Communications on Applied Mathematics and Computation

Hyperbolic Conservation Laws, Integral Balance Laws and Regularity of Fluxes

--Manuscript Draft--

Manuscript Number:	CAMC-D-23-00108R1
Full Title:	Hyperbolic Conservation Laws, Integral Balance Laws and Regularity of Fluxes
Article Type:	Original Paper
Funding Information:	National Outstanding Youth Science Fund Project of National Natural Science Foundation of China (11771054, 12072042,91852207) — Prof. Jiequan Li National Key Scientific Instrument and Equipment Development Projects of China (GJXM92579) — Prof. Jiequan Li Chinesisch-Deutsche Zentrum für Wissenschaftsförderung (GZ1465) — Prof. Jiequan Li
Abstract:	Hyperbolic conservation laws arise in the context of continuum physics, and are mathematically presented in differential form and understood in the distributional (weak) sense. The formal application of the Gauss-Green theorem results in integral balance laws, in which the concept of flux plays a central role. This paper addresses the spacetime viewpoint of flux regularity, providing a rigorous treatment of integral balance laws. The established Lipschitz regularity of fluxes (over time intervals) leads to a consistent flux approximation. Thus fully discrete finite volume schemes of high order may be consistently justified with reference to the spacetime integral balance laws.
Corresponding Author:	Matania Ben-Artzi Hebrew University of Jerusalem ISRAEL
Corresponding Author Secondary Information:	
Corresponding Author's Institution:	Hebrew University of Jerusalem
Corresponding Author's Secondary Institution:	
First Author:	Matania Ben-Artzi
First Author Secondary Information:	
Order of Authors:	Matania Ben-Artzi
	Jiequan Li
Order of Authors Secondary Information:	
Author Comments:	

Hyperbolic Conservation Laws, Integral Balance Laws and Regularity of Fluxes

Matania Ben-Artzi[1*] and Jiequan Li[2,3]

[1]Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem, 91904, Israel.
[2] Academy for Multidisciplinary Studies, Capital Normal University, Beijing, 100048, China.
[3]State Key Laboratory for Turbulence Research and Complex System, Peking University, Beijing, 100871, China.

*Corresponding author(s). E-mail(s): mbartzi@math.huji.ac.il;
Contributing authors: jiequan@cnu.edu.cn;

Abstract

Hyperbolic conservation laws arise in the context of continuum physics, and are mathematically presented in differential form and understood in the distributional (weak) sense. The formal application of the Gauss-Green theorem results in integral balance laws, in which the concept of flux plays a central role. This paper addresses the spacetime viewpoint of flux regularity, providing a rigorous treatment of integral balance laws. The established Lipschitz regularity of fluxes (over time intervals) leads to a consistent flux approximation. Thus fully discrete finite volume schemes of high order may be consistently justified with reference to the spacetime integral balance laws.

Keywords: balance laws, hyperbolic conservation laws, finite volume approximations, flux regularity, consistency.

Dedicated to Professor Gerald Warnecke on his 65-th birthday.

1 Introduction

Hyperbolic conservation laws arise in the context of continuum physics, and are mathematically presented in differential form [10]. Since the solution may contain discontinuities no matter how smooth the initial data are, the equations are often understood in the distributional (weak) sense. Quite often the Gauss-Green theorem is applied in a formal way, thus obtaining integral balance laws that are at the foundation of finite volume methods. However, this formal treatment needs rigorous justification.

We refer to [8, 9, 16, 18, 19] for general abstract treatments of the Gauss-Green theorem in the context of geometric measure theory [13].

This paper reviews our recent progress: (i) justifying integral balance laws by the verification of continuity of spacetime fluxes; (ii) defining the consistency of high order finite volume methods by the Lax-Wendroff approach; (iii) establishing the Lax-Wendroff type convergence of finite volume approximations.

2 Integral balance laws

Hyperbolic conservation laws are often written in the divergence form of partial differential equations,

$$\mathbf{u}_t + \nabla_{\mathbf{x}} \cdot \mathbf{f}(\mathbf{u}) = 0, \tag{1}$$

where t is the time of variable, $\nabla_{\mathbf{x}}\cdot$ is the divergence operator in terms of space variable $\mathbf{x} = (x_1, \cdots, x_m)$, $\mathbf{u} = (u_1, \cdots, u_D)^\top \in \mathbb{R}^D$ is the vector of conserved quantities and $\mathbf{f}(\mathbf{u})$ is the matrix of fluxes

$$\mathbf{f}(\mathbf{u}) = (f_1(\mathbf{u}), \cdots, f_D(\mathbf{u})) \in \mathbb{R}^m \times \mathbb{R}^D, \tag{2}$$

and each $\mathbf{f}_i(\mathbf{u})$ is an m-vector. We only assume that the flux $\mathbf{f}(\mathbf{u})$ is locally bounded as the function of $\mathbf{u}$.

