Divergence-type operators: Spectral theory and spacetime estimates

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Abstract. The paper is concerned with various aspects of the spectral structure of the operator

$$H = -\sum_{j,k=1}^{n} \partial_{x_j} a_{j,k}(x) \partial_{x_k}.$$

It is assumed to be formally self-adjoint in $L^2(\mathbb{R}^n)$, $n \geq 2$. The real coefficients $a_{j,k}(x) = a_{k,j}(x)$ are assumed to be bounded and H is assumed to be uniformly elliptic and to coincide with $-\Delta$ outside of a ball. A Limiting Absorption Principle (LAP) is proved in the framework of weighted Sobolev spaces. It is then used for (i) A general eigenfunction expansion theorem and (ii) Global spacetime estimates for the associated (inhomogeneous) generalized wave equation.

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1. Introduction

Let H be a self-adjoint (bounded or unbounded) operator in a Hilbert space $\mathcal{H}$. The classical spectral theorem [59] gives a representation of H,

$$H = \int_{\mathbb{R}} \lambda \, dE(\lambda),$$

in terms of its (uniquely determined) spectral family (of projections) $\{E(\lambda)\}$.

The knowledge of $\{E(\lambda)\}$ yields valuable information on the spectral structure of H; the location of its singular or absolutely continuous spectrum, as well as its eigenvalues.

On the other hand, there are important issues (typically related to partial differential operators) that cannot be resolved simply on the basis of the spectral theorem. We pick here one important topic and expound it in more detail, so as to illustrate the point at hand.

Assuming that $\{E(\lambda)\}$ is (strongly) continuous from the left, one might think of $E(\lambda+0)-E(\lambda)$ as a projection on the *eigenspace* associated with λ. However, if λ is not an eigenvalue, this projection clearly vanishes. On the other hand, the mathematical foundation of quantum mechanics has turned the *expansion by generalized eigenfunctions* (such as the Fourier transform with respect to the Laplacian) into a basic tool of the theory (see e.g. [86] for an early treatment). So the question is how (if at all possible) to incorporate such an expansion into the abstract framework of the spectral theorem. We shall address this question in Section 5, where we show how the basic premise of this review, namely, the *smoothness* concept of the spectral family, leads to an eigenfunction expansion theorem for the class of *divergence-type* operators.

Using a formal point of view we can say that the bridge between the spectral theorem and the aforementioned *eigenfunction expansion theorem* is obtained by replacing the above difference $E(\lambda+0)-E(\lambda)$ by its *scaled version*, the (formal, at this stage) derivative $\frac{d}{d\lambda}E(\lambda)$. In fact, this derivative is the cornerstone of the present review.

Certainly, this derivative is far from being a *new object*. In the physical literature it is known as the *density of states* [29, Chapter XIII]. It has appeared implicitly in many mathematical studies of quantum mechanics.

After introducing our basic notational conventions and functional spaces in Section 2, we present the basic abstract setting in Section 3. This structure was first established in a joint work with the late A. Devinatz [15]. It relies on the fundamental hypothesis that the spectral derivative is Hölder continuous in a suitable functional setting. The primary aim is to establish a *Limiting Absorption Principle* (LAP), namely, that the resolvents (from either side of the spectrum) remain continuous up to the (absolutely continuous) spectrum in this setting. Once established for an operator H, we show in Subsection 3.2 that it persists to functions $f(H)$, for a wide family of functions f, with interesting results for operators of mathematical physics, such as the *relativistic Schrödinger* operator. It is pointed out that without the smoothness assumption, the validity of the LAP for H does not necessarily imply its validity even for H^2.

The next three sections are devoted to the main application considered in this review, namely, a detailed study of the operator

$$H=-\sum_{j,k=1}^{n}\frac{\partial}{\partial x_j}a_{j,k}(x)\frac{\partial}{\partial x_k},$$

which is assumed to be formally self-adjoint in $L^2(\mathbb{R}^n)$, $n\geq 2$. The real coefficients $a_{j,k}(x)=a_{k,j}(x)$ are assumed to be bounded and H is assumed to be uniformly elliptic and to coincide with $-\Delta$ outside of a ball. In particular, the coefficients can be discontinuous. It is readily seen that these assumptions imply that $\sigma(H)$, the spectrum of H, is the half-axis $[0,\infty)$, and is entirely continuous. The *threshold* $z=$

0 plays a special role in this setting. The absolute continuity of the spectrum was established in [11]. It is a straightforward result of the LAP, which is established in Section 4 . In particular, we show that the limiting values of the resolvent remain continuous across the threshold (which is therefore not a resonance).

Since its appearance in the classical works of Eidus [40] and Agmon [1], the LAP has proven to be a fundamental tool in the study of spectral and scattering theory . The method of Eidus (for second-order elliptic operators) relied on careful elliptic estimates while the method of Agmon used Fourier analysis (division by symbols with simple zeros), followed by a perturbative ("bootstrap") argument to deal with lower order terms. This latter method, extended to simply characteristic operators of any order, is expounded in [49, Chapter 14]. The method of Mourre (also known as the "conjugate operator method") [68] paved the way to the breakthrough in the study of the (quantum) $N-$body problem [70]. We refer to the monographs [4, 36] for the presentation of Mourre's method in an abstract framework. We also refer to the recent paper [41], where the LAP is proved by using a combination of Mourre's method and energy estimates.

The LAP for the divergence-type operator H introduced above cannot be obtained by a straightforward application of either one of these methods. Firstly, the presence of the non-constant coefficients $a_{j,k}(x)$ means that H is not a relatively compact perturbation of the Laplacian, and the perturbation method of Agmon cannot be applied. Secondly, if one insists (as we do here) on assuming only boundedness (and not smoothness) of these coefficients, the method of Mourre, as used in the semiclassical literature [76], cannot be applied (the conjugate operator is related to a generator of the corresponding flow that , in turn, relies on smoothness). In contrast, our approach to the LAP enables us to obtain resolvent estimates for the Laplacian *beyond* the L^2 setting, by using $H^{-1,s}$ weighted Sobolev spaces (see Subsection 4.1). In this context the operator H can be handled as a perturbation of the Laplacian.

We note in addition that both Agmon's and Mourre's methods cannot be applied across the threshold at $z = 0$. Here we obtain continuity of the limiting values of the resolvent across the threshold, at the expense of using a more restrictive weight function. This fact is essential in the the treatment of global spacetime estimates in Section 6.

A more detailed discussion of the relevant literature is given in Section 4.

Section 5 is devoted to the eigenfunction expansion theorem (by generalized eigenfunctions) associated with the operator H. We have already touched upon this topic above, illustrating the differences between the general (abstract) spectral theorem and the detailed *Fourier-type* expansion needed in applications. We expand on this issue in the section.

A global spacetime estimate for the associated (inhomogeneous) generalized wave equation is proved in Section 6. We chose to bring this example (instead of the simpler Schrödinger-type equation) in order to stress the various possibilities available with the tool of the spectral derivative. In doing so we need to restrict much further our class of coefficient matrices. In fact, in order to obtain good control on the behavior of the limiting values of the spectral derivative at *high energy*, we need to use geometric assumptions (*non-trapping trajectories*), which are common in semiclassical theory.

2. Functional spaces and notation

We collect here some basic notations and functional spaces to be used throughout this paper.

The closure of a set Ω (either in the real line $\mathbb{R}$ or in the complex plane $\mathbb{C}$) is denoted by $\overline{\Omega}$.

For any two normed spaces X, Y, we denote by $B(X,Y)$ the space of bounded linear operators from X to Y, equipped with the operator norm $\| \ \|_{B(X,Y)}$ topology (to which we refer as the *uniform operator topology*). In the case $X = Y$ we simplify to $B(X)$.

The following weighted L^2 and Sobolev spaces will appear frequently. First, for $s \in \mathbb{R}$ and m a non-negative integer, we define

$$L^{2,s}(\mathbb{R}^n) := \Big\{ u(x) \ \Big| \ \|u\|_{0,s}^2 = \int_{\mathbb{R}^n} (1+|x|^2)^s |u(x)|^2 dx < \infty \Big\}$$

$$H^{m,s}(\mathbb{R}^n) := \Big\{ u(x) \ \Big| \ D^\alpha u \in L^{2,s}, \ \ |\alpha| \le m, \ \ \|u\|_{m,s}^2 = \sum_{|\alpha \le m} \|D^\alpha u\|_{0,s}^2 \Big\}$$

(we write L^2 for $L^{2,0}$ and $\|u\|_0 = \|u\|_{0,0}$). More generally, for any $\sigma \in \mathbb{R}$, let $H^\sigma \equiv H^{\sigma,0}$ be the Sobolev space of order σ, namely,

$$H^\sigma = \big\{ \hat{u} \ \big| \ u \in L^{2,\sigma} \big\},$$

$\|\hat{u}\|_{\sigma,0} = \|u\|_{0,\sigma}$, where the Fourier transform is defined as usual by

$$\hat{u}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} u(x) \exp(-i\xi x) dx. \tag{2.1}$$

For negative indices, we denote by $\big\{ H^{-m,s}, \| \cdot \|_{-m,s} \big\}$ the dual space of $H^{m,-s}$. In particular, observe that any function $f \in H^{-1,s}$ can be represented (not uniquely) as

$$f = f_0 + \sum_{k=1}^n i^{-1} \frac{\partial}{\partial x_k} f_k, \quad f_k \in L^{2,s}, \ \ 0 \le k \le n. \tag{2.2}$$

In the case $n = 2$ and $s > 1$, we define

$$L_0^{2,s}(\mathbb{R}^2) = \big\{ u \in L^{2,s}(\mathbb{R}^2) \ \big| \ \hat{u}(0) = 0 \big\},$$

and set $H_0^{-1,s}(\mathbb{R}^2)$ to be the space of functions $f \in H^{-1,s}(\mathbb{R}^2)$ which have a representation (2.2), where $f_k \in L_0^{2,s}$, $k = 0, 1, 2$.

3. The basic abstract structure

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$ (the complex numbers), whose scalar product and norm we denote, respectively, by $(\ ,\ )$ and $\|\ \|$.

Let $\mathcal{X}$ be another Hilbert space such that $\mathcal{X} \subseteq \mathcal{H}$, where the embedding is dense and continuous. In other words, $\mathcal{X}$ can be considered as a dense subspace of $\mathcal{H}$, equipped with a stronger norm. Then, of course, $\mathcal{X} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{X}^*$, where $\mathcal{X}^*$ is the anti-dual of $\mathcal{X}$, i.e., the continuous additive functionals l on $\mathcal{X}$, such that $l(\alpha v) = \overline{\alpha}\, l(v)$, $\alpha \in \mathbb{C}$. The (linear) embedding $h \in \mathcal{H} \hookrightarrow x^* \in \mathcal{X}^*$ is obtained as usual by the scalar product (in $\mathcal{H}$), $x^*(x) = (h, x)$.

We use $\|x\|_{\mathcal{X}}$, $\|x^*\|_{\mathcal{X}^*}$ for the norms in $\mathcal{X}$, $\mathcal{X}^*$, respectively, and designate by $<\ ,\ >$ the $(\mathcal{X}^*, \mathcal{X})$ pairing.

Let H be a self-adjoint (in general unbounded) operator on $\mathcal{H}$ and let $\{E(\lambda)\}$ be its spectral family. Let

$$R(z) = (H - z)^{-1}, \quad z \in \mathbb{C}^{\pm} = \big\{ z \;\big|\; \pm \operatorname{Im} z > 0 \big\},$$

be the associated resolvent operator. We denote by $\sigma(H) \subseteq \mathbb{R}$ the spectrum of H.

Clearly, if $\lambda \in \sigma(H)$, then $R(z)$ cannot converge to a limit in the uniform operator topology of $B(\mathcal{H})$ as $z \to \lambda$. However, a basic notion in our treatment is the fact that such *continuity up to the spectrum* of the resolvent can be achieved in a weaker topology. We begin with the following definition.

Definition 3.1. Let $[\kappa_1, \kappa_2] \subseteq \mathbb{R}$. We say that H satisfies the *Limiting Absorption Principle* (LAP) in $[\kappa_1, \kappa_2]$ if $R(z)$, $z \in \mathbb{C}^{\pm}$, can be extended continuously to $\operatorname{Im} z = 0$, $\operatorname{Re} z \in [\kappa_1, \kappa_2]$, in the uniform operator topology of $B(\mathcal{X}, \mathcal{X}^*)$. In this case we denote the limiting values by $R^{\pm}(\lambda)$, $\kappa_1 \leq \lambda \leq \kappa_2$.

Remark 3.2. By the well-known Stieltjes formula [59], for all $x \in \mathcal{X}$,

$$((E(\delta) - E(\gamma))x, x) = \frac{1}{2\pi i} \int_{\gamma}^{\delta} \big\langle \big(R^+(\lambda) - R^-(\lambda)\big)\, x, x \big\rangle\, d\lambda, \quad [\gamma, \delta] \subseteq [\kappa_1, \kappa_2],$$

it follows that H is absolutely continuous in $[\kappa_1, \kappa_2]$.

Remark that our assumptions readily imply that the uniform operator topology of $B(\mathcal{X}, \mathcal{X}^*)$ is weaker than that of $B(\mathcal{H})$. Also note that the limiting values $R^-(\lambda)$ are, generally speaking, different from $R^+(\lambda)$.

For reasons to become clear later, we introduce still another Hilbert space $\mathcal{X}_H^*$, which is a dense subspace of $\mathcal{X}^*$, equipped with a stronger norm (so that the embedding $\mathcal{X}_H^* \hookrightarrow \mathcal{X}^*$ is continuous). However, we do not require that $\mathcal{H}$ be embedded in $\mathcal{X}_H^*$. As indicated by the notation, $\mathcal{X}_H^*$ may depend on H (see Example 3.5 below). A typical case would be when H can be extended as a densely defined operator in $\mathcal{X}^*$ and $\mathcal{X}_H^*$ would be its domain there, equipped with the graph norm. This will be the case in Theorem 3.11 below.

