mathematical Physics, Vol. 22, No. 10 (2010) 1209–1240]

Matania Ben-Artzi
*Institute of Mathematics, Hebrew University,
Jerusalem 91904, Israel
mbartzi@math.huji.ac.il*

Received 3 July 2014
Published 17 July 2014

Let $H = -\sum_{j,k=1}^{n} \partial_j a_{j,k}(x)\partial_k$, where $a_{j,k}(x) = a_{k,j}(x)$, be a formally self-adjoint operator in $L^2(\mathbb{R}^n)$, $n \geq 2$. The notations $\partial_j = \frac{\partial}{\partial x_j}$ and $\partial_t = \frac{\partial}{\partial t}$ are used throughout the paper.

We assume that the real measurable matrix function $a(x) = \{a_{j,k}(x)\}_{1\leq j,k\leq n}$ satisfies, with some positive constants $a_1 > a_0 > 0, \Lambda_0 > 0$,

$$a_0 I \leq a(x) \leq a_1 I, \quad x \in \mathbb{R}^n, \tag{1}$$

$$a(x) = I \qquad \text{for } |x| > \Lambda_0. \tag{2}$$

The first theorem proved in the paper (see Sec. 3) was the following.

Theorem A. *Suppose that $a(x)$ satisfies* (1), (2)*. Then the operator H satisfies the Limiting Absorption Principle* (*LAP*) *in $\mathbb{R}$. More precisely, let $s > 1$ and consider the resolvent $R(z) = (H - z)^{-1}, \operatorname{Im} z \neq 0$, as a bounded operator from $L^{2,s}(\mathbb{R}^n)$ to $H^{1,-s}(\mathbb{R}^n)$.*

Then:

(a) *$R(z)$ is bounded with respect to the $H^{-1,s}(\mathbb{R}^n)$ norm. Using the density of $L^{2,s}$ in $H^{-1,s}$, we can therefore view $R(z)$ as a bounded operator from $H^{-1,s}(\mathbb{R}^n)$ to $H^{1,-s}(\mathbb{R}^n)$.*

(b) *The operator-valued functions, defined respectively in the lower and upper half-planes,*

$$z \to R(z) \in B(H^{-1,s}(\mathbb{R}^n), H^{1,-s}(\mathbb{R}^n)), \quad s > 1, \ \pm\operatorname{Im} z > 0, \tag{3}$$

can be extended continuously from $\mathcal{C}^{\pm} = \{z/\pm \operatorname{Im} z > 0\}$ *to* $\overline{\mathcal{C}^{\pm}} = \mathcal{C}^{\pm} \cup \mathbb{R}$ (*with respect to the operator-norm topology of* $B(H^{-1,s}(\mathbb{R}^n), H^{1,-s}(\mathbb{R}^n))$.

In the case $n = 2$, *replace* $H^{-1,s}$ *by* $H_0^{-1,s}$.

In this generality (with respect to the coefficients $a_{j,k}(x)$), however, it is still possible to have a *discrete sequence of eigenvalues* embedded in the continuous spectrum. In order to exclude such eigenvalues (as in the statement of the theorem), one needs the coefficients to be *Lipschitz continuous*. Hence the correct theorem is the following.

Theorem A-Modified. *Suppose that* $a(x)$ *satisfies* (1), (2). *Then the operator* H *satisfies the Limiting Absorption Principle* (*LAP*) *in* $\mathbb{R}$. *More precisely, let* $s > 1$ *and consider the resolvent* $R(z) = (H - z)^{-1}, \operatorname{Im} z \neq 0$, *as a bounded operator from* $L^{2,s}(\mathbb{R}^n)$ *to* $H^{1,-s}(\mathbb{R}^n)$.

Then:

(a) $R(z)$ *is bounded with respect to the* $H^{-1,s}(\mathbb{R}^n)$ *norm. Using the density of* $L^{2,s}$ *in* $H^{-1,s}$, *we can therefore view* $R(z)$ *as a bounded operator from* $H^{-1,s}(\mathbb{R}^n)$ *to* $H^{1,-s}(\mathbb{R}^n)$.