Since classical solutions of (1) in general break down and discontinuities appear in the solutions even when subject to very smooth initial data

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \tag{3}$$

we resort to the notion of a weak solution of (1) and (3), namely , the solution is defined in distributional sense:

Definition 1 (Weak solutions) Let $\mathbf{u} \in L^\infty \cap L^1(\mathbb{R}^m;\ [0, T))$ be a weak solution of (1) and (3) if it satisfies

$$\int_0^T \int_{\mathbb{R}^m} \mathbf{u}\phi_t + \mathbf{f}(\mathbf{u}) \cdot \nabla_{\mathbf{x}}\phi(\mathbf{x}, t) d\mathbf{x}dt + \int_{\mathbb{R}^m} \mathbf{u}_0(\mathbf{x})\phi(\mathbf{x}, 0) d\mathbf{x} = 0, \tag{4}$$

for all smooth test functions $\phi \in C^\infty(\mathbb{R}^m \times [0, T))$. Note that ϕ is a D-vector.

Alternatively, following the physical point-of-view, one has two approaches to *integral balance laws* in making sense for a solution of (1). We proceed to present these approaches.

Let $\Omega \subseteq \mathbb{R}^m$ be a bounded domain, $\Gamma = \partial\Omega$, and $0 \leq t_1 < t < t_2 < T$. Let ν be the outward unit normal. We formally apply the Gauss-Green theorem and carry out the integration of (1) in space to have:

Definition 2 (Instant Integral balance law) Let the function $\mathbf{u} \in C((0,T);\ L^\infty(\mathbb{R}^n)) \cap C([0,T);\ L^1(\mathbb{R}^n))$. Then

$$\frac{d}{dt}\int_\Omega \mathbf{u}(\mathbf{x},t)d\mathbf{x} = -\int_\Gamma \mathbf{f}(\mathbf{u})\cdot\nu dS_{\mathbf{x}} \tag{5}$$

is called the *instant balance law* of (1), where $dS_{\mathbf{x}}$ is surface Lebesgue measure, if the following two conditions are satisfied,

(i) For every $t \in [0,T)$, and every bounded domain $\Omega \subseteq \mathbb{R}^m$, the total mass $M(\Omega,t) = \int_\Omega \mathbf{u}(\mathbf{x},t)d\mathbf{x}$ over Ω is well defined and continuously differentiable function of t.

(ii) the Cauchy flux across the boundary Γ

$$h(\Gamma;\ t) := \int_\Gamma \mathbf{f}(\mathbf{u})\cdot\nu dS_{\mathbf{x}} \tag{6}$$

is well-defined and continuous in time t, and Equation (5) is satisfied. Note that h is a D-vector.

Remark 1 In the context of theoretical continuum mechanics the quantity $\int_A \mathbf{f}(\mathbf{u})\cdot\nu dS_{\mathbf{x}}$ across a section $A \subseteq \Gamma$ is called *the Cauchy flux (across A)* and $\mathbf{f}(\mathbf{u})\cdot\nu$ is its density.

Remark 2 In the context of finite volume formulation, (5) is called the semi-discrete form of (1), for which Ω is regarded as a computational control volume. Temporal advancing techniques could be used to derive practical schemes provided that the Cauchy flux is well-defined and effectively approximated.

We can go further and integrate (5) (formally) over any time interval $[t_1,t_2]$, obtaining the following spacetime integral balance law.

Definition 3 (Spacetime integral balance laws) Let the function $\mathbf{u} \in C((0,T);\ L^\infty(\mathbb{R}^n)) \cap C([0,T);\ L^1(\mathbb{R}^n))$. Then

$$\int_\Omega \mathbf{u}(\mathbf{x},t_2)d\mathbf{x} - \int_\Omega \mathbf{u}(\mathbf{x},t_1)d\mathbf{x} = -\int_{t_1}^{t_2}\int_\Gamma \mathbf{f}(\mathbf{u})\cdot\nu dS_{\mathbf{x}}dt, \quad 0 \leq t_1 < t_2 < T, \tag{7}$$

is called the *spacetime balance law* of (1) if Equation (7) holds, subject to the following conditions.

1. $M(\Omega,t)$ (see (i) in Definition 2) is continuous in t.

2. The *spacetime flux* across the boundary Γ over time interval $[t_1, t_2]$,

$$h(\Gamma;\ t_1, t_2) := \int_{t_1}^{t_2} \int_{\Gamma} \mathbf{f}(\mathbf{u}) \cdot \nu dS_{\mathbf{x}} dt \tag{8}$$

is well-defined and continuous with respect to suitable perturbations of the boundary Γ.

Remark 3 In analogy to the Cauchy flux, we call

$$H(A;\ t_1, t_2) = \int_{t_1}^{t_2} \int_{A} \mathbf{f}(\mathbf{u}) \cdot \nu dS_{\mathbf{x}} dt, \quad A \subseteq \Gamma, \tag{9}$$

the *spacetime flux* through the boundary section A.

The validity of the conditions in Definitions 2 and 3, especially in what concerns the fluxes (6) and (8) is far from obvious. Note that the solution of (1) is typically discontinuous and $\mathbf{f}(\mathbf{u})$ is nonlinear so that the traces of fluxes need to be attended. Recall the following comment concerning this issue: "the drawback of this, functional analytic, demonstration is that it does not provide any clues on how the $q_{\mathcal{D}}$ may be computed from A" [10, Section 1.3], where $q_{\mathcal{D}}$ refers to the flux density, and A is the Cauchy flux. In the next section we justify the continuity of the spacetime flux with respect to space perturbation and establish the validity of the spacetime integral balance law (7) for a weak solution.