Let $\{E(\lambda)\}$ be the spectral family of H. When there is no risk of confusion, we also use $E(B)$ to denote the spectral projection on any Borel set B (so that $E(\lambda) = E(-\infty, \lambda)$).

Definition 3.3. Let $U \subseteq \mathbb{R}$ be open and let $0 < \alpha \leq 1$. Assume that U is of *full spectral measure*, namely, $E(\mathbb{R} \setminus U) = 0$. Then H is said to be of *type* $(\mathcal{X}, \mathcal{X}_H^*, \alpha, U)$ if the following conditions are satisfied:

1. The operator-valued function
$$\lambda \to E(\lambda) \in B(\mathcal{X}, \mathcal{X}^*), \quad \lambda \in U,$$
is weakly differentiable with a locally Hölder continuous derivative in $B(\mathcal{X}, \mathcal{X}_H^*)$; that is, there exists an operator-valued function
$$\lambda \to A(\lambda) \in B(\mathcal{X}, \mathcal{X}_H^*), \quad \lambda \in U,$$
so that (recall that $(\ ,\ )$ is the scalar product in $\mathcal{H}$ while $<\ ,\ >$ is the $(\mathcal{X}^*, \mathcal{X})$ pairing)
$$\frac{d}{d\lambda}(E(\lambda)x, y) = < A(\lambda)x, y >, \quad x,\, y \in \mathcal{X},\, \lambda \in U,$$
and such that for every compact interval $K \subseteq U$, there exists an $M_K > 0$ satisfying
$$\|A(\lambda) - A(\mu)\|_{B(\mathcal{X}, \mathcal{X}_H^*)} \leq M_K \, |\lambda - \mu|^{\alpha}, \quad \lambda,\, \mu \in K.$$
2. For every bounded open set $J \subseteq U$ and for every compact interval $K \subseteq J$, the operator-valued function (defined in the weak sense)
$$z \to \int_{U \setminus J} \frac{A(\lambda)}{\lambda - z}\, d\lambda, \quad z \in \mathbb{C},\, \operatorname{Re} z \in K,\, |\operatorname{Im} z| \leq 1,$$
takes values and is Hölder continuous in the uniform operator topology of $B(\mathcal{X}, \mathcal{X}_H^*)$, with exponent α.

Remark 3.4. We could *localize* this definition and, in particular, relax the assumption that $E(\mathbb{R} \setminus U) = 0$. However, this is not needed for the operators discussed in this review, typically perturbations of operators with absolutely continuous spectrum (see the following example below).

Example 3.5 ($H_0 = -\Delta$). (This example will be continued in Subsections **??** and 4.1).

We take the operator H_0 to be the unique self-adjoint extension of the restriction of $-\Delta$ to smooth compactly supported functions [59]. Let $\{E_0(\lambda)\}$ be the spectral family associated with H_0 so that, using the Fourier notation introduced in Section 2,
$$(E_0(\lambda)h, h) = \int\limits_{|\xi|^2 \leq \lambda} |\hat{h}|^2 \, d\xi, \quad \lambda \geq 0, \;\; h \in L^2(\mathbb{R}^n). \tag{3.1}$$
We refer to Section 2 for definitions of the weighted L^2 and Sobolev spaces involved in the sequel. Recall that by the standard trace lemma, we have
$$\int\limits_{|\xi|^2 = \lambda} |\hat{h}|^2 \, d\tau \leq C \big\|\hat{h}\big\|_{H^s}^2, \quad s > \frac{1}{2},\, \lambda > 0, \tag{3.2}$$

where $C > 0$ is independent of λ and $d\tau$ is the restriction of the Lebesgue measure (see [15] for the argument that it can be used for the full half-axis, not just compact intervals).

We conclude that the weak derivative $A_0(\lambda) = \frac{d}{d\lambda}E_0(\lambda)$ exists in the space $B(L^{2,s}, L^{2,-s})$ for any $s > \frac{1}{2}$ and $\lambda > 0$ and satisfies

$$< A_0(\lambda)h, k >= (2\sqrt{\lambda})^{-1} \int\limits_{|\xi|^2=\lambda} \hat{h}\overline{\hat{k}}\, d\tau, \quad h,\, k \in L^{2,s}, \tag{3.3}$$

where $< \, , >$ is the $(L^{2,-s}, L^{2,s})$ pairing (conjugate linear with respect to the second term) and $d\tau$ is the Lebesgue surface measure (we write $L^{2,s}$ for $L^{2,s}(\mathbb{R}^n)$).

Furthermore, by taking s large in (3.2) (it suffices to take $s > \frac{n}{2}+2$) and using the Sobolev imbedding theorem we infer that $A_0(\lambda)$ is locally Lipschitz continuous in the uniform operator topology, so that by interpolation it is locally Hölder continuous in the uniform operator topology of $B(L^{2,s}, L^{2,-s})$ for any $s > \frac{1}{2}$.

Finally, since the (distributional) Fourier transform of $A_0(\lambda)h$ is the surface density $(2\sqrt{\lambda})^{-1}\delta_{|\xi|^2-\lambda}\hat{h}(\xi)\, d\tau$, we conclude that actually $A_0(\lambda)h \in H^{m,-s}$, $s > \frac{1}{2}$, for any $m > 0$, and $A_0(\lambda)$ is locally Hölder continuous in the uniform operator topology of $B(L^{2,s}, H^{m,-s})$ for any $s > \frac{1}{2}$.

Thus, all the requirements of Definition 3.3 are satisfied with $\mathcal{X} = L^{2,s}(\mathbb{R}^n)$, $\mathcal{X}^*_{H_0} = H^{2,-s}(\mathbb{R}^n)$, $s > \frac{1}{2}$.

3.1. The limiting absorption prinicple – LAP

Recall first the classical Privaloff-Korn theorem (see [31] for a proof).

Theorem. *Let $f\colon \mathbb{R} \to \mathbb{C}$ be a compactly supported Hölder continuous function so that, for some $N > 0$ and $0 < \alpha < 1$,*

$$|f(\lambda_2) - f(\lambda_1)| \le N\,|\lambda_2 - \lambda_1|^{\alpha}, \quad \lambda_2,\, \lambda_1 \in \mathbb{R}.$$

Let

$$F^{\pm}(z) = \int_{\mathbb{R}} \frac{f(\lambda)}{\lambda - z}\, d\lambda, \quad z \in \mathbb{C}^{\pm}.$$

Then, for every $\mu \in \mathbb{R}$, the limits

$$F^{\pm}(\mu) = \lim F(z) = \pm i\pi f(\mu) + \mathrm{P.V.} \int_{\mathbb{R}} \frac{f(\lambda)}{\lambda - \mu} d\lambda \quad \text{as } z \to \mu,\ \pm \operatorname{Im} z > 0,$$

exist and moreover, for every compact $K \subseteq \overline{\mathbb{C}^+}$ (or $K \subseteq \overline{\mathbb{C}^-}$), there exists a constant M_K so that

$$|F^{\pm}(z_2) - F^{\pm}(z_1)| \le N M_K\,|z_2 - z_1|^{\alpha}, \quad z_1,\, z_2 \in K.$$

We can now state our basic theorem, concerning the LAP in the abstract setting. We remark that a slightly different version will appear in Subsection 4.3.

Theorem 3.6. *Let H be of* type $(\mathcal{X}, \mathcal{X}^*_H, \alpha, U)$ (*where $U \subseteq \mathbb{R}$ is open and $0 < \alpha \le 1$*). *Then H satisfies the* LAP *in U. More explicitly, the limits*

$$R^{\pm}(\lambda) = \lim_{\epsilon \downarrow 0} R(\lambda \pm i\epsilon), \quad \lambda \in U,$$

exist in the uniform operator topology of $B(\mathcal{X}, \mathcal{X}_H^*)$ *and the extended operator-valued function*

$$R(z) = \begin{cases} R(z), & z \in \mathbb{C}^+, \\ R^+(z), & z \in U, \end{cases}$$

is locally Hölder continuous in the same topology (*with exponent* α).

A similar statement applies when $\mathbb{C}^+$ *is replaced by* $\mathbb{C}^-$, *but note that the limiting values* $R^{\pm}(\lambda)$ *are in general different.*

Proof. Let $J \subseteq U$ be a bounded open set such that $\overline{J} \subseteq U$ and $K \subseteq J$ be a compact interval. Let $\varphi \in C_0^\infty(U)$ be a cutoff function with $\varphi \equiv 1$ on J. Taking $x, y \in \mathcal{X}$, we have, for $\operatorname{Re} z \in K,\ \operatorname{Im} z \neq 0$,

$$\begin{aligned}(R(z)x, y) &= \int_U \frac{\varphi(\mu) < A(\mu)x, y >}{\mu - z}\, d\mu + \int_{U \setminus J} \frac{(1 - \varphi(\mu)) < A(\mu)x, y >}{\mu - z}\, d\mu \\ &= (R_1(z)x, y) + (R_2(z)x, y)\,.\end{aligned}$$

By hypothesis (see Definition 3.3) the operator-valued function

$$R_2(z) = \int_{U \setminus J} \frac{(1 - \varphi(\mu))A(\mu)}{\mu - z}\, d\mu,$$

belongs to $B(\mathcal{X}, \mathcal{X}_H^*)$, and it is locally Hölder continuous for $\operatorname{Re} z \in K$. Thus, we are reduced to considering R_1.

Observe that the integral

$$R_1(z) = \int_{U \setminus J} \frac{\varphi(\mu)A(\mu)}{\mu - z}\, d\mu,$$

is well-defined as a Riemann integral, since the integrand is continuous in the uniform norm topology of $B(\mathcal{X}, \mathcal{X}_H^*)$. Thus $R_1(z) \in B(\mathcal{X}, \mathcal{X}_H^*)$. It remains to prove the assertion concerning its Hölder continuity.

Note that the embeddings $\mathcal{X} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{X}^*$ and $\mathcal{X}_H^* \hookrightarrow \mathcal{X}^* \hookrightarrow \mathcal{X}_H^{**}$ are dense and continuous. Thus, we can view $\mathcal{X}$ as embedded in $\mathcal{X}_H^{**}$, so that the pairing $< A(\mu)x, y >$ can be regarded as an $(\mathcal{X}_H^*, \mathcal{X}_H^{**})$ pairing.

Suppose now that $\operatorname{Im} z_i > 0$, $\operatorname{Re} z_i \in K,\ i = 1, 2$, so that the Privaloff-Korn theorem yields, for $x, y \in \mathcal{X}$,

$$\begin{aligned}&|([R_1(z_2) - R_1(z_1)]x, y)| \\ &\qquad \leq M_K \sup_{\mu_1 \neq \mu_2} \frac{| < [\varphi(\mu_2)A(\mu_2) - \varphi(\mu_2)A(\mu_2)]x, y > |}{|\mu_2 - \mu_1|^\alpha} |z_2 - z_1|^\alpha,\end{aligned}$$

and as observed above

$$\begin{aligned}&| < [\varphi(\mu_2)A(\mu_2) - \varphi(\mu_2)A(\mu_2)]x, y > | \\ &\qquad \leq \left\| [\varphi(\mu_2)A(\mu_2) - \varphi(\mu_2)A(\mu_2)]x \right\|_{\mathcal{X}_H^*} \|y\|_{\mathcal{X}_H^{**}} \\ &\qquad \leq \left\| \varphi(\mu_2)A(\mu_2) - \varphi(\mu_2)A(\mu_2) \right\|_{B(\mathcal{X}, \mathcal{X}_H^*)} \|x\|_{\mathcal{X}} \|y\|_{\mathcal{X}_H^{**}}.\end{aligned}$$

Thus,

$$\left|([R_1(z_2) - R_1(z_1)]x, y)\right| \le N M_K \left|z_2 - z_1\right|^{\alpha} \, \|x\|_{\mathcal{X}} \, \|y\|_{\mathcal{X}_H^{**}},$$

where

$$N = \sup_{\mu_1 \neq \mu_2} \frac{\|\varphi(\mu_2)A(\mu_2) - \varphi(\mu_2)A(\mu_2)\|_{B(\mathcal{X},\mathcal{X}_H^*)}}{|\mu_2 - \mu_1|^{\alpha}}.$$

Since $\mathcal{X}$ is dense in $\mathcal{X}_H^{**}$, the last estimate yields

$$\|R_1(z_2) - R_1(z_1)\|_{B(\mathcal{X},\mathcal{X}_H^*)} \le N M_K \left|z_2 - z_1\right|^{\alpha},$$

and the proof is complete. □

Corollary 3.7. *In view of the Stieltjes formula* (*see* Remark 3.2 *above*) *we have*

$$A(\lambda) = \frac{1}{2\pi i}\left(R^+(\lambda) - R^-(\lambda)\right), \quad \lambda \in U.$$

In particular, H is absolutely continuous in U and $R^+(\lambda) - R^-(\lambda)$ cannot vanish on a subset of $\sigma(H) \cap U$ of positive (*Lebesgue*) *measure.*

Remark 3.8. The operator $A(\lambda)$, $\lambda \in [0, \infty)$, is known in the physical literature as the *density of states* [29, Chapter XIII].

Also, combining the theorem with the observations in Example 3.5 we obtain the following corollary, which is Agmon's classical LAP theorem [1].

Corollary 3.9. *Let $H_0 = -\Delta$ and set $R_0(z) = (H_0 - z)^{-1}$, $\operatorname{Im} z \neq 0$. Then the limits*

$$R_0^{\pm}(\lambda) = \lim_{\epsilon \downarrow 0} R_0(\lambda \pm i\epsilon), \quad \lambda \in (0, \infty),$$

exist in the uniform operator topology of $B(L^{2,s}, H^{2,-s})$, $s > \frac{1}{2}$. Furthermore, these limiting values are Hölder continuous in this topology.

Remark 3.10. The considerations of Example 3.5, based on trace estimates, can be applied to a wide range of constant coefficient partial differential operators (so called *simply characteristic* operators, including all principal-type operators). Hence, a suitable LAP can be established for such operators. We shall not pursue this direction further in this review, but refer the reader to [15].