(b) *There exists* (*at most*) *a discrete sequence* $\Lambda \subseteq (0, \infty)$, *having no finite accumulation points* (*and, in particular, does not accumulate at zero*), *of eigenvalues of* H.

The operator-valued functions, defined respectively in the lower and upper half-planes,

$$z \to R(z) \in B(H^{-1,s}(\mathbb{R}^n), H^{1,-s}(\mathbb{R}^n)), \quad s > 1, \ \pm\operatorname{Im} z > 0, \tag{4}$$

can be extended continuously from $\mathcal{C}^{\pm} = \{z/\pm\operatorname{Im} z > 0\}$ *to* $\overline{\mathcal{C}^{\pm}} \backslash \Lambda = \mathcal{C}^{\pm} \cup (\mathbb{R} \backslash \Lambda)$ (*with respect to the operator-norm topology of* $B(H^{-1,s}(\mathbb{R}^n), H^{1,-s}(\mathbb{R}^n))$).

In the case $n = 2$ *replace* $H^{-1,s}$ *by* $H_0^{-1,s}$.

(c) *If in addition to* (1), (2) *the coefficients* $a_{j,k}(x)$ *are Lipschitz continuous, then there are no positive embedded eigenvalues, namely,* $\Lambda = \emptyset$.

Outline of modified proof. We need to modify the argument in Sec. 5, in the paragraph following Eq. (5.11).

Thus, suppose first that $z_0 > 0$. Note that u vanishes for $|x| > \Lambda_0 + 2$. If the coefficients are Lipschitz continuous, then the unique continuation property holds [4] and we get $u \equiv 0$, which leads to a contradiction as in the paper. Thus part (c) is proved.

However, if the coefficients only satisfy (1), (2), then an example by Filonov [3] shows that the unique continuation property does not hold and the function u can still be a nontrivial eigenfunction of H, with the eigenvalue z_0. Note that it is compactly supported.

Let $K \subseteq (0, \infty)$ be a compact interval with $z_0 \in K$.

From the equation $Hu = z_0 u$, we obtain (no weights are needed due to the compact support),

$$\|u\|_1 \leq C\|u\|_0.$$

The constant $C > 0$ depends only on K. By a classical argument of Agmon [1] the last estimate implies that any set of eigenfunctions of H, with eigenvalues in K, is compact in $L^2(\mathbb{R}^n)$. It must therefore be finite, and that establishes part (b).

Also, $z_0 = 0$ cannot be an eigenvalue, since $Hu = 0$ implies $\nabla u = 0$, which in turn entails $u = 0$.

Finally, we show that there are no eigenvalues in $(0, \delta)$, for $\delta > 0$ sufficiently small. Indeed, by the Poincaré inequality there exists $c > 0$ such that, for all H^1 functions supported in $|x| < \Lambda_0 + 2$,

$$\|\nabla u\|_0^2 \geq c\|u\|_0^2.$$

Let $0 < \delta < ca_0$. Now if $z_0 \in (0, \delta)$ is an eigenvalue, and u is the associated normalized eigenfunction, then $(Hu, u) = z_0(u, u) < \delta$. On the other hand we have by (1),

$$(Hu, u) \geq a_0\|\nabla u\|_0^2 \geq ca_0 > \delta,$$

which is a contradiction to the preceding inequality.

Acknowledgment

I thank Professor Hendrik Vogt for pointing out the error discussed in this erratum and bringing to my attention the paper of Filonov cited here.

References

[1] S. Agmon, Spectral properties of Schrödinger operators and scattering theory, *Ann. Sc. Norm. Super. Pisa* **2** (1975) 151–218.

[2] G. Alessandrini, Strong unique continuation for general elliptic equations in 2D, *J. Math. Anal. App.* **386** (2012) 669–676.

[3] N. Filonov, Second-order elliptic equation of divergence form having a compactly supported solution, *J. Math. Sci.* **106** (2001) 3078–3086.

[4] H. Koch and D. Tataru, Carleman estimates and unique continuation for second order elliptic equations with nonsmooth coefficients, *Comm. Pure Appl. Math.* **54** (2001) 339–360.