3 Regularity of fluxes and spacetime integral balance laws

In this section we follow [6] in establishing (7) for weak solutions in the multi-dimensional case. In the case of one spatial coordinate this result was obtained in [5]. We start from $\Gamma_0 = \Gamma$ and construct a tubular neighborhood [20] with the following properties. For some small $0 < \delta < 1$ there exists a family of expanding smooth bounded domains $\{\Omega_\mu \subseteq \mathbb{R}^m, \mu \in (-\delta, 1-\delta)$ so that their respective boundaries $\{\Gamma_\mu, \mu \in (-\delta, 1-\delta)\}$ form a foliation of a tubular neighborhood of Γ_0. The coordinate μ is normal to Γ_μ so that $\partial/\partial\mu = \nu_{\mathbf{x}}$ is the unit normal. Denote by dS_μ the Labesgue surface measure on Γ_μ, $\mu \in (-\delta, 1-\delta)$.

Theorem 1 *Let* $\mathbf{u} \in L^\infty_{loc}(\mathbb{R}^m \times (0,T)) \cap L^1_{loc}(\mathbb{R}^m \times [0,T))$ *be a weak solution of* (1) *in the sense of Definition 1. Then we have*

(i) For every $t \in [t_1, t_2]$, *the function* $\mathbf{g}(\mathbf{x};\ t_1, t_2) = \int_{t_1}^{t_2} \mathbf{f}(\mathbf{u}(\mathbf{x}, t)) dt$ *satisfies*

$$\nabla_{\mathbf{x}} \cdot \mathbf{g}(\mathbf{x};\ t_1, t_2) \in L^\infty_{loc}(\mathbb{R}^m). \tag{10}$$

(ii) For every smooth domain and the geometric construction $\{\Omega_\mu\}$ *the trace function defined by*

$$h(\mu;\ t_1,t_2) = \int_{t_1}^{t_2} \left[\int_{\Gamma_\mu} \mathbf{f}(\mathbf{u}) \cdot \nu_\mu dS_\mu \right] dt, \quad \mu \in (-\delta, 1-\delta) \tag{11}$$

is Lipschitz continuous with respect to μ.

In this case Eq. (7) holds for every $0 \le t_1 < t_2 \le T$.

Since a considerable part of the theoretical and numerical studies are still carried out in the one-dimensional case ($m = 1$), it is useful to state the form of the theorem in this case.

Theorem 2 *Let* $\mathbf{u}(x,t) \in L^\infty_{loc}(\mathbb{R} \times (0,T)) \cap L^1_{loc}(\mathbb{R} \times [0,T))$ *be the weak solution to one-dimensional conservation laws*

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0, \quad x \in \mathbb{R}, t > 0. \tag{12}$$

Then we have:

(i) For every fixed $[t_1, t_2]$*, the spacetime flux* $\mathbf{g}(x) = \int_{t_1}^{t_2} \mathbf{f}(\mathbf{u}(x,t))dt$ *is locally Lipschitz continuous in* $x \in \mathbb{R}$.

(ii) the spacetime integral balance law holds over a spacetime domain $Q = [x_1, x_2] \times [t_1, t_2]$

$$\int_{x_1}^{x_2} \mathbf{u}(x,t_2)dx - \int_{x_1}^{x_2} \mathbf{u}(x,t_1)dx = \int_{t_1}^{t_2} \mathbf{f}(\mathbf{u}(x_1,t))dt - \int_{t_1}^{t_2} \mathbf{f}(\mathbf{u}(x_2,t))dt. \tag{13}$$

The proof can be found in [6] for Theorem 1 and in [5] for Theorem 2, relying on Sobolev estimates in $W^{1,p}$ to get Lipschitz continuity. This regularity property of spacetime fluxes is in sharp contrast to that of the Cauchy flux [7] since the discontinuous property of solution gives rise to the difficulty in defining the trace of $\mathbf{f}(\mathbf{u})$ on $A \subseteq \Gamma$.

4 Finite volume approximation and its consistency

In view of Theorems 1 and 2 the spacetime flux (8) is indeed continuous, while the instantaneous Cauchy flux (6), which is formally the time derivative of the spacetime flux is in general not well defined. We conclude that *the spacetime flux should be used for the approximation*, implying that the resulting finite volume scheme is fully discrete.

4.1 1D finite volume schemes

The integral balance law (7) is at the basis of the finite volume approximation to the conservation law (1). We first discuss the discretization in the one-dimensional setting, using a uniform grid. Let $\tau = \Delta t$ be a fixed time step. The spatial control volumes (intervals) are $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$, $j \in \mathbb{N}$, $\Delta x = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, and the spacetime control volumes are

$$Q_j^n = I_j \times (t_n, t_{n+1}), \quad t_{n+1} = t_n + \tau. \tag{14}$$

We denote by $\mathcal{U}$ the functional space of solutions of (12) and by $V^k \subseteq \mathcal{U}$ a finite dimensional subspace of order k when restricted to each I_j. In order to define the finite volume approximation, we need first to define approximate fluxes. We assume that there is a unique "entropy" solution, denoted by $\mathbf{u}(x,t;\ \xi) = S(t)\xi \in \mathcal{U}$, $0 < t < \tau$, subject to the initial data $\xi \in V^k$. Due to the semigroup property of solutions to (12), we can focus our discussion on the interval $[0, \tau)$.