In general, it is easier to verify the conditions of Definition 3.3 for the operator space $B(\mathcal{X}, \mathcal{X}^*)$ than for $B(\mathcal{X}, \mathcal{X}_H^*)$. However, in some circumstances it is enough to establish the conditions in the latter space. This is expressed in the following theorem.

Theorem 3.11. *Let H be densely defined and closable in $\mathcal{X}^*$, with closure $\overline{H}$. Take $\mathcal{X}_H^* = D(\overline{H})$* (*its domain*), *equipped with the graph norm*

$$\|x\|^2_{\mathcal{X}_H^*} = \|x\|^2_{\mathcal{X}^*} + \|\overline{H}x\|^2_{\mathcal{X}^*}.$$

Suppose that H is of type $(\mathcal{X}, \mathcal{X}^*, \alpha, U)$ (*see* Definition 3.3)*. Then in fact H is of type $(\mathcal{X}, \mathcal{X}_H^*, \alpha, U)$.*

Proof. In view of Theorem 3.6 (where all assumptions hold in $B(\mathcal{X}, \mathcal{X}^*)$) we know that the limits

$$R^{\pm}(\lambda) = \lim_{\epsilon \downarrow 0} R(\lambda \pm i\epsilon), \quad \lambda \in U,$$

exist in the uniform operator topology, are locally Hölder continuous and, furthermore, for all $x \in \mathcal{X}$,

$$\lim_{\epsilon \downarrow 0} \overline{H} R(\lambda \pm i\epsilon)\, x = x + \lambda R^{\pm}(\lambda) x, \quad \lambda \in U.$$

Since $\overline{H}$ is closed in $\mathcal{X}^*$, we obtain

$$\overline{H} R^{\pm}(\lambda)\, x = x + \lambda R^{\pm}(\lambda) x \in \mathcal{X}^*,$$

so that $R^{\pm}(\lambda) x \in \mathcal{X}^*_H$. From the definition of the graph norm topology we see that $R^{\pm}(\lambda)$ is locally Hölder continuous in $B(\mathcal{X}, \mathcal{X}^*_H)$. Thus, using Eq. (3.7), we conclude that the same is true for $A(\lambda)$, so that the first condition in Definition 3.3 is satisfied.

To establish the second condition, let $J \subseteq U$ be an open set and $K \subseteq J$ compact. Let $z \in \mathbb{C}$ with $\operatorname{Re} z \in K$, and let $F(\lambda; z) = \frac{\chi_{U \setminus J}(\lambda)}{\lambda - z}$ (as usual, χ is the characteristic function of the indicated set). By the standard spectral calculus

$$HF(H; z) = \int_U \lambda F(\lambda; z)\, dE(\lambda) = \int_{U \setminus J} \frac{\lambda A(\lambda)}{\lambda - z}\, d\lambda,$$

so that both $F(H; z) = \int_{U \setminus J} \frac{A(\lambda)}{\lambda - z} d\lambda$ and $\overline{H} F(H; z)$ are in $B(\mathcal{X}, \mathcal{X}^*)$ and are, in fact, locally Lipschitz continuous in the uniform operator topology. Thus $z \to F(H; z)$ is locally Lipschitz continuous in $B(\mathcal{X}, \mathcal{X}^*_H)$, which concludes the proof. $\square$

3.2. Persistence of smoothness under functional operations

For a wide class of functions $f : \mathbb{R} \to \mathbb{R}$ the (self-adjoint) operator $f(H)$ is defined via the calculus associated with the spectral theorem [59], namely,

$$f(H) = \int_{\mathbb{R}} f(\lambda)\, dE(\lambda),$$

where $\{E(\lambda)\}$ is the spectral family of H.

Various spectral properties of $f(H)$ (whose spectrum is $\operatorname{Ran} f_{\sigma(H)}$) can be read off from the structure of f. (We use the notation $\operatorname{Ran} f_W$ for the image of $W \subseteq \mathbb{R}$ under f).

However, one important aspect which is missing is the fact that if H satisfies the Limiting Absorption Principle in U, there is no guarantee that $f(H)$ satisfies the same principle in $\operatorname{Ran} f_U$ or any part thereof. This remains true even if f is very smooth, monotone, etc.

In contrast, if H is of type $(\mathcal{X}, \mathcal{X}^*_H, \alpha, U)$, then also $f(H)$ is of that type (with U replaced by $\operatorname{Ran} f_U$ and perhaps a different Hölder exponent), for a rather broad family of functions. This is the content of the next theorem. In particular, in view of Theorem 3.6, also $f(H)$ satisfies the LAP.

We do not attempt to make the most general statement, but instead refer the reader to [20] for further details.

Theorem 3.12. *Let H be of type $(\mathcal{X}, \mathcal{X}_H^*, \alpha, U)$ (where $U \subseteq \mathbb{R}$ is open and $0 < \alpha \leq 1$). Let $f\colon \mathbb{R} \to \mathbb{R}$ be a locally Hölder continuous function. Assume, in addition, that the restriction of f to U is continuously differentiable, and that its derivative f' is positive and locally Hölder continuous on U.*

Then the operator $f(H)$ is of type $(\mathcal{X}, \mathcal{X}_H^*, \alpha', \operatorname{Ran} f_U)$, for some $0 < \alpha' \leq 1$.

Proof. Let $\{F(\lambda)\}$ be the spectral family of $f(H)$. If $B \subseteq \mathbb{R}$ is a Borel set, the spectral theorem yields

$$F(B) = E(f^{-1}(B)),$$

and since $E(\mathbb{R} \setminus U) = 0$ (see Definition 3.3), we can further write

$$F(B) = E(f^{-1}(B) \cap U).$$

Since $f' > 0$ in U, an easy calculation gives for the (weak) derivative

$$\frac{d}{d\mu} F(\mu) = f'(\lambda)^{-1} \frac{d}{d\lambda} E(\lambda), \quad \lambda = f(\mu) \in U.$$

The assertion of the theorem follows directly from this formula. □

In view of Theorem 3.6 we infer that $f(H)$ satisfies the LAP in $\operatorname{Ran} f_U$.

Remark 3.13. It should be remarked that if H satisfies the LAP in the sense of Definition 3.1 (including all the functional setting mentioned there), there is no guarantee that H^2 satisfies the LAP in $\{\mu = \lambda^2 \mid \lambda \in U\}$. For this to be false, however, one needs to find an example where the limiting values of the resolvent are not Hölder continuous.

Continuing Corollary 3.9 and taking $f(\lambda) = \sqrt{|\lambda|+1}$, we obtain a LAP for the *relativistic Schrödinger operator* [20].

Corollary 3.14. *Let $L = \sqrt{-\Delta + I}$ and set $P(z) = (L - z)^{-1}$, $\operatorname{Im} z \neq 0$. The spectrum of L is $\sigma(L) = [1, \infty)$ and is absolutely continuous. The limits*

$$P^{\pm}(\lambda) = \lim_{\epsilon \downarrow 0} P(\lambda \pm i\epsilon), \quad \lambda \in (1, \infty),$$

exist in the uniform operator topology of $B(L^{2,s}, H^{2,-s})$, $s > \frac{1}{2}$. Furthermore, these limiting values are Hölder continuous in this topology.

4. The limiting absorption principle for second-order divergence-type operators

In the following sections we consider *divergence-type* second-order operators. As perturbations of the Laplacian they do not belong to any of the above categories; the difference between such an operator and the Laplacian is not even compact. However, our aim is to show that we can still deal with such operators, starting from the smoothness properties of (the spectral derivative of) the Laplacian.

Let $H = -\sum_{j,k=1}^{n} \partial_j a_{j,k}(x)\partial_k$, where $a_{j,k}(x) = a_{k,j}(x)$, be a formally self-adjoint operator in $L^2(\mathbb{R}^n)$, $n \geq 2$. The notation $\partial_j = \frac{\partial}{\partial x_j}$ is used throughout the following sections.

We assume that the real measurable matrix function $a(x) = \{a_{j,k}(x)\}_{1\leq j,k\leq n}$ satisfies, with some positive constants $a_1 > a_0 > 0$, $\Lambda_0 > 0$,

$$a_0 I \leq a(x) \leq a_1 I, \qquad x \in \mathbb{R}^n, \tag{4.1}$$

$$a(x) = I, \qquad |x| > \Lambda_0. \tag{4.2}$$

In what follows we shall use the notation $H = -\nabla \cdot a(x)\nabla$.

We retain the notation H for the self-adjoint (Friedrichs) extension associated with the form $(a(x)\nabla\varphi, \nabla\psi)$, where $(\ ,\ )$ is the scalar product in $L^2(\mathbb{R}^n)$. When $a(x) \equiv I$, we get $H = H_0 = -\Delta$.

We refer to Section 2 for definitions of the various functional spaces that will appear in what follows.

Let

$$R_0(z) = (H_0 - z)^{-1}, \;\; R(z) = (H - z)^{-1}, \quad z \in \mathcal{C}^{\pm} = \big\{z \;\big|\; \pm \operatorname{Im} z > 0\big\},$$

be the associated resolvent operators.

We note that the operator H can be extended in an obvious way (retaining the same notation) as a bounded operator $H\colon H^1_{\mathrm{loc}} \to H^{-1}_{\mathrm{loc}}$. In particular, $H\colon H^{1,-s} \to H^{-1,-s}$, for all $s \geq 0$. Furthermore, the graph norm of H in $H^{-1,-s}$ is equivalent to the norm of $H^{1,-s}$.

Similarly, we can consider the resolvent $R(z)$ as defined on $L^{2,s}$, $s \geq 0$, where $L^{2,s}$ is densely and continuously embedded in $H^{-1,s}$.

The fundamental result presented in this section is that H satisfies the LAP over the *whole real axis*. The exact formulation is as follows:

Theorem 4.1. *Suppose that $a(x)$ satisfies* (4.1),(4.2)*. Then the operator H satisfies the* LAP *in $\mathbb{R}$. More precisely, let $s > 1$ and consider the resolvent $R(z) = (H - z)^{-1}$, $\operatorname{Im} z \neq 0$, as a bounded operator from $L^{2,s}(\mathbb{R}^n)$ to $H^{1,-s}(\mathbb{R}^n)$.*

Then:

(a) *$R(z)$ is bounded with respect to the $H^{-1,s}(\mathbb{R}^n)$ norm. Using the density of $L^{2,s}$ in $H^{-1,s}$, we can therefore view $R(z)$ as a bounded operator from $H^{-1,s}(\mathbb{R}^n)$ to $H^{1,-s}(\mathbb{R}^n)$.*

(b) *The operator-valued functions, defined respectively in the lower and upper half-planes,*

$$z \to R(z) \in B(H^{-1,s}(\mathbb{R}^n), H^{1,-s}(\mathbb{R}^n)), \quad s > 1, \pm \operatorname{Im} z > 0, \tag{4.3}$$

can be extended continuously from $\mathcal{C}^{\pm} = \big\{z \;\big|\; \pm \operatorname{Im} z > 0\big\}$ to $\overline{\mathcal{C}^{\pm}} = \mathcal{C}^{\pm} \bigcup \mathbb{R}$ (with respect to the uniform operator topology of $B(H^{-1,s}(\mathbb{R}^n), H^{1,-s}(\mathbb{R}^n))$.

In the case $n = 2$ replace $H^{-1,s}$ by $H_0^{-1,s}$.

We denote the limiting values of the resolvent on the real axis by

$$R^{\pm}(\lambda) = \lim_{z\to\lambda,\, \pm \operatorname{Im} z>0} R(z). \tag{4.4}$$

Remark 4.2. Since $L^{2,s}$ (resp. $H^{1,-s}$) is densely and continuously embedded in $H^{-1,s}$ (resp. $L^{2,-s}$), we conclude that the resolvents $R_0(z)$, $R(z)$ can be extended continuously to $\overline{\mathcal{C}^{\pm}}$ in the $B(L^{2,s}(\mathbb{R}^n), L^{2,-s}(\mathbb{R}^n))$ uniform operator topology.

The spectrum of H is therefore entirely absolutely continuous. In particular, it follows that the limiting values $R^{\pm}(\lambda)$ are continuous at $\lambda = 0$ and H has no resonance there.

The study of the resolvent near the threshold $\lambda = 0$ is sometimes referred to as *low energy estimates*. Following the proof of the theorem, at the end of Subsection 4.2, we review some of the existing literature concerning such estimates, as well as some other results pertaining to the LAP in *non short-range* settings.

Before proceeding to the proof of the theorem, we need to obtain more information on the resolvent of the Laplacian.

4.1. The operator $H_0 = -\Delta$ – revisited

The basic properties of this operator have already been discussed in Example 3.5 and Corollary 3.9. In particular, the explicit form of $\{E_0(\lambda)\}$, its spectral family, is given in Eq. (3.1), and the spectral derivative A_0 is given explicitly in Eq. (3.3).

The weighted L^2 estimates for A_0 were obtained by using the trace estimate (3.2).

However, we can refine this estimate near $\lambda = 0$ as follows.

Proposition 4.3. *Let $\frac{1}{2} < s < \frac{3}{2}$, $h \in L^{2,s}$. For $n = 2$ assume further that $s > 1$ and $h \in L_0^{2,s}$. Then*

$$\int\limits_{|\xi|^2=\lambda} |\hat{h}|^2\, d\tau \le C \min\{\lambda^{\gamma}, 1\} \big\|\hat{h}\big\|^2_{H^s}, \tag{4.5}$$

where

$$\begin{aligned} &0 < \gamma = s - \frac{1}{2}, \quad n \ge 3, \\ &0 < \gamma < s - \frac{1}{2}, \quad n = 2, \end{aligned} \tag{4.6}$$

and $C = C(s, \gamma, n)$.

Proof. If $n \ge 3$, the proof follows as in [19, Appendix], when we take into account the fact (*generalized Hardy inequality*) that multiplication by $|\xi|^{-s}$ is bounded from H^s into L^2 [45] (see also [64, Section 9.4]).