Definition 4 (1-D Approximate flux) Let $\{\mathbf{F}^{\xi}_{j+\frac{1}{2}}(t),\ 0 \le t < \tau\}_{j=-\infty}^{\infty}$ be a family of D-dimensional functions of t. They are **approximate fluxes** (in the time interval $[0, \tau)$) corresponding to the initial function $\xi \in V^k$, if the following **finite propagation property** is satisfied for all $j \in \mathbb{N}$.

(i) $\mathbf{F}^{\xi}_{j+\frac{1}{2}}(t),\ 0 \le t < \tau$, depends only on the restriction of ξ to $I_j \cup I_{j+1}$.

(ii) If $\xi \equiv c = const.$ in $I_j \cup I_{j+1}$ then $\mathbf{F}^{\xi}_{j+\frac{1}{2}}(t) \equiv \mathbf{f}(c)$.

Next we define the **consistency of the approximate fluxes**.

Definition 5 (Consistency in 1D) The approximate flux $\mathbf{F}^{\xi}_{j+\frac{1}{2}}(t)$ is consistent of order $q > 0$ with the balance law (13) if there holds, for any $\xi \in V^k$,

$$\begin{aligned} &\left[\int_0^{\tau} \mathbf{F}^{\xi}_{j+\frac{1}{2}}(t)dt - \int_0^{\tau} \mathbf{F}^{\xi}_{j-\frac{1}{2}}(t)dt\right] \\ &- \left[\int_0^{\tau} \mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}}, t;\ \xi)dt - \int_0^{\tau} \mathbf{f}(\mathbf{u}(x_{j-\frac{1}{2}}, t;\ \xi))dt\right] = \mathcal{O}(\tau^{2+q}). \end{aligned} \tag{15}$$

Remark 4 Observe that the order of consistency strongly depends on the order of the approximating subspace V^k. This is clearly demonstrated in the case of the fundamental Godunov flux below.

The finite volume approximation to (13) is now presented in terms of the approximate fluxes.

Definition 6 (1D Finite Volume Approximation) (i) Let $\{\mathbf{F}^{\xi}_{j+\frac{1}{2}}(t);\ 0 < t < \tau\}$ be approximate fluxes consistent with (13) of order $q > 0$ in the sense of Definition 5.

(ii) Let $\widetilde{S}(\tau) : V^k \to \mathcal{U}$ be an approximate evolution operator associated with the approximate fluxes such that

$$\int_{I_j} \widetilde{S}(\tau)\xi dx - \int_{I_j} \xi dx + \int_0^{\tau} \mathbf{F}^{\xi}_{j+\frac{1}{2}}(t)dt - \int_0^{\tau} \mathbf{F}^{\xi}_{j-\frac{1}{2}}(t)dt = 0, \tag{16}$$

for $j \in \mathbb{N}$.

(iii) There exists a projection map $P^k : \mathcal{U} \to V^k$ such that the average is preserved

$$\int_{I_j} P^k\xi(x)dx = \int_{I_j} \xi(x)dx, \quad j \in \mathbb{N}. \tag{17}$$

Then a family of maps $\{\Phi^k : V^k \to V^k\}$ is finite volume scheme for the conservation law (12) if

$$\Phi^k = P^k\widetilde{S}(\tau). \tag{18}$$

Thus given initial data $\mathbf{u}(x, 0) = \mathbf{u}_0(x) \in \mathcal{U}$ to (12), we construct the sequence of finite volume approximate solutions by taking first $\mathbf{u}^0(x) = P^k\mathbf{u}_0(x)$ and then proceed for $n = 0, 1, 2\cdots$, by

$$\mathbf{u}^{n+1}(x) = \Phi^k\mathbf{u}^n(x). \tag{19}$$

4.2 Practical consistency of flux approximation

It is clear that the error of a finite volume approximation comes from two parts: the flux approximation and the projection. The literature concerning the projection error (i,e. slope-limiters) is quite extensive. Here we concentrate on the flux approximation, which strongly depends on the space of approximation. In this section, we suppress the dependence of notation on ξ if no confusion can occur.

Godunov flux. We first assume that the initial data $\xi(x) \in V^0$ is piecewise constant

$$\mathbf{u}_0(x) = \xi(x) = \mathbf{u}^0_j, \quad x \in I_j. \tag{20}$$

Then (assuming a CFL condition) the solution $\mathbf{u}(x_{j+\frac{1}{2}}, t;\ \xi)$ is constant for $0 < t < \tau$ and can be obtained by solving the local Riemann problem. The value is denoted by $\mathbf{u}_{j+\frac{1}{2}} := R(0;\ \mathbf{u}^0_j, \mathbf{u}^0_{j+1})$. The Godunov flux is defined as [15]

$$\mathbf{F}_{j+\frac{1}{2}}(t) = \mathbf{f}(\mathbf{u}_{j+\frac{1}{2}}). \tag{21}$$

Hence, if $\xi \in V^0$, the Godunov flux fully agrees with the exact flux for the piecewise constant initial data (20) and no error exists. Formally, it means that the order of consistency of the Godunov flux is $q = \infty$!

In general, as $\xi(x) \in V^k$, the Godunov flux uses the leading term for the approximation. To be more precise, the solution $\mathbf{u}(x_{j+\frac{1}{2}}, t;\ \xi)$ is no longer constant for $0 < t < \tau$ and the solution $\mathbf{u}(x, t;\ \xi)$ along $x = x_{j+\frac{1}{2}}$ can be expanded as

$$\mathbf{u}(x_{j+\frac{1}{2}}, t;\ \xi) = \mathbf{u}(x_{j+\frac{1}{2}}, 0+;\ \xi) + t \cdot \frac{\partial \mathbf{u}}{\partial t}(x_{j+\frac{1}{2}}, 0+;\ \xi) + \mathcal{O}(t^2), \tag{22}$$

and

$$\mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}}, t;\ \xi)) = \mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}}, 0+;\ \xi)) + \mathbf{f}'(\mathbf{u}(x_{j+\frac{1}{2}}, 0+;\ \xi))\frac{\partial \mathbf{u}}{\partial t}(x_{j+\frac{1}{2}}, 0+;\ \xi)t + \mathcal{O}(t^2). \tag{23}$$

The Godunov flux uses the leading term of the expansion (23),

$$\mathbf{F}^G_{j+\frac{1}{2}}(t) = \mathbf{f}(\mathbf{u}_{j+\frac{1}{2}}), \quad \mathbf{u}_{j+\frac{1}{2}} = \mathbf{u}(x_{j+\frac{1}{2}}, 0+;\ \xi). \tag{24}$$

Then we have

$$\int_0^\tau \mathbf{F}^G_{j+\frac{1}{2}}(t)dt - \int_0^\tau \mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}}, t;\ \xi))dt = \frac{\tau^2}{2}\mathbf{f}'(\mathbf{u}_{j+\frac{1}{2}})\frac{\partial \mathbf{u}}{\partial t}(x_{j+\frac{1}{2}}, 0+;\ \xi) + \mathcal{O}(\tau^3). \tag{25}$$

Taking the difference of the two boundary values

$$\begin{aligned}&\int_0^\tau \mathbf{F}^G_{j+\frac{1}{2}}(t)dt - \int_0^\tau \mathbf{F}^G_{j-\frac{1}{2}}(t)dt - \left[\int_0^\tau \mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}}, t;\ \xi))dt - \int_0^\tau \mathbf{f}(\mathbf{u}(x_{j-\frac{1}{2}}, t;\ \xi))dt\right] \\ &= \left[\mathbf{f}'(\mathbf{u}_{j+\frac{1}{2}})\frac{\partial \mathbf{u}}{\partial t}(x_{j+\frac{1}{2}}, 0+;\ \xi) - \mathbf{f}'(\mathbf{u}_{j-\frac{1}{2}})\frac{\partial \mathbf{u}}{\partial t}(x_{j-\frac{1}{2}}, 0+;\ \xi)\right]\frac{\tau^2}{2} + \mathcal{O}(\tau^3).\end{aligned} \tag{26}$$

If the solution $\mathbf{u}(x, t;\ \xi)$ is smooth the difference in the right-hand side of (26) contributes (via the CFL condition) another factor of τ. Otherwise, the error is $\mathcal{O}(\tau^2)$. We therefore arrive at the following conclusion.

Proposition 3 (Godunov Flux) *Assume that $\xi(x) \in V^k, k \geq 1$. Then the Godunov scheme has first order accuracy for smooth solutions but just zero order if the solution contains discontinuities.*

First order flux approximation. In practice, for the given initial data $\xi(x) \in V^k$, there is an alternative way of defining first order flux approximations [14],

$$\mathbf{F}_{j+\frac{1}{2}}(t) = \frac{1}{2}(\mathbf{f}(\mathbf{u}_-) + \mathbf{f}(\mathbf{u}_+)) - \frac{\alpha}{2\lambda}(\mathbf{u}_+ - \mathbf{u}_-), \quad \mathbf{u}_\pm := \xi(x_{j+\frac{1}{2}}\pm), \tag{27}$$

for some $\alpha > 0$, $\lambda = \tau/\Delta x$. If we try to obtain its order of consistency as in Definition 5 we get

$$\begin{aligned} &\Big[\int_0^\tau \mathbf{F}_{j+\frac{1}{2}}(t)dt - \int_0^\tau \mathbf{F}_{j-\frac{1}{2}}(t)dt\Big] \\ -&\Big[\int_0^\tau \mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}}, t;\ \xi))dt - \int_0^\tau \mathbf{f}(\mathbf{u}(x_{j-\frac{1}{2}}, t;\ \xi))dt\Big] = \mathcal{O}(|\mathbf{u}_+ - \mathbf{u}_-|)\tau. \end{aligned} \tag{28}$$

Thus the error is estimated in terms of the total variation $TV(\mathbf{u})$. This is true even for $\xi(x) \in V^0$. In general it cannot be converted to estimates in terms of τ due to discontinuities. Furthermore, while for scalar conservation laws the total variation is not increasing [10], this is not true for hyperbolic systems, where solutions involve very complex wave interactions. We conclude that for such approximate fluxes the order of consistency (even the notion of consistency) cannot be addressed in our framework.