If $n = 2$ and $1 < s < \frac{3}{2}$ we have, for $h \in L_0^{2,s}$,

$$|\hat{h}(\xi)| = |\hat{h}(\xi) - \hat{h}(0)| \le C_{s,\delta}\, |\xi|^{\delta}\, \|\hat{h}\|_{H^s},$$

for any $0 < \delta < \min\{1, s-1\}$. Using this estimate in the integral in the right-hand side of (4.5), the claim follows also in this case. $\square$

Combining Eqs. (3.3),(3.2) and (4.5), we conclude that

$$\begin{aligned}|< A_0(\lambda)f, g >| &\le < A_0(\lambda)f, f >^{\frac{1}{2}} < A_0(\lambda)g, g >^{\frac{1}{2}} \\ &\le C \min\{\lambda^{-\frac{1}{2}}, \lambda^{\eta}\} \|f\|_{0,s} \|g\|_{0,\sigma}, \quad f \in L^{2,s},\ g \in L^{2,\sigma},\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}&\text{(i)} \quad n \ge 3, \quad \frac{1}{2} < s, \sigma < \frac{3}{2}, \quad s + \sigma > 2 \quad \text{and} \quad 0 < 2\eta = s + \sigma - 2, \\ &\text{or} \\ &\text{(ii)} \quad n = 2, \quad 1 < s < \frac{3}{2}, \quad \frac{1}{2} < \sigma < \frac{3}{2}, \quad s + \sigma > 2, \quad 0 < 2\eta < s + \sigma - 2 \\ &\qquad\qquad\qquad \text{and} \quad \hat{f}(0) = 0.\end{aligned} \tag{4.8}$$

In both cases, $A_0(\lambda)$ is Hölder continuous and vanishes at 0, ∞, so we obtain as in [15]:

Proposition 4.4. *The operator-valued function*

$$z \to R_0(z) \in \begin{cases} B(L^{2,s}, L^{2,-\sigma}), & n \ge 3, \\ B(L_0^{2,s}, L^{2,-\sigma}), & n = 2, \end{cases} \tag{4.9}$$

where s, σ *satisfy* (4.8), *can be extended continuously from* $\mathcal{C}^{\pm}$ *to* $\overline{\mathcal{C}^{\pm}}$, *in the respective uniform operator topologies.*

Remark 4.5. We note that the conditions (4.8) yield the continuity of $A_0(\lambda)$ *across the threshold* $\lambda = 0$ and hence the continuity property of the resolvent as in Proposition 4.4. However, for the local continuity at any $\lambda_0 > 0$, it suffices to take s, $\sigma > \frac{1}{2}$, as has been stated in Corollary 3.9, which is Agmon's original result [1].

This remark applies equally to the statements below, where the resolvent is considered in other functional settings.

We shall now extend this proposition to more general function spaces. Let $g \in H^{1,\sigma}$, where s, σ satisfy (4.8). Let $f \in H^{-1,s}$ have a representation of the form (2.2). Eq. (3.3) can be extended to yield an operator (for which we retain the same notation)

$$A_0(\lambda) \in B(H^{-1,s}, H^{-1,-\sigma}),$$

defined by (where now $<\,,\,>$ is used for the $(H^{-1,s}, H^{1,\sigma})$ pairing),

$$\begin{aligned}&< A_0(\lambda) \left[f_0 + i^{-1} \sum_{k=1}^{n} \frac{\partial}{\partial x_k} f_k \right], g > \\ &= (2\sqrt{\lambda})^{-1} \int\limits_{|\xi|^2=\lambda} \left[\hat{f}_0(\xi) + \sum_{k=1}^{n} \xi_k \hat{f}_k(\xi) \right] \overline{\hat{g}(\xi)}\, d\tau, \quad f \in H^{-1,s},\ g \in H^{1,\sigma}.\end{aligned} \tag{4.10}$$

(replace $H^{-1,s}$ by $H_0^{-1,s}$ if $n = 2$).

Observe that this definition makes good sense even though the representation (2.2) is not unique, since

$$f = f_0 + \sum_{k=1}^{n} i^{-1} \frac{\partial}{\partial x_k} f_k = \tilde{f}_0 + \sum_{k=1}^{n} i^{-1} \frac{\partial}{\partial x_k} \tilde{f}_k,$$

implies

$$\hat{f}_0(\xi) + \sum_{k=1}^{n} \xi_k \hat{f}_k(\xi) = \hat{\tilde{f}}_0(\xi) + \sum_{k=1}^{n} \xi_k \hat{\tilde{f}}_k(\xi)$$

(as tempered distributions).

To estimate the operator-norm of $A_0(\lambda)$ in this setting we use (4.10) and the considerations preceding Proposition 4.4, to obtain, instead of (4.7), for $k = 1, 2, \ldots, n$,

$$\begin{aligned} &\left| < A_0(\lambda) \frac{\partial}{\partial x_k} f_k, g > \right| \\ &\qquad \le C \min\{\lambda^{-\frac{1}{2}}, \lambda^{\eta}\} \, \|f\|_{-1,s} \, \|g\|_{1,\sigma}, \quad f \in H^{-1,s},\ g \in H^{1,\sigma}, \end{aligned} \tag{4.11}$$

where s, σ satisfy (4.8) (replace $H^{-1,s}$ by $H_0^{-1,s}$ if $n = 2$).

We now define the extension of the resolvent operator by

$$R_0(z) = \int_0^{\infty} \frac{A_0(\lambda)}{\lambda - z} \, d\lambda, \quad \operatorname{Im} z \neq 0. \tag{4.12}$$

The convergence of the integral (in the operator norm) follows from the estimate (4.11).

The LAP in this case is given in the following proposition.

Proposition 4.6. *The operator-valued function $R_0(z)$ is well-defined* (*and analytic*) *for non-real z in the following functional setting.*

$$z \to R_0(z) \in \begin{cases} B(H^{-1,s}, H^{1,-\sigma}), & n \ge 3, \\ B(H_0^{-1,s}, H^{1,-\sigma}), & n = 2, \end{cases} \tag{4.13}$$

where s, σ satisfy (4.8). *Furthermore, it can be extended continuously from $\mathcal{C}^{\pm}$ to $\overline{\mathcal{C}^{\pm}}$, in the respective uniform operator topologies. The limiting values are denoted by $R_0^{\pm}(\lambda)$.*

The extended function satisfies

$$(H_0 - z)\, R_0(z) f = f, \quad f \in H^{-1,s},\ z \in \overline{\mathcal{C}^{\pm}}, \tag{4.14}$$

where for $z = \lambda \in \mathbb{R}$, $R_0(z) = R_0^{\pm}(\lambda)$.

Proof. For simplicity we assume $n \ge 3$. By Definition (4.12) and estimate (4.11), we get readily $R_0(z) \in B(H^{-1,s}, H^{-1,-\sigma})$ if $\operatorname{Im} z \neq 0$ as well as the analyticity of the map $z \to R_0(z)$, $\operatorname{Im} z \neq 0$. Furthermore, the extension to $\operatorname{Im} z = 0$ is carried out as in [15].

Eq. (4.14) is obvious if $\operatorname{Im} z \neq 0$ and $f \in L^{2,s}$. By the density of $L^{2,s}$ in $H^{-1,s}$, the continuity of $R_0(z)$ on $H^{-1,s}$ and the continuity of $H_0 - z$ (in the sense of distributions) we can extend it to all $f \in H^{-1,s}$.

As $z \to \lambda \pm i \cdot 0$ we have $R_0(z)f \to R_0^{\pm}(\lambda)f$ in $H^{-1,-\sigma}$. Applying the (constant coefficient) operator $H_0 - z$ yields, in the sense of distributions, $f = (H_0 - z)\, R_0(z)f \to (H_0 - \lambda)\, R_0^{\pm}(\lambda)f$ which establishes (4.14) also for $\operatorname{Im} z = 0$.

Finally, the established continuity of $z \to R_0(z) \in B(H^{-1,s}, H^{-1,-\sigma})$ (up to the real boundary) and Eq. (4.14) imply the continuity of the map $z \to H_0\, R_0(z) \in B(H^{-1,s}, H^{-1,-\sigma})$.

The stronger continuity claim (4.13) follows, since the norm of $H^{1,-\sigma}$ is equivalent to the graph norm of H_0 as a map of $H^{-1,-\sigma}$ to itself. $\square$

Remark 4.7. The main point here is the fact that the limiting values can be extended continuously to the threshold at $\lambda = 0$.

In the neighborhood of any $\lambda > 0$ this proposition follows from [79, Theorem 2.3], where a very different proof is used. In fact, using the terminology there, the limit functions $R_0^{\pm}(\lambda)f$ are the unique (on either side of the positive real axis) radiative functions and they satisfy a suitable *Sommerfeld radiation condition*. We recall it here for the sake of completeness, since we will need it in the next section.

Let $z = k^2 \in \mathcal{C} \setminus \{0\}$, $\operatorname{Im} k \geq 0$. For $f \in H^{-1,s}$ let $u = R_0(z)f \in H^{1,-\sigma}$ be as defined above. Then

$$\mathcal{R}u = \int\limits_{|x|>\Lambda_0} \left| r^{-\frac{n-1}{2}} \frac{\partial}{\partial r}(r^{\frac{n-1}{2}} u) - iku \right|^2 dx < \infty, \tag{4.15}$$

where $r = |x|$. We shall refer to $\mathcal{R}u$ as the radiative norm of u.

Furthermore, we can take s, $\sigma > \frac{1}{2}$, as in Remark 4.5.

4.2. Proof of the LAP for the operator H

We start with some considerations regarding the behavior of the resolvent near the spectrum.

Fix $[\alpha, \beta] \subset \mathbb{R}$ and let

$$\Omega = \big\{ z \in \mathcal{C}^+ \;\big|\; \alpha < \operatorname{Re} z < \beta,\ 0 < \operatorname{Im} z < 1 \big\}. \tag{4.16}$$

Let $z = \mu + i\varepsilon \in \Omega$ and consider the equation

$$(H - z)\, u = f \in H^{-1,s}, \quad u \in H^{1,-\sigma}\ (f \in H_0^{-1,s} \text{ if } n = 2). \tag{4.17}$$

(Observe that in the case $n = 2$ also $u \in L_0^{2,\sigma}$).
With Λ_0 as in (4.2), let $\chi(x) \in C^\infty(\mathbb{R}^n)$ be such that

$$\chi(x) = \begin{cases} 0, & |x| < \Lambda_0 + 1, \\ 1, & |x| > \Lambda_0 + 2. \end{cases} \tag{4.18}$$

Eq. (4.17) can be written as

$$(H_0 - z)\,(\chi u) = \chi f - 2\nabla\chi \cdot \nabla u - u\Delta\chi. \tag{4.19}$$

Letting $\psi(x) = 1-\chi(\frac{x}{2}) \in C_0^\infty(\mathbb{R}^n)$ and using Proposition 4.6 and standard elliptic estimates, we obtain from (4.19)

$$\|u\|_{1,-\sigma} \le C\left[\|f\|_{-1,s} + \|\psi u\|_{0,-s}\right], \tag{4.20}$$

where s, σ satisfy (4.8), $\sigma' > \sigma$ and $C > 0$ depends only on Λ_0, σ, s, n.

We note that, since ψ is compactly supported, the term $\|\psi u\|_{0,-s}$ can be replaced by $\|\psi u\|_{0,-s'}$ for any real s'.

In fact, the second term in the right-hand side can be dispensed with, as is demonstrated in the following proposition.

Proposition 4.8. *The solution to* (4.17) *satisfies,*

$$\|u\|_{1,-\sigma} \le C\,\|f\|_{-1,s}\,, \tag{4.21}$$

where s, σ satisfy (4.8) *and $C > 0$ depends only on σ, s, n, Λ_0.*

Proof. In view of (4.20) we only need to show that

$$\|\psi u\|_{0,-s} \le C\,\|f\|_{-1,s}\,. \tag{4.22}$$

Since $L^{2,s}(\mathbb{R}^n)$ is dense in $H^{-1,s}(\mathbb{R}^n)$, it suffices to prove this inequality for $f \in L^{2,s}(\mathbb{R}^n) \cap H^{-1,s}(\mathbb{R}^n)$ (using the norm of $H^{-1,s}$).

We argue by contradiction. Let

$$\left\{z_k\right\}_{k=1}^\infty \subseteq \Omega, \quad \left\{f_k\right\}_{k=1}^\infty \subseteq L^{2,s}(\mathbb{R}^n) \cap H^{-1,s}(\mathbb{R}^n)$$

(with $\hat{f}_k(0) = 0$ if $n = 2$) and

$$\left\{u_k = R(z_k) f_k\right\}_{k=1}^\infty \subseteq H^{1,-\sigma}(\mathbb{R}^n)$$

be such that

$$\begin{gathered} \|\psi u_k\|_{0,-s} = 1, \quad \|f_k\|_{-1,s} \le k^{-1}, \quad k = 1,2,\dots, \\ z_k \to z_0 \in \overline{\Omega} \quad \text{as } k \to \infty. \end{gathered} \tag{4.23}$$

By (4.20), $\left\{u_k\right\}_{k=1}^\infty$ is bounded in $H^{1,-\sigma}$. Replacing the sequence by a suitable subsequence (without changing notation) and using the Rellich compactness theorem we may assume that there exists a function $u \in L^{2,-\sigma'}$, $\sigma' > \sigma$, such that

$$u_k \to u \quad \text{in } L^{2,-\sigma'} \text{ as } k \to \infty. \tag{4.24}$$

Furthermore, by weak compactness we actually have (restricting again to a subsequence if needed)

$$u_k \xrightarrow{w} u \quad \text{in } H^{1,-\sigma} \text{ as } k \to \infty. \tag{4.25}$$

Since H maps continuously $H^{1,-\sigma}$ into $H^{-1,-\sigma}$, we have

$$Hu_k \xrightarrow{w} Hu \quad \text{in } H^{-1,-\sigma} \text{ as } k \to \infty,$$

so that from $(H - z_k)\,u_k = f_k$ we infer that

$$(H - z_0)\,u = 0. \tag{4.26}$$

In view of (4.19) and Remark 4.7 the functions χu_k are *radiative functions*. Since they are uniformly bounded in $H^{1,-\sigma}$, their *radiative norms* (4.15) are uniformly bounded.