High order flux approximations As discussed above, in order to achieve high order accuracy, we have to adopt high order flux approximation $\mathbf{F}_{j+\frac{1}{2}}(t)$. **It is precisely here that we can use the Lipschitz continuity of fluxes** as expressed in Theorem 2 . Indeed, the theorem guarantees that the difference

$$\int_0^\tau \mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}}, t;\ \xi))dt - \int_0^\tau \mathbf{f}(\mathbf{u}(x_{j-\frac{1}{2}}, t;\ \xi))dt$$

provides (using the CFL condition) a factor $\mathcal{O}(\tau)$. In view of Definition 5 this means that we can focus on one endpoint and attempt to obtain a high value of α in the estimate of

$$\int_0^\tau \mathbf{F}_{j+\frac{1}{2}}(t)dt - \int_0^\tau \mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}}, t;\ \xi))dt = \mathcal{O}(\tau^{1+\alpha}), \tag{29}$$

for some $\alpha > 0$. The error is measured in terms of the temporal increment τ or equivalently the spatial grid size Δx. Thanks to (22) and (23), we have

$$\mathbf{F}_{j+\frac{1}{2}}(t) = \mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}}, 0+;\ \xi)) + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial t}(x_{j+\frac{1}{2}}, 0+;\ \xi)t + \cdots + \mathcal{O}(t^\alpha). \tag{30}$$

This is equivalent to the Taylor method for ordinary differential equations and requires the knowledge of the instantaneous values $\frac{\partial \mathbf{u}}{\partial t}, \cdots, \frac{\partial^\alpha \mathbf{u}}{\partial t^\alpha}$. In numerical approximations this approach is replaced by multi-stage methods [23], in order to avoid high order temporal derivatives.

In Section 5 we will see how the Taylor method can be implemented by introducing the generalized Riemann problem (GRP) methodology. In a suitable sense, it can be considered as a Lax-Wendroff approach (normally associated with analytic setting)) in a discontinuous nonlinear framework. The evaluation of temporal derivatives is carried out by using spatial slopes on the two sides of a discontinuity [2] and careful inspection of the propagation of the solution along characteristics (including shock formation).

4.3 Multi-D extensions

We briefly discuss how the $1-D$ methodology can be adopted in order to establish a finite volume approximation to multidimensional conservation laws (7). Let Ω be a computational domain covered by a set of closed control volume Ω_j, $\Omega = \cup_{j\in J}\Omega_j$, $\Omega_j \cap \Omega_\ell = \Gamma_{j\ell}$. They are assumed to be pairwise disjoint except for common boundaries. Then the spacetime integral balance law (7), when applied to Ω_j, becomes,

$$\int_{\Omega_j} \mathbf{u}(\mathbf{x},\tau;\ \xi)d\mathbf{x} - \int_{\Omega_j} \mathbf{u}(\mathbf{x},0;\ \xi)d\mathbf{x} + \sum_{\ell}\int_0^\tau \int_{\Gamma_{j\ell}} \mathbf{f}(\mathbf{u})\cdot\nu_{j\ell} dS_{\mathbf{x}}dt = 0, \quad (31)$$

where $\mathbf{u}(\mathbf{x},0;\ \xi) = \xi$ is the initial data.

The general approach, as in the $1-D$ case, emphasizes the role of approximate fluxes.

Definition 7 (Multi-D approximate flux) The functions of the family $\{\mathbf{F}^{\xi}_{j\ell}(t),\ 0 \le t < \tau\}_{j=-\infty}^{\infty}$ are **approximate fluxes** (in the time interval $[0,\tau)$) corresponding to the initial function $\xi \in V^k$, if the following **finite propagation property** is satisfied.

(i) $\mathbf{F}^{\xi}_{j\ell}(t)$, $0 \le t < \tau$, depends only on the restriction of ξ to $\Omega_j \cup \Omega_\ell$, where the index ℓ is taken such that $\Gamma_{j\ell} \neq \emptyset$.

(ii) If $\xi \equiv c = const.$ in $\Omega_j \cup \Omega_\ell$ then $\mathbf{F}^{\xi}_{j\ell}(t) \equiv \mathbf{f}(c)\|\Gamma_{j\ell}\|$.

The multi-dimensional case is complicated since the exact flux depends on $x \in \Gamma_{j\ell}$, and needs to be approximated at every boundary point. Remark that $\mathbf{F}^{\xi}_{j\ell}(t)$ implicitly contains the approximate integration along the common boundary $\Gamma_{j\ell}$.

Numerically, the boundary integrals are handled by using suitably high order integration formulas, such as Gaussian quadrature:

$$\int_0^\tau \int_{\Gamma_{j\ell}} \mathbf{f}(\mathbf{u})\cdot\nu_{j\ell} dS_{\mathbf{x}}dt \approx \sum_{\ell,r}\int_0^\tau \omega_r \mathbf{f}(\mathbf{u})\cdot\nu_{j\ell}(\mathbf{x}_r)dt, \quad (32)$$

within desired order of accuracy, where ω_r is the weight at the Gaussian point $(\mathbf{x}_r, t)$ on $\Gamma_{j\ell}$. Then we can construct the approximate flux at each point $\mathbf{x}_r$.