Suppose first that $z_0 \neq 0$. In view of Remark 4.7 we can take $s,\, \sigma > \frac{1}{2}$. Then the limit function u is a radiative solution to $(H_0 - z_0)\, u = 0$ in $|x| > \Lambda_0 + 2$ and hence must vanish there (see [79]). By the unique continuation property of solutions to (4.26) we conclude that $u \equiv 0$. Thus by (4.24) we get $\|\psi u_k\|_{0,-\sigma'} \to 0$ as $k \to \infty$, which contradicts (4.23).

We are therefore left with the case $z_0 = 0$. In this case $u \in H^{1,-\sigma}$ satisfies the equation

$$\nabla \cdot (a(x)\nabla u) = 0. \tag{4.27}$$

In particular, $\Delta u = 0$ in $|x| > \Lambda_0$ and

$$\int_{\Lambda_0}^{\infty} \int_{|x|=r} r^{-2\sigma} \left(|u|^2 + |\frac{\partial u}{\partial r}|^2 \right) d\tau dr < \infty. \tag{4.28}$$

Consider first the case $n \geq 3$. We may then use the representation of u by spherical harmonics so that, with $x = r\omega$, $\omega \in S^{n-1}$,

$$u(x) = r^{-\frac{n-1}{2}} \left\{ \sum_{j=0}^{\infty} b_j r^{\mu_j} h_j(\omega) + \sum_{j=0}^{\infty} c_j r^{-\nu_j} h_j(\omega) \right\}, \quad r > \Lambda_0, \tag{4.29}$$

where

$$\mu_j(\mu_j - 1) = \nu_j(\nu_j + 1) = \lambda_j + \frac{(n-1)(n-3)}{4}, \tag{4.30}$$
$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$$

being the eigenvalues of the Laplace-Beltrami operator on S^{n-1}, and $h_j(\omega)$ the corresponding spherical harmonics. Since $\lambda_1 = n - 1$, it follows that

$$\mu_0 = \frac{n-1}{2}, \quad \mu_0 + 1 \leq \mu_1 \leq \mu_2 \leq \dots, \quad \frac{n-3}{2} = \nu_0 < \nu_1 \leq \nu_2 \leq \dots \tag{4.31}$$

We now observe that (4.28) forces

$$b_0 = b_1 = \dots = 0.$$

Also, by (4.29)

$$\int_{|x|=r} \frac{\partial u}{\partial r}\, d\tau = -\,(n-2)\left|S^{n-1}\right| c_0, \quad r > \Lambda_0, \tag{4.32}$$

($\left|S^{n-1}\right|$ is the surface measure of S^{n-1}), while integrating (4.27) we get

$$\int_{|x|=r} \frac{\partial u}{\partial r}\, d\tau = 0, \quad r > \Lambda_0. \tag{4.33}$$

Thus $c_0 = 0$. It now follows from (4.29) that, for $r > \Lambda_0$,

$$\int\limits_{|x|=r} \left(|u|^2 + \left|\frac{\partial u}{\partial r}\right|^2 \right) d\tau \le \left(\frac{r}{\Lambda_0}\right)^{-2\nu_1} \int\limits_{|x|=\Lambda_0} \left(|u|^2 + \left|\frac{\partial u}{\partial r}\right|^2 \right) d\tau. \qquad (4.34)$$

Multiplying (4.27) by $\overline{u}$ and integrating by parts over the ball $|x| \le r$, we infer from (4.34) that the boundary term vanishes as $r \to \infty$. Thus $\nabla u \equiv 0$, in contradiction to (4.23)–(4.24).

It remains to deal with the case $n = 2$. Instead of (4.29) we now have

$$u(x) = r^{-\frac{1}{2}} \left\{ \widetilde{b_0} r^{\frac{1}{2}} \log r + \sum_{j=0}^{\infty} b_j r^{\mu_j} h_j(\omega) + \sum_{j=1}^{\infty} c_j r^{-\nu_j} h_j(\omega) \right\}, \quad r > \Lambda_0, \quad (4.35)$$

where $\mu_0 = \frac{1}{2}$, $\mu_1 = \frac{3}{2}$, $\nu_1 = \frac{1}{2}$. As in the derivation above, the condition (4.28) yields $b_0 = b_1 = \ldots = 0$. Also, we get $\widetilde{b_0} = 0$ in view of (4.33). It now follows that

$$\int\limits_{|x|=r} \overline{u} \frac{\partial u}{\partial r} d\tau = -2\pi \sum_{j=1}^{\infty} \left(\nu_j + \frac{1}{2}\right) |c_j|^2 r^{-2\nu_j - 1}, \quad r \ge \Lambda_0, \qquad (4.36)$$

from which, as in the argument following (4.34), we deduce that $u \equiv 0$, again in contradiction to (4.23)–(4.24). □

Proof of Theorem 4.1. Part (a) of the theorem is actually covered by Proposition 4.8. Moreover, the proposition implies that the operator-valued function

$$z \to R(z) \in B(H^{-1,s}(\mathbb{R}^n), H^{1,-\sigma}(\mathbb{R}^n)), \quad s > 1,\ z \in \Omega,$$

is uniformly bounded, where s, σ satisfy (4.8). Here and below replace $H^{-1,s}$ by $H_0^{-1,s}$ if $n = 2$.

We next show that the function $z \to R(z)$ can be continuously extended to $\overline{\Omega}$ in the *weak toplogoy* of $B(H^{-1,s}(\mathbb{R}^n), H^{1,-\sigma}(\mathbb{R}^n))$. To this end, we take $f \in H^{-1,s}(\mathbb{R}^n)$ and $g \in H^{-1,\sigma}(\mathbb{R}^n)$ and consider the function

$$z \to < g, R(z) f >, \quad z \in \Omega,$$

where $< , >$ is the $(H^{-1,\sigma}, H^{1,-\sigma})$ pairing. We need to show that it can be extended continuously to $\overline{\Omega}$.

In view of the uniform boundedness established in Proposition 4.8, we can take f, g in dense sets (of the respective spaces). In particular, we can take $f \in L^{2,s}(\mathbb{R}^n)$ and $g \in L^{2,\sigma}(\mathbb{R}^n)$, so that the continuity property in Ω is obvious.

Consider therefore a sequence $\{z_k\}_{k=1}^{\infty} \subseteq \Omega$ such that $z_k \xrightarrow[k\to\infty]{} z_0 \in [\alpha, \beta]$. The sequence $\{u_k = R(z_k) f\}_{k=1}^{\infty}$ is bounded in $H^{1,-\sigma}(\mathbb{R}^n)$. Therefore there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ which converges to a function $u \in L^{2,-\sigma'}$, $\sigma' > \sigma$.

We can further assume that $u_{k_j} \xrightarrow[j\to\infty]{w} u$ in $H^{1,-\sigma}$. It follows that

$$< g, u_{k_j} > \xrightarrow[j\to\infty]{} < g, u > .$$

Passing to the limit in $\left(H - z_{k_j}\right) u_{k_j} = f$ we see that the limit function satisfies

$$(H - z_0)\, u = f.$$

We now repeat the argument employed in the proof of Proposition 4.8. If $z_0 \neq 0$ we note that the functions $\{\chi u_k\}_{k=1}^{\infty}$ are radiative functions with uniformly bounded *radiative norms* (4.15) in $|x| > \Lambda_0 + 2$. The same is therefore true for the limit function u.

If $z_0 = 0$, then the function $u \in H^{1,-\sigma}$ solves $Hu = f$.

In both cases this function is unique and we get the convergence

$$< g, R(z_k) f > = < g, u_k > \xrightarrow[k\to\infty]{} < g, u > .$$

We can now define

$$R^+(z_0) f = u, \tag{4.37}$$

with an analogous definition for $R^-(z_0)$.

At this point we can readily deduce the following extension of the resolvent $R(z)$ as the inverse of $H - z$.

$$(H - z)\, R(z) f = f, \quad f \in H^{-1,s},\, z \in \overline{\mathcal{C}^{\pm}}, \tag{4.38}$$

where $R(z) = R^{\pm}(\lambda)$ when $z = \lambda \in \mathbb{R}$.

Indeed, observe that if $\operatorname{Im} z \neq 0$ then $(H - z) R(z) f = f$ for $f \in L^{2,s}(\mathbb{R}^n)$ and $(H - z) R(z) \in B(H^{-1,s}, H^{-1,-\sigma})$, so the assertion follows from the density of $L^{2,s}(\mathbb{R}^n)$ in $H^{-1,s}(\mathbb{R}^n)$. For $z = \lambda \in \mathbb{R}$ we use the (just established) weak continuity of the map $z \hookrightarrow (H - z) R(z)$ from $H^{-1,s}$ into $H^{-1,-\sigma}$ in $\overline{\mathcal{C}^{\pm}}$.

The passage *from weak to uniform continuity* (in the operator topology) is a classical argument due to Agmon [1]. In [9] we have applied it in the case $n = 1$. Here we outline the proof in the case $n > 1$.

We establish first the continuity of the operator-valued function $z \to R(z)$, $\overline{\Omega}$, in the *uniform operator topologoy* of $B(H^{-1,s}(\mathbb{R}^n), L^{2,-\sigma}(\mathbb{R}^n))$.

Let $\left\{z_k\right\}_{k=1}^{\infty} \subseteq \overline{\Omega}$ and $\left\{f_k\right\}_{k=1}^{\infty} \subseteq H^{-1,s}(\mathbb{R}^n)$ be sequences such that $z_k \xrightarrow[k\to\infty]{}$ $z \in \overline{\Omega}$ and f_k converges weakly to f in $H^{-1,s}(\mathbb{R}^n)$. It suffices to prove that the sequence $u_k = R(z_k) f_k$, which is bounded in $H^{1,-\sigma}(\mathbb{R}^n)$, converges strongly in $L^{2,-\sigma}(\mathbb{R}^n)$. Since this is clear if $\operatorname{Im} z \neq 0$, we can take $z \in [\alpha, \beta]$.

Note first that we can take $\frac{1}{2} < \sigma' < \sigma$ so that s, σ' satisfy (4.8). Then the sequence $\{u_k\}_{k=1}^{\infty}$ is bounded in $H^{1,-\sigma'}(\mathbb{R}^n)$ and there exists a subsequence $\left\{u_{k_j}\right\}_{j=1}^{\infty}$ which converges to a function $u \in L^{2,-\sigma}$.

We can further assume that $u_{k_j} \xrightarrow[j\to\infty]{w} u$ in $H^{1,-\sigma}$.

It follows that the limit function satisfies (see Eq. (4.38))

$$(H - z)\, u = f.$$

Once again we consider separately the cases $z \neq 0$ and $z = 0$.

In the first case, in view of (4.38) and Remark 4.7, the functions χu_k are *radiative functions*. Since they are uniformly bounded in $H^{1,-\sigma}$ their *radiative norms* (4.15) are uniformly bounded, and we conclude that also $\mathcal{R} u < \infty$.

In the second case, we simply note that $u \in H^{1,-\sigma}$ solves $Hu = f$.

As in the proof of Proposition 4.8 we conclude that in both cases the limit is unique, so that the whole sequence $\{u_k\}_{k=1}^{\infty}$ converges to u in $L^{2,-\sigma}(\mathbb{R}^n)$.

Thus, the continuity in the uniform operator topologoy of $B(H^{-1,s}(\mathbb{R}^n), L^{2,-\sigma}(\mathbb{R}^n))$ is proved.

Finally, we claim that the operator-valued function $z \to R(z)$ is continuous in the *uniform operator topology* of $B(H^{-1,s}(\mathbb{R}^n), H^{1,-\sigma}(\mathbb{R}^n))$. Indeed, if we invoke Eq. (4.38), we get that also $z \to HR(z)$ is continuous in the uniform operator topology of $B(H^{-1,s}(\mathbb{R}^n), H^{-1,-\sigma}(\mathbb{R}^n))$.

Since the domain of H in $H^{-1,-\sigma}(\mathbb{R}^n)$ is $H^{1,-\sigma}(\mathbb{R}^n)$, the claim follows. The conclusion of the theorem follows by taking $\sigma = s$. $\square$

Remark 4.9. In view of (4.19) and Remark 4.7 it follows that for $\lambda > 0$ the functions $R^{\pm}(\lambda)f$, $f \in H^{-1,s}$, are *radiative*, i.e., satisfy a Sommerfeld radiation condition.

The fact that the limiting values of the resolvent are continuous across the threshold at $\lambda = 0$ has been established in the case $H = H_0$ [14, Appendix A], and in the one-dimensional case ($n = 1$) in [9, 12, 30]. The paper [74] deals with the two-dimensional ($n = 2$) case, but the resolvent $R(z)$ is restricted to continuous compactly supported functions f, thus enabling the use of pointwise decay estimates of $R(z)f$ at infinity. In the case of the closely related *acoustic propagator*, where the matrix $a(x) = b(x_1)I$ is scalar and dependent on a single coordinate, there are in general countably many thresholds embedded in the continuous spectrum. Any study of the LAP must therefore deal with this difficulty. We mention here the papers [12, 24, 23, 39, 32, 34, 57, 58, 63, 85], as well as the *anisotropic* case where $b(x_1)$ is a general positive matrix [13].

We mention next some related studies concerning the LAP where, however, the threshold has been avoided. Our discussion is restricted, however, to operators that can be characterized as "perturbations of the Laplacian" . The extensive literature concerning the $N-$body operators is omitted , apart from the monographs [4, 36] that have already been mentioned in the Introduction in connection with Mourre's approach to the LAP.

The pioneering works of Eidus and Agmon have already been mentioned in the Introduction. Under assumptions close to ours here (but also assuming that $a(x)$ is continuously differentiable) a weaker version (roughly, *strong* instead of *uniform* convergence of the resolvents) was obtained by Eidus [40, Theorem 4 and Remark 1]. For $H = H_0$ the LAP has been established by Agmon [1]. Indeed, it was established for operators of the type $H_0 + V$, where V is a short-range perturbation. The short-range potential V was later replaced by a long-range or Stark-like potential [53, 6], a potential in $L^p(\mathbb{R}^n)$ [44, 55], a potential depending only on direction $x/|x|$ [46] and a perturbation of such a potential [71, 72]. In these latter cases the condition $\alpha > 0$ is replaced by $\alpha > \limsup_{|x|\to\infty} V(x)$.

We refer to [20] for the LAP for operators of the type $f(-\Delta)+V$ for a certain class of functions f.

We refer to [76] and references therein for the case of perturbations of the Laplace-Beltrami operator Δ_g on noncompact manifolds. The LAP (still in $(0,\infty)$) holds under the assumption that g is a smooth metric on $\mathbb{R}^n$ that vanishes at infinity. We make use of this result in the proof of Theorem 6.1 (see Section 6).

The LAP for the periodic case (namely, $a(x)$ is symmetric and periodic) has recently been established in [69]. Note that in this case the spectrum is absolutely continuous and consists of a union of intervals (*bands*).

4.3. An application: Existence and completeness of the wave operators $W_\pm(H, H_0)$

A nice consequence of Theorem 4.1 is the existence and completeness of the wave operators. We recall first the definition [59, Chapter X].

Consider the family of unitary operators

$$W(t) = \exp(itH)\exp(-itH_0), \quad -\infty < t < \infty.$$

The strong limits $W_\pm(H,H_0) = s\text{-}\lim_{t\to\pm\infty} W(t)$, if they exist, are called the *wave operators* (relating H, H_0). They are clearly isometries. If their ranges are equal, we say that they are *complete*.

Using a well-known theorem of Kato and Kuroda [61], we have the following corollary.

Corollary 4.10. *The wave operators $W_\pm(H,H_0)$ exist and are complete.*

Indeed, all that is needed is that H, H_0 satisfy the LAP in $\mathbb{R}$, with respect to the same operator topologies.

We refer to [54], where the existence and completeness of the wave operators $W_\pm(H,H_0)$ is established under suitable smoothness assumptions on $a(x)$. (However, $a(x)-I$ is not assumed to be compactly supported and H can include also magnetic and electric potentials.)

5. An eigenfunction expansion theorem

In the Introduction we mentioned the connection (as well as the *gap*) between the spectral theorem (for self-adjoint operators) in its functional-analytic formulation and the *generalized eigenfunction theorem*, a fundamental tool in the study of partial differential operators (and scattering theory). It was mentioned there that these theorems should be connected through the Limiting Absorption Principle. This is indeed the purpose of this section.

We derive an eigenfunction expansion theorem for a divergence-type operator H, the operator considered in Section 4.

Let $\{E(\lambda),\, \lambda\in\mathbb{R}\}$ be the spectral family associated with H and $A(\lambda) = \frac{d}{d\lambda}E(\lambda)$ be its weak derivative. We use the formula (3.7),

$$A(\lambda) = \frac{1}{2\pi i}\lim_{\epsilon\downarrow 0}\left(R(\lambda+i\epsilon)-R(\lambda-i\epsilon)\right) = \frac{1}{2\pi i}\left(R^+(\lambda)-R^-(\lambda)\right).$$

By Theorem 4.1 we know that $A(\lambda) \in B(L^{2,s}(\mathbb{R}^n), L^{2,-s}(\mathbb{R}^n))$, for values of s as given in the theorem.

The formal relation $(H - \lambda)\, A(\lambda) = 0$ can be given a rigorous meaning if, for example, we can find a bounded operator T such that $T^* A(\lambda) T$ is bounded in $L^2(\mathbb{R}^n)$ and has a complete set (necessarily at most countable) of eigenvectors. These will serve as *generalized eigenvectors* for H. We refer to [22, Chapters V, VI] and [25] for a development of this approach for self-adjoint elliptic operators. Note that by this approach we have at most a countable number of such generalized eigenvectors for any fixed λ. In the case of $H_0 = -\Delta$ they correspond to

$$|x|^{-\frac{n-3}{2}} J_{\sqrt{\kappa_j}}\big(\sqrt{\lambda}\,|x|\big)\psi_j(\omega),$$

where $\kappa_j = \lambda_j + \frac{(n-1)(n-3)}{4}$, λ_j being the j^{th} eigenvalue of the Laplace-Beltrami operator on the unit sphere S^{n-1}, ψ_j the corresponding eigenfunction and J_ν is the Bessel function of order ν.

On the other hand, the inverse Fourier transform

$$g(x) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} \hat{g}(\xi) e^{i\xi x}\, d\xi, \tag{5.1}$$

can be viewed as expressing a function in terms of the *generalized eigenfunctions* $\exp(i\xi x)$ of H_0. Observe that now there is a continuum of such functions corresponding to $\lambda > 0$, namely, $|\xi|^2 = \lambda$.

From the physical point-of-view this expansion in terms of *plane waves* proves to be more useful for many applications. In particular, replacing $-\Delta$ by the Schrödinger operator $-\Delta + V(x)$ one can expect, under certain hypotheses on the potential V, a similar expansion in terms of *distorted plane waves*. This has been accomplished, in increasing order of generality (more specifically, decay assumptions on $V(x)$ as $|x| \to \infty$) in [73, 52, 1, 79, 2]. See also [87] for an eigenfunction expansion for relativistic Schrödinger operators.

Here we use the LAP result of Theorem 4.1 in order to derive a similar expansion for the operator H. In fact, our generalized eigenfunctions are given by the following definition.

Definition 5.1. For every $\xi \in \mathbb{R}^n$, let

$$\begin{aligned} \psi_\pm(x,\xi) &= -R^\mp\left(|\xi|^2\right)\big((H - |\xi|^2)\exp(i\xi x)\big) \\ &= R^\mp(|\xi|^2)\left(\sum_{l,j=1}^{n} \partial_l (a_{l,j}(x) - \delta_{l,j})\partial_j\right)\exp(i\xi x). \end{aligned} \tag{5.2}$$

The *generalized eigenfunctions* of H are defined by

$$\varphi_\pm(x,\xi) = \exp(i\xi x) + \psi_\pm(x,\xi). \tag{5.3}$$

We assume $n \geq 3$ in order to simplify the statement of the theorem. As we show below (see Proposition 5.3) the generalized eigenfunctions are (at least) continuous in x, so that the integral in the statement makes sense.

Theorem 5.2. *Suppose that* $n \geq 3$ *and that* $a(x)$ *satisfies* (4.1), (4.2). *For any compactly supported* $f \in L^2(\mathbb{R}^n)$ *define*

$$(\mathbb{F}_\pm f)(\xi) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} f(x)\overline{\varphi_\pm(x,\xi)}\, dx, \quad \xi \in \mathbb{R}^n. \tag{5.4}$$

Then the transformations $\mathbb{F}_\pm$ *can be extended as unitary transformations (for which we retain the same notation) of* $L^2(\mathbb{R}^n)$ *onto itself. Furthermore, these transformations diagonalize* H *in the following sense*:

$f \in L^2(\mathbb{R}^n)$ *is in the domain* $D(H)$ *if and only if* $|\xi|^2(\mathbb{F}_\pm f)(\xi) \in L^2(\mathbb{R}^n)$ *and*

$$H = \mathbb{F}_\pm^* M_{|\xi|^2} \mathbb{F}_\pm, \tag{5.5}$$

where $M_{|\xi|^2}$ *is the multiplication operator by* $|\xi|^2$.

Before starting the proof of the theorem, we collect some basic properties of the generalized eigenfunctions in the following proposition.

Proposition 5.3. *The generalized eigenfunctions*

$$\varphi_\pm(x,\xi) = \exp(i\xi x) + \psi_\pm(x,\xi)$$

(*see* (5.3)) *are in* $H^1_{loc}(\mathbb{R}^n)$ *for each fixed* $\xi \in \mathbb{R}^n$ *and satisfy the equation*

$$\left(H - |\xi|^2\right)\varphi_\pm(x,\xi) = 0. \tag{5.6}$$

In addition, these functions have the following properties:

(i) *The map*

$$\mathbb{R}^n \ni \xi \to \psi_\pm(\cdot,\xi) \in H^{1,-s}(\mathbb{R}^n), \quad s > 1,$$

is continuous.

(ii) *For any compact* $K \subseteq \mathbb{R}^n$, *the family of functions* $\left\{\varphi_\pm(x,\xi) \;\middle|\; \xi \in K\right\}$ *is uniformly bounded and uniformly Hölder continuous in* $x \in \mathbb{R}^n$.

Proof. Since $(H - |\xi|^2)\exp(i\xi x) \in H^{-1,s}$, $s > 1$, Eq. (5.6) follows from the definition (5.2) in view of Eq. (4.38).

Furthermore, the map

$$\mathbb{R}^n \ni \xi \to \left(H - |\xi|^2\right)\exp(i\xi x) \in H^{-1,s}(\mathbb{R}^n), \quad s > 1,$$

is continuous, so the continuity assertion (i) follows from Theorem 4.1.

For $s > 1$, the set of functions $\left\{\psi_\pm(\cdot,\xi) \;\middle|\; \xi \in K\right\}$ is uniformly bounded in $H^{1,-s}$. Thus, in view of (5.6), it follows from the De Giorgi-Nash-Moser Theorem [42, Chapter 8] that the set $\left\{\varphi_\pm(x,\xi) \;\middle|\; \xi \in K\right\}$ is uniformly bounded and uniformly Hölder continuous in $\{|x| < R\}$ for every $R > 0$. In particular, we can take $R > \Lambda_0$ (see Eq. (4.2)). In the exterior domain $\{|x| > R\}$ the set $\left\{\psi_\pm(x,\xi) \;\middle|\; \xi \in K\right\}$ is bounded in $H^{1,-s}$, $s > 1$, and we have $\left(H_0 - |\xi|^2\right)\psi_\pm(x,\xi) = 0$.

In addition, the boundary values $\left\{\psi_\pm(x,\xi) \;\middle|\; |x| = R,\, \xi \in K\right\}$ are uniformly bounded. From well-known properties of solutions of the Helmholtz equation we conclude that this set is uniformly bounded and therefore, invoking once again the De Giorgi-Nash-Moser Theorem, uniformly Hölder continuous. □

Proof of Theorem 5.2. We use the LAP proved in Theorem 4.1, adapting the methodology of Agmon's proof [1] for the eigenfunction expansion in the case of Schrödinger operators with short-range potentials. To simplify notation, we prove for $\mathbb{F}_+$.

Let $u \in H^1$ be compactly supported. For any z such that $\operatorname{Im} z \neq 0$ we can write its Fourier transform as

$$\hat{u}(\xi) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} u(x) \exp(-i\xi x)\, dx = \frac{(2\pi)^{-\frac{n}{2}}}{|\xi|^2 - z} \int\limits_{\mathbb{R}^n} u(x)\, (H_0 - z) \exp(-i\xi x)\, dx.$$

Let $\theta \in C_0^\infty(\mathbb{R}^n)$ be a (real) cutoff function such that $\theta(x) = 1$ for x in a neighborhood of the support of u.

We can rewrite the above equality as

$$\hat{u}(\xi) = \frac{(2\pi)^{-\frac{n}{2}}}{|\xi|^2 - z} < (H_0 - z)\, u(x), \theta(x) \exp(i\xi x) >,$$

where $<\ ,\ >$ is the $(H^{-1,s}, H^{1,-s})$-sesquilinear pairing (conjugate linear with respect to the second term).

We have therefore, with $f = (H - z)\, u$,

$$\begin{aligned} &\hat{u}(\xi) \qquad (5.7)\\ &= \frac{(2\pi)^{-\frac{n}{2}}}{|\xi|^2 - z} \left(< (H - z)\, u(x), \theta(x) \exp(i\xi x) > + \overline{< (H_0 - H) \exp(i\xi x), u(x) >} \right) \\ &= \frac{(2\pi)^{-\frac{n}{2}}}{|\xi|^2 - z} \left(< f(x), \theta(x) \exp(i\xi x) > + < f(x), R(\bar{z})\, (H_0 - H) \exp(i\xi x) > \right). \end{aligned}$$

Introducing the function

$$\widetilde{f}(\xi, z) = \hat{f}(\xi) + (2\pi)^{-\frac{n}{2}} < f(x), R(\bar{z})\, (H_0 - H) \exp(i\xi x) >,$$

we have

$$\hat{u}(\xi) = \widehat{R(z)f}(\xi) = \frac{\widetilde{f}(\xi, z)}{|\xi|^2 - z}, \quad \operatorname{Im} z \neq 0. \tag{5.8}$$

We now claim that this equation is valid for all compactly supported $f \in H^{-1}$.

Indeed, let $u = R(z)f \in H^{1,-s}$, $s > 1$. Let $\psi(x) = 1 - \chi(x)$, where $\chi(x)$ is defined in (4.18). We set

$$u_k(x) = \psi(k^{-1}x) u(x), \quad f_k(x) = (H - z) \left(\psi(k^{-1}x) u(x) \right), \quad k = 1, 2, 3, \ldots$$

The equality (5.8) is satisfied with u, f replaced, respectively, by u_k, f_k. Since

$$\psi(k^{-1}x) u(x) \xrightarrow[k \to \infty]{} u(x)$$

in $H^{1,-s}$, we have

$$(H - z)\, (\psi(k^{-1}x) u(x)) \xrightarrow[k \to \infty]{} (H - z)\, u = f(x)$$

in $H^{-1,-s}$, where in the last step we have used Eq. (4.38).

In addition, since $(H_0 - H)\exp(i\xi x)$ is compactly supported

$$< f_k(x), R(\bar{z})\,(H_0 - H)\exp(i\xi x) >= \overline{< (H_0 - H)\exp(i\xi x), R(z) f_k(x) >}$$
$$\xrightarrow[k\to\infty]{} \overline{< (H_0 - H)\exp(i\xi x), R(z) f >} =< f, R(\bar{z})\,(H_0 - H)\exp(i\xi x) > .$$

Combining these considerations with the continuity of the Fourier transform (on tempered distributions) we establish that (5.8) is valid for all compactly supported $f \in H^{-1}$.