Definition 8 (Consistency in Multi-D) The approximate flux $\mathbf{F}^{\xi}_{j\ell}(t)$ is consistent of order $q > 0$ with the balance law (31) if there holds

$$\sum_\ell \int_0^\tau \mathbf{F}^{\xi}_{j\ell}(t)dt - \sum_\ell \int_0^\tau \int_{\Gamma_{j\ell}} \mathbf{f}(\mathbf{u}(\mathbf{x},t;\ \xi))\cdot\nu_{j\ell} dS_{\mathbf{x}}dt = \mathcal{O}(\tau^{2+q}). \quad (33)$$

Observe that the boundary integral $\int_0^\tau \int_{\Gamma_{j\ell}} \mathbf{f}(\mathbf{u}(\mathbf{x},t;\ \xi)) \cdot \nu_{j\ell} dS_{\mathbf{x}}$ is well defined precisely due to Theorem 1. We refer to [6] for more details.

5 Lax-Wendroff type approach for flux approximations

The Lax-Wendroff approach was proposed in a finite difference version for hyperbolic conservation laws [21], assuming very regular solutions. Essentially it can be viewed as the numerical realization of Cauchy-Kovalevskaya theorem for partial differential equations [11, Chapter 4]. In this section we place it in the context of our approximate fluxes with high order of consistency. We shall do it only in the $1-D$ setting. We therefore consider the $1-D$ version of Equation (1):

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0, \quad x \in \mathbb{R}. \tag{34}$$

In general, solutions $\mathbf{u}(x,t;\ \xi)$ are known to develop discontinuities even for very smooth initial data ξ. In particular, the same is true for the fluxes $\mathbf{f}(\mathbf{u}(x,t;\ \xi))$. Nevertheless, in light of Theorem 2 the integral $\int_0^\tau \mathbf{f}(\mathbf{u}(x,t;\ \xi))dt$ is a Lipschitz function of x, hence it is legitimate to consider its point value at every fixed point, in particular the point $x = x_{j+\frac{1}{2}}$, that is a point of discontinuity of the initial data ξ. Then we are led to study the behavior of $\mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}}, t;\ \xi))$ as a function of $t \in (0, \tau)$.

Specifically it boils down to solving the Generalized Riemann Problem (GRP) [2, 4]) which we proceed to discuss.

Let $\mathbf{u}(x_{j+\frac{1}{2}}, 0+;\ \xi)$ be the instantaneous value of the solution (obtained by solving a Riemann problem) and let

$$F^{\xi}_{j+\frac{1}{2}}(t) = \mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}}, 0+;\ \xi)), \tag{35}$$

be the approximate flux.

Let $\mathbf{u}_t(x_{j+\frac{1}{2}}, 0+;\ \xi)$ be the instantaneous value of the time-derivative of the solution. From

$$\begin{aligned} &\mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}}, t;\ \xi)) \\ &= \mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}}, 0+;\ \xi)) + \mathbf{f}'(\mathbf{u}(x_{j+\frac{1}{2}}, 0+;\ \xi))\mathbf{u}_t(x_{j+\frac{1}{2}}, 0+;\ \xi)t + \mathcal{O}(t^2), \end{aligned} \tag{36}$$

it follows that

$$\begin{aligned} &\int_0^\tau F^{\xi}_{j+\frac{1}{2}}(t)dt - \int_0^\tau \mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}}, t;\ \xi))dt \\ &= \frac{1}{2}\mathbf{f}'(\mathbf{u}(x_{j+\frac{1}{2}}, 0+;\ \xi))\mathbf{u}_t(x_{j+\frac{1}{2}}, 0+;\ \xi)\tau^2 + \mathcal{O}(\tau^3). \end{aligned}$$

Hence

$$\int_0^{\tau}\left[F^{\xi}_{j+\frac{1}{2}}(t)-F^{\xi}_{j-\frac{1}{2}}(t)\right]dt-\int_0^{\tau}\left[\mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},t;\ \xi))-\mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},t;\ \xi))\right]dt$$
$$=\frac{1}{2}[\mathbf{f}'(\mathbf{u}(x_{j+\frac{1}{2}},0+;\ \xi))\mathbf{u}_t(x_{j+\frac{1}{2}},0+;\ \xi)-\mathbf{f}'(\mathbf{u}(x_{j-\frac{1}{2}},0+;\ \xi))\mathbf{u}_t(x_{j-\frac{1}{2}},0+;\ \xi)]\tau^2$$
$$+\mathcal{O}(\tau^3). \tag{37}$$

If no regularity of the solution $\mathbf{u}(x,t;\ \xi)$ is assumed (in particular, if it is discontinuous) then the approximate flux is only consistent of order zero ($q=0$ in (15)). However, in regions where the solution is smooth the difference

$$\mathbf{f}'(\mathbf{u}(x_{j+\frac{1}{2}},0+;\ \xi))\mathbf{u}_t(x_{j+\frac{1}{2}},0+;\ \xi)-\mathbf{f}'(\mathbf{u}(x_{j-\frac{1}{2}},0+;\ \xi))\mathbf{u}_t(x_{j-\frac{1}{2}},0+;\ \xi)=\mathcal{O}(\tau), \tag{38}$$

thus raising the order of consistency to $q=1$.

The remedy here is to upgrade the approximate flux (35) by adding the GRP solution, thus introducing the GRP fluxes.

Definition 9 (GRP Approximate Flux) The GRP approximate flux is given by

$$F^{\xi}_{j+\frac{1}{2}}(t)=\mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},0+;\ \xi))+\mathbf{f}'(\mathbf{u}(x_{j+\frac{1}{2}},0+;\ \xi))\mathbf{u}_t(x_{j+\frac{1}{2}},0+;\ \xi)t. \tag{39}$$

Now

$$\int_0^{\tau}F^{\xi}_{j+\frac{1}{2}}(t)dt-\int_0^{\tau}\mathbf{f}(\mathbf{u}(x_{j+\frac{1}{2}},t;\ \xi))dt=\mathcal{O}(\tau^3),$$

so that the order of consistency is $q=1$ in all cases. For smooth solutions we obtain second-order consistency ($q=2$), since in analogy with (38)

$$\mathbf{f}''(\mathbf{u}(x_{j+\frac{1}{2}},0+;\ \xi))\mathbf{u}_t(x_{j+\frac{1}{2}},0+;\ \xi)-\mathbf{f}''(\mathbf{u}(x_{j-\frac{1}{2}},0+;\ \xi))\mathbf{u}_t(x_{j-\frac{1}{2}},0+;\ \xi)=\mathcal{O}(\tau). \tag{40}$$

Thus, when reduced to the smooth setting, the common statement about the second order consistency of this approximate flux (as well as the MUSCL flux) is recovered.

In the following, we clarify the methodology by two types of equations. The first is the simplest of all hyperbolic equations(scalar, linear, constant speed) while the second deals with Euler's system for compressible, nonisentropic flows, sort of a "flagship" representing nonlinear systems of conservation laws.

5.1 Linear advection equation

We consider one-dimensional linear advection equations

$$u_t+au_x=0,\quad a>0, \tag{41}$$

as example for the flux approximation. The finite volume formula takes the form

$$\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x,\tau)dx - \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u(x,0)dx + a\int_0^\tau u(x_{j+\frac{1}{2}},t)dt - a\int_0^\tau u(x_{j-\frac{1}{2}},t)dt. \tag{42}$$

The initial data $\xi(x)$ consists of piecewise polynomials of degree k,

$$\xi(x) = p_j^k(x), \quad x \in [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]. \tag{43}$$

We use the Taylor approach for the flux computation

$$u(x_{j+\frac{1}{2}},t) = u(x_{j+\frac{1}{2}},0+) + \frac{\partial u}{\partial t}(x_{j+\frac{1}{2}},0+)t + \cdots + \frac{\partial^k u}{\partial t^k}(x_{j+\frac{1}{2}},0+)\frac{t^k}{k!}, \tag{44}$$

thanks to the linearity. As the Lax-Wendroff approach is applied to this case, we have

$$\frac{\partial u}{\partial t}(x_{j+\frac{1}{2}},0+) = -a\frac{\partial u}{\partial x}(x_{j+\frac{1}{2}},0+), \cdots, \frac{\partial^k u}{\partial t^k}(x_{j+\frac{1}{2}},0+) = (-a)^k\frac{\partial^k u}{\partial x^k}(x_{j+\frac{1}{2}},0+). \tag{45}$$

Then we use the initial data $p_j^k(x)$ to upwind (for $a > 0$) obtain the value $\frac{\partial^\alpha u}{\partial x^\alpha}(x_{j+\frac{1}{2}},0+) = \frac{d^\alpha}{dx^\alpha}p_j^k(x_{j+\frac{1}{2}}-)$, $\alpha = 1, \cdots, k$, and obtain the approximate flux

$$F_{j+\frac{1}{2}}(t) = a\left[u(x,0+) + \frac{\partial u}{\partial t}(x,0+)\cdot t + \cdots + \frac{\partial^\alpha u}{\partial t^\alpha}(x,0+)\cdot\frac{t^q}{q!}\right]_{x=x_{j+\frac{1}{2}}}, \tag{46}$$

where $1 \leq q \leq k$. It is evident that the truncation error is

$$\int_0^\tau F_{j+\frac{1}{2}}(t)dt - \int_0^\tau au(x_{j+\frac{1}{2}},t)dt = \mathcal{O}(\tau^{q+2}). \tag{47}$$

If $p_k(x)$ consists of piecewise polynomials of degree k, the linearity of (41) implies

$$\int_0^\tau F_{j+\frac{1}{2}}(t)dt - \int_0^\tau au(x_{j+\frac{1}{2}},t)dt \equiv 0 \tag{48}$$

if $q > k$. In particular, as $k = 1$, we have

$$\int_0^\tau au(x_{j+\frac{1}{2}},t)dt = au(x_{j+\frac{1}{2}},\tau/2)\cdot