Let $\{E(\lambda),\ \lambda \in \mathbb{R}\}$ be the spectral family associated with H. Let $A(\lambda) = \frac{d}{d\lambda}E(\lambda)$ be its weak derivative. More precisely, we use the relation (3.7), to get (using Theorem 4.1), for any $f \in H^{-1,s}$, $s > 1$,

$$< f, A(\lambda) f >= \frac{1}{2\pi i} < f, \left(R^+(\lambda) - R^-(\lambda)\right) f > .$$

We now take $f \in L^2$ and compactly supported. From the resolvent equation we infer

$$R(\lambda + i\epsilon) - R(\lambda - i\epsilon) = 2i\epsilon R(\lambda + i\epsilon) R(\lambda - i\epsilon), \quad \epsilon > 0,$$

so that

$$< f, A(\lambda) f >= \lim_{\epsilon\downarrow 0} \frac{\epsilon}{\pi} \left\| R(\lambda + i\epsilon) f \right\|_0^2, \quad \epsilon > 0.$$

Using Eq. (5.8) and Parseval's theorem, we therefore have

$$< f, A(\lambda) f >= \lim_{\epsilon\to 0+} \frac{\epsilon}{\pi} \left\| \left(|\xi|^2 - (\lambda + i\epsilon)\right)^{-1} \widetilde{f}(\xi, \lambda + i\epsilon) \right\|_0^2, \quad \epsilon > 0. \tag{5.9}$$

Note that $\widetilde{f}(\xi, z)$ can be extended continuously as $z \to \lambda + i \cdot 0$ by

$$\widetilde{f}(\xi, \lambda) = \hat{f}(\xi) + (2\pi)^{-\frac{n}{2}} < f(x), R^-(\lambda)\,(H_0 - H)\exp(i\xi x) > . \tag{5.10}$$

In order to study properties of $\widetilde{f}(\xi, z)$ as a function of ξ we compute

$$\begin{aligned} \widetilde{f}(\xi, z) &= \hat{f}(\xi) + (2\pi)^{-\frac{n}{2}} \overline{< \left(\sum_{l,j=1}^{n} \partial_l (a_{l,j}(x) - \delta_{l,j}) \partial_j \right) \exp(i\xi x), R(z) f(x) >} \\ &= \hat{f}(\xi) + (2\pi)^{-\frac{n}{2}} i \sum_{l,j=1}^{n} \xi_j \int_{\mathbb{R}^n} (a_{l,j}(x) - \delta_{l,j})\, \partial_l (R(z) f(x)) \exp(-i\xi x)\, dx, \end{aligned} \tag{5.11}$$

where in the last step we have used that both $\partial_l(R(z)f(x))$ and $(a_{l,j}(x) - \delta_{l,j})$ $\exp(-i\xi x)$ are in L^2.

Consider now the integral

$$g(\xi, z) = \int_{\mathbb{R}^n} (a_{l,j}(x) - \delta_{l,j})\, \partial_l (R(z) f(x)) \exp(-i\xi x)\, dx, \quad z \in \Omega,$$

where Ω is as in (4.16).

In view of Theorem 4.1, the family $\{\partial_l R(z)f(x)\}_{z\in\Omega}$ is uniformly bounded in $L^{2,-s}$, $s>1$, so by Parseval's theorem we get

$$\|g(\cdot,z)\|_0 < C, \quad z\in\Omega,$$

where C only depends on f.

This estimate and (5.11) imply that, if $f\in L^2$ is compactly supported,

(i) The function

$$\mathbb{R}^n\times\overline{\Omega}\ni(\xi,z)\to\widetilde{f}(\xi,z)$$

is continuous. For real z it is given by (5.10).

(ii)

$$\lim_{k\to\infty}\int\limits_{|\xi|>k}\left(|\xi|^2-z\right)^{-1}|\widetilde{f}(\xi,z)|^2\,d\xi=0,$$

uniformly in $z\in\Omega$.

As $z\to|\xi|^2+i\cdot 0$, we have by Theorem 4.1 and Eq. (5.3),

$$\lim_{z\to|\xi|^2+i\cdot 0}\widetilde{f}(\xi,z)=(2\pi)^{-\frac{n}{2}}\int\limits_{\mathbb{R}^n}f(x)\,\overline{\varphi_+(x,\xi)}\,dx=\mathbb{F}_+f(\xi),$$

so that, taking (i) and (ii) into account, we obtain from (5.9), for any compactly supported $f\in L^2$,

$$<f,A(\lambda)f>=\frac{1}{2\sqrt{\lambda}}\int\limits_{|\xi|^2=\lambda}|\mathbb{F}_+f(\xi)|^2\,d\sigma, \quad \lambda>0, \tag{5.12}$$

where $d\sigma$ is the surface Lebesgue measure.

It follows that, for any $[\alpha,\beta]\subset(0,\infty)$,

$$((E(\beta)-E(\alpha))f,f)=\int_\alpha^\beta<f,A(\lambda)f>d\lambda=\int\limits_{\alpha\le|\xi|^2\le\beta}|\mathbb{F}_+f(\xi)|^2\,d\xi. \tag{5.13}$$

Letting $\alpha\to 0$, $\beta\to\infty$, we get

$$\|f\|_0=\|\mathbb{F}_+f\|_0\,. \tag{5.14}$$

Thus $f\to\mathbb{F}_+f\in L^2(\mathbb{R}^n)$ is an isometry for compactly supported functions, which can be extended by density to all $f\in L^2(\mathbb{R}^n)$.

Furthermore, since the spectrum of H is entirely absolutely continuous, it follows that for every $f\in L^2$, Eq. (5.12) holds for almost all $\lambda>0$ (with respect to the Lebesgue measure).

Let $f\in D(H)$. By the spectral theorem

$$<Hf,A(\lambda)Hf>=\lambda^2<f,A(\lambda)f>=\frac{1}{2\sqrt{\lambda}}\int\limits_{|\xi|^2=\lambda}\left||\xi|^2\,\mathbb{F}_+f(\xi)\right|^2\,d\sigma, \quad \lambda>0.$$

In particular,

$$\|Hf\|_0^2 = \int_{\mathbb{R}^n} \left||\xi|^2\, \mathbb{F}_+ f(\xi)\right|^2\, d\xi. \tag{5.15}$$

Conversely, if the right-hand side of (5.15) is finite, then $\int_0^\infty \lambda^2 < f, A(\lambda)f > d\lambda < \infty$, so $f \in D(H)$.

The adjoint operator $\mathbb{F}_+^*$ is a partial isometry (on the range of $\mathbb{F}_+$). If $f(x) \in L^2(\mathbb{R}^n)$ is compactly supported and $g(\xi) \in L^2(\mathbb{R}^n)$ is likewise compactly supported, then

$$\begin{aligned}(\mathbb{F}_+ f, g) &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x)\, \overline{\varphi_+(x,\xi)}\, dx \right) \overline{g(\xi)}\, d\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) \left(\int_{\mathbb{R}^n} \overline{g(\xi)\, \varphi_+(x,\xi)}\, d\xi \right) dx,\end{aligned}$$

where in the change of order of integration Proposition 5.3 was taken into account.

It follows that, for a compactly supported $g(\xi) \in L^2(\mathbb{R}^n)$,

$$(\mathbb{F}_+^* g)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} g(\xi)\, \varphi_+(x,\xi)\, d\xi, \tag{5.16}$$

and the extension to all $g \in L^2(\mathbb{R}^n)$ is obtained by the fact that $\mathbb{F}_+^*$ is a partial isometry.

Now if $f \in D(H)$, $g \in L^2(\mathbb{R}^n)$, we have

$$(Hf, g) = \int_{\mathbb{R}^n} |\xi|^2\, \mathbb{F}_+ f(\xi)\, \overline{\mathbb{F}_+ g(\xi)}\, d\xi = \int_{\mathbb{R}^n} \mathbb{F}_+^* \left(|\xi|^2\, \mathbb{F}_+ f(\xi)\right) \overline{g(\xi)}\, d\xi,$$

which is the statement (5.5) of the theorem.

It follows from the spectral theorem that, for every interval $J = [\alpha, \beta] \subseteq [0, \infty)$ and for every $f \in L^2(\mathbb{R}^n)$, we have, with $E_J = E(\beta) - E(\alpha)$ and χ_J the characteristic function of J,

$$E_J f(x) = \mathbb{F}_+^* \left(\chi_J(|\xi|^2)\, \mathbb{F}_+ f(\xi)\right)$$

or

$$\mathbb{F}_+ E_J f(\xi) = \chi_J(|\xi|^2)\, \mathbb{F}_+ f(\xi).$$

It remains to prove that the isometry $\mathbb{F}_+$ is onto (and hence unitary). So, suppose to the contrary that, for some nonzero $g(\xi) \in L^2(\mathbb{R}^n)$,

$$(\mathbb{F}_+^* g)(x) = 0.$$

In particular, for any $f \in L^2(\mathbb{R}^n)$ and any interval J as above,

$$0 = \left(E_J f, \mathbb{F}_+^* g\right) = (\mathbb{F}_+ E_J f, g) = \left(\chi_J(|\xi|^2)\mathbb{F}_+ f(\xi), g(\xi)\right) = \left(\mathbb{F}_+ f(\xi), \chi_J(|\xi|^2) g(\xi)\right),$$

so that $\mathbb{F}_+^* \left(\chi_J(|\xi|^2)\, g(\xi)\right) = 0$.

By Eq. (5.16) we have, for any $0 \le \alpha < \beta$,

$$\int\limits_{\alpha<|\xi|^2<\beta} g(\xi)\, \varphi_+(x,\xi)\, d\xi = 0$$

so that, in view of the continuity properties of $\varphi_+(x,\xi)$ (see Proposition 5.3), for a.e. $\lambda \in (0,\infty)$,

$$\int\limits_{|\xi|^2=\lambda} g(\xi)\, \varphi_+(x,\xi)\, d\sigma = 0. \tag{5.17}$$

From the definition (5.3) we get

$$\int\limits_{|\xi|^2=\lambda} g(\xi) \exp(i\xi x)\, d\sigma - \int\limits_{|\xi|^2=\lambda} g(\xi) R^-(\lambda)\, ((H-\lambda)\exp(i\xi x))\, d\sigma = 0. \tag{5.18}$$

Since $(H-\lambda)\exp(i\xi x)$ is compactly supported (when $|\xi|^2 = \lambda$), the continuity property of $R^-(\lambda)$ enables us to write

$$\int\limits_{|\xi|^2=\lambda} g(\xi)\, R^-(\lambda)\, ((H-\lambda)\exp(i\xi x))\, d\sigma = R^-(\lambda) \int\limits_{|\xi|^2=\lambda} g(\xi)\, (H-\lambda) \exp(i\xi x)\, d\sigma,$$

which, by Remark 4.9, satisfies a Sommerfeld radiation condition. We conclude that the function

$$G(x) = \int\limits_{|\xi|^2=\lambda} g(\xi) \exp(i\xi x)\, d\sigma \in H^{1,-s}, \quad s > \frac{1}{2},$$

is a radiative solution (see Remark 4.7) of $(-\Delta - \lambda)\, G = 0$ and hence must vanish. Since this holds for a.e. $\lambda > 0$, we get $\hat{g}(\xi) = 0$, hence $g = 0$. □

6. Global spacetime estimates for a generalized wave equation

The Strichartz estimates [83] have become a fundamental ingredient in the study of nonlinear wave equations. They are L^p spacetime estimates that are derived for operators whose leading part has constant coefficients. We refer to the books [81, 82] and [5] for detailed accounts and further references.

Here we focus on spacetime estimates pertinent to the framework of this review, namely, weighted L^2 estimates.

We recall first some results related to the Cauchy problem for the classical wave equation,

$$\Box u = \frac{\partial^2 u}{\partial t^2} - \Delta u = 0, \tag{6.1}$$

subject to the initial data

$$u(x,0) = u_0(x), \quad \partial_t u(x,0) = v_0(x), \quad x \in \mathbb{R}^n. \tag{6.2}$$

The Morawetz estimate [66] yields

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} |x|^{-3}|u(x,t)|^2\,dxdt \le C\left(\|\nabla u_0\|_0^2 + \|v_0\|_0^2\right), \quad n \ge 4,$$

while in [8] we gave the estimate

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} |x|^{-2\alpha-1}\,|u(x,t)|^2\,dxdt \le C_\alpha\left(\||\nabla|^\alpha u_0\|_0^2 + \left\||\nabla|^{\alpha-1}v_0\right\|_0^2\right), \quad n \ge 3,$$

for every $\alpha \in (0,1)$.

Related results were obtained in [65] (allowing also dissipative terms), [50] (with some gain in regularity), [88] (with short-range potentials) and [47] for spherically symmetric solutions.

Here we consider the equation

$$\frac{\partial^2 u}{\partial t^2} + Hu = \frac{\partial^2 u}{\partial t^2} - \sum_{i,j=1}^{n} \partial_i a_{i,j}(x)\partial_j u = f(x,t), \tag{6.3}$$

subject to the initial data (6.2).

We first replace the assumptions (4.1), (4.2) by stronger ones as follows:

(H1) $$a(x) = g^{-1}(x) = \left(g^{i,j}(x)\right)_{1\le i,j\le n}, \tag{6.4}$$

where $g(x) = (g_{i,j}(x))_{1\le i,j\le n}$ is a smooth Riemannian metric on $\mathbb{R}^n$ such that

$$g(x) = I, \quad |x| > \Lambda_0.$$

(H2) The Hamiltonian flow associated with $h(x,\xi) = (g(x)\xi,\xi)$ is nontrapping for any (positive) value of h. (6.5)

Recall that (H2) means that the flow associated with the Hamiltonian vectorfield $\mathcal{H} = \frac{\partial h}{\partial \xi}\frac{\partial}{\partial x} - \frac{\partial h}{\partial x}\frac{\partial}{\partial \xi}$ leaves any compact set in $\mathbb{R}^n_x$.

Identical hypotheses are imposed in the study of resolvent estimates in semiclassical theory [26, 27].

In our estimates we use *homogeneous Sobolev spaces* associated with the operator H.

We let $G = H^{\frac{1}{2}}$ which is a positive self-adjoint operator. Note that $\|G\theta\|_0$ is equivalent to the homogeneous Sobolev norm $\|\nabla\theta\|_0$.

Theorem 6.1. *Suppose that $n \ge 3$ and that $a(x)$ satisfies* Hypotheses (H1)–(H2). *Let $s > 1$.*

(a) (*local energy decay*) *There exists a constant* $C_1 = C_1(s,n) > 0$ *such that the solution to* (6.3), (6.2) *satisfies*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} \left(1+|x|^2\right)^{-s} \left[|Gu(x,t)|^2 + |u_t(x,t)|^2\right] dxdt$$
$$\leq C_1 \left\{ \|Gu_0\|_0^2 + \|v_0\|_0^2 + \int_{\mathbb{R}} \int_{\mathbb{R}^n} |f(x,t)|^2 \, dxdt \right\}. \quad (6.6)$$

(b) (*amplitude decay*) *Assume that* $f = 0$. *There exists a constant* $C_2 = C_2(s,n) > 0$ *such that the solution to* (6.3), (6.2) *satisfies,*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} \left(1+|x|^2\right)^{-s} |u(x,t)|^2 \, dxdt \leq C_2 \left[\|u_0\|_0^2 + \|G^{-1}v_0\|_0^2\right]. \quad (6.7)$$

This estimate generalizes similar estimates obtained for the classical ($g = I$) wave equation [8, 65].

Remark 6.2. The estimate (6.6) is an *energy decay estimate* for the wave equation (6.3). A localized (in space) version of the estimate has served to obtain global (small amplitude) existence theorems for the corresponding nonlinear equation [27, 48].

The weighted L^2 spacetime estimates for the *dispersive* equation

$$i^{-1} \frac{\partial}{\partial t} u = Lu,$$

have been extensively treated in recent years. In general, in this case there is also a gain of derivatives (so called *smoothing*) in addition to the energy decay. For the Schrödinger operator $L = -\Delta + V(x)$, with various assumptions on the potential V, we refer to [3, 7, 8, 17, 19, 50, 62, 78, 80, 89] and references therein. In [33] the case of magnetic potentials is considered. The Schrödinger operator on a Riemannian manifold is treated in [26, 38]. For more general operators see [16, 20, 28, 51, 67, 77, 84] and references therein.

Proof of Theorem 6.1. (a) Define, with $G = H^{\frac{1}{2}}$,

$$u_{\pm} = \frac{1}{2}\left(Gu \pm iu_t\right).$$

Then

$$\partial_t u_{\pm} = \mp i \, Gu_{\pm} \pm \frac{i}{2} f. \quad (6.8)$$

Defining

$$U(t) = \begin{pmatrix} u_+(t) \\ u_-(t) \end{pmatrix}, \quad (6.9)$$

we have

$$i^{-1} U'(t) = -KU + F, \quad (6.10)$$

where

$$K = \begin{pmatrix} G & 0 \\ 0 & -G \end{pmatrix}, \quad F(t) = \begin{pmatrix} \frac{1}{2} f(\cdot, t) \\ -\frac{1}{2} f(\cdot, t) \end{pmatrix}.$$

Note that, as is common when treating evolution equations, we write $U(t)$, $F(t)$, etc. for $U(x,t)$, $F(x,t)$, etc. when there is no risk of confusion.

The operator K is a self-adjoint operator on $\mathcal{D} = L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$. Its spectral family $E_K(\lambda)$ is given by $E_K(\lambda) = E_G(\lambda) \oplus (I - E_G(-\lambda))$, $\lambda \in \mathbb{R}$, where E_G is the spectral family of G.

Let $E(\lambda)$ be the spectral family of H, and let $A(\lambda) = \frac{d}{d\lambda} E(\lambda)$ be its weak derivative (3.7). By the definition of G we have

$$E_G(\lambda) = E(\lambda^2),$$

hence its weak derivative is given by

$$A_G(\lambda) = \frac{d}{d\lambda} E_G(\lambda) = 2\lambda\, A(\lambda^2), \quad \lambda > 0. \tag{6.11}$$

In view of the LAP (Theorem A) we therefore have that the operator-valued function

$$A_G(\lambda) \in B(L^{2,s}(\mathbb{R}^n), L^{2,-s}(\mathbb{R}^n))$$

is continuous for $\lambda \geq 0$.

Denoting $\mathcal{D}^s = L^{2,s}(\mathbb{R}^n) \oplus L^{2,s}(\mathbb{R}^n)$, it follows that

$$A_K(\lambda) = \frac{d}{d\lambda} E_K(\lambda) = A_G(\lambda) \oplus A_G(-\lambda), \quad \lambda \in \mathbb{R},$$

is continuous with values in $B(\mathcal{D}^s, \mathcal{D}^{-s})$ for $s > 1$.

Making use of Hypotheses (H1)–(H2), we invoke [76, Theorem 5.1] to conclude that $\limsup_{\mu \to \infty} \mu^{\frac{1}{2}} \|A(\mu)\|_{B(L^{2,s}, L^{2,-s})} < \infty$, so that by (6.11) there exists a constant $C > 0$ such that

$$\|A_G(\lambda)\|_{B(L^{2,s}, L^{2,-s})} < C, \quad \lambda \geq 0. \tag{6.12}$$

It follows that also

$$\|A_K(\lambda)\|_{B(\mathcal{D}^s, \mathcal{D}^{-s})} < C, \quad \lambda \in \mathbb{R},\ s > 1,\ \lambda \in \mathbb{R}. \tag{6.13}$$

Let $<\ ,\ >$ be the sesquilinear pairing between $\mathcal{D}^{-s}$ and $\mathcal{D}^s$ (conjugate linear with respect to the second term).

For any ψ, $\chi \in \mathcal{D}^s$ we have, in view of the fact that $A_K(\lambda)$ is a weak derivative of a spectral measure,

$$\begin{aligned} &\text{(i)} \qquad |< A_K(\lambda)\psi, \chi >|^2 \leq < A_K(\lambda)\psi, \psi > \cdot < A_K(\lambda)\chi, \chi >, \\ &\text{(ii)} \qquad \int_{-\infty}^{\infty} < A_K(\lambda)\psi, \psi > d\lambda = \|\psi\|^2_{L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)}. \end{aligned} \tag{6.14}$$

We first treat the pure Cauchy problem, i.e., $f \equiv 0$.

To estimate $U(x,t) = e^{-itK} U(x,0)$ we use a duality argument. Some of the following computations will be rather formal, but they can easily be justified by a density argument, as in [8, 20]. We shall use $((\ ,\ ))$ for the scalar product in $L^2(\mathbb{R}^{n+1}) \oplus L^2(\mathbb{R}^{n+1})$.

Take $w(x,t) \in C_0^\infty(\mathbb{R}^{n+1}) \oplus C_0^\infty(\mathbb{R}^{n+1})$. Then,

$$\begin{aligned}
((U,w)) &= \int_{-\infty}^{\infty} e^{-itK} U(x,0) \cdot \overline{w(x,t)}\, dxdt \\
&= \int_{-\infty}^{\infty} < A_K(\lambda) U(x,0), \int_{-\infty}^{\infty} e^{it\lambda} w(\cdot,t) dt > d\lambda \\
&= (2\pi)^{1/2} \int_{-\infty}^{\infty} < A_K(\lambda) U(x,0), \tilde{w}(\cdot,\lambda) > d\lambda,
\end{aligned}$$

where

$$\tilde{w}(x,\lambda) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} w(x,t) e^{it\lambda} dt.$$

Noting (6.14), (6.13) and using the Cauchy-Schwarz inequality

$$\begin{aligned}
\left|((U,w))\right| &\le (2\pi)^{1/2} \left\| U(x,0) \right\|_0 \left(\int_{-\infty}^{\infty} < A_K(\lambda) \tilde{w}(\cdot,\lambda), \tilde{w}(\cdot,\lambda) > d\lambda \right)^{1/2} \\
&\le C \left\| U(x,0) \right\|_0 \left(\int_{-\infty}^{\infty} \left\| \tilde{w}(\cdot,\lambda) \right\|^2_{\mathcal{D}^s}\ d\lambda \right)^{\frac{1}{2}}.
\end{aligned}$$

It follows from the Plancherel theorem that

$$\left|((U,w))\right| \le C \left\| U(x,0) \right\|_0 \left(\int_{\mathbb{R}} \left\| w(\cdot,t) \right\|^2_{\mathcal{D}^s}\ dt \right)^{1/2}.$$

Let $\phi(x,t) \in C_0^\infty(\mathbb{R}^{n+1}) \oplus C_0^\infty(\mathbb{R}^{n+1})$ and take $w(x,t) = \left(1+|x|^2\right)^{-\frac{s}{2}} \phi(x,t)$ so that

$$\left| \left(\left(\left(1+|x|^2\right)^{-\frac{s}{2}} U, \phi \right) \right) \right| \le C\ \left\| U(x,0) \right\|_0 \cdot \left\| \phi \right\|_{L^2(\mathbb{R}^{n+1})}.$$

This concludes the proof of the part involving the Cauchy data in (6.6), in view of (6.9).

To prove the part concerning the inhomogeneous equation, it suffices to take $u_0 = v_0 = 0$. In this case the Duhamel principle yields, for $t > 0$,

$$U(t) = \int_0^t e^{-i(t-\tau)K} F(\tau) d\tau,$$

where we have used the form (6.10) of the equation.

Integrating the inequality

$$\|U(t)\|_{\mathcal{D}^{-s}} \leq \int_0^t \left\| e^{-i(t-\tau)K} F(\tau) \right\|_{\mathcal{D}^{-s}} d\tau,$$

we get

$$\int_0^\infty \|U(t)\|_{\mathcal{D}^{-s}} \, dt \leq \int_0^\infty \int_\tau^\infty \left\| e^{-i(t-\tau)K} F(\tau) \right\|_{\mathcal{D}^{-s}} dt \, d\tau.$$

Invoking the first part of the proof we obtain

$$\int_0^\infty \|U(t)\|_{\mathcal{D}^{-s}} \, dt \leq C \int_0^\infty \|F(\tau)\|_0 \, d\tau,$$

which proves the part related to the inhomogeneous term in (6.6).

(b) Define

$$v_\pm(x,t) = \exp(\pm itG) \, \phi_\pm(x),$$

where

$$\phi_\pm(x) = \frac{1}{2} \left[u_0(x) \mp G^{-1} v_0(x) \right].$$

Then clearly

$$u(x,t) = v_+(x,t) + v_-(x,t).$$

We establish the estimate (6.7) for v_+.

Taking $w(x,t) \in C_0^\infty(\mathbb{R}^{n+1})$ we proceed as in the first part of the proof. Let $<\,,\,>$ be the $(L^{2,-s}(\mathbb{R}^n), L^{2,s}(\mathbb{R}^n))$ pairing. Then

$$\begin{aligned}(v_+, w) &= \int\limits_{-\infty}^{\infty} e^{itG} \phi_+(x) \, \overline{w(x,t)} \, dx dt \\ &= \int\limits_0^\infty < A_G(\lambda)\phi_+, \int\limits_{-\infty}^{\infty} e^{-it\lambda} w(\cdot, t) dt > d\lambda \\ &= (2\pi)^{1/2} \int\limits_0^\infty < A_G(\lambda)\phi_+, \tilde{w}(\cdot, \lambda) > d\lambda,\end{aligned}$$

where

$$\tilde{w}(x,\lambda) = (2\pi)^{-\frac{1}{2}} \int\limits_{\mathbb{R}} w(x,t) e^{-it\lambda} \, dt.$$

Noting (6.12) as well as the inequalities (6.14) (with A_G replacing A_K) and using the Cauchy-Schwarz inequality

$$|(v_+, w)| \le (2\pi)^{1/2} \|\phi_+\|_0 \left(\int_0^\infty < A_G(\lambda)\tilde{w}(\cdot, \lambda), \tilde{w}(\cdot, \lambda) > d\lambda \right)^{1/2}$$

$$\le C \|\phi_+\|_0 \left(\int_0^\infty \|\tilde{w}(\cdot, \lambda)\|_{0,s}^2 \, d\lambda \right)^{\frac{1}{2}} .$$

The Plancherel theorem yields

$$|(v_+, w)| \le C \|\phi_+\|_0 \left(\int_{\mathbb{R}} \|w(\cdot, t)\|_{0,s}^2 \, dt \right)^{1/2} .$$

Let $\omega \in C_0^\infty(\mathbb{R}^{n+1})$ and take $w(x,t) = \left(1 + |x|^2\right)^{-\frac{s}{2}} \omega(x,t)$ so that

$$\left| \left(\left(1 + |x|^2\right)^{-\frac{s}{2}} v_+, \omega \right) \right| \le C \|\phi_+\|_0 \|\omega\|_{L^2(\mathbb{R}^{n+1})} .$$

This (with the similar estimate for v_-) concludes the proof of the estimate (6.7). □

Remark 6.3 (optimality of the requirement $s > 1$). A key point in the proof was the use of the uniform bound (6.13). In view of the relation (6.11), this is reduced to the uniform boundedness of $\lambda\, A(\lambda^2)$, $\lambda \ge 0$, in $B(L^{2,s}, L^{2,-s})$. By [76, Theorem 5.1] the boundedness at infinity, $\limsup_{\mu \to \infty} \mu^{\frac{1}{2}} \|A(\mu)\| < \infty$, holds already with $s > \frac{1}{2}$. Thus the further restriction $s > 1$ is needed in order to ensure the boundedness at $\lambda = 0$ (Theorem A).

Remark 6.4. Clearly we can take $[0, T]$ as the time interval, instead of $\mathbb{R}$, for any $T > 0$.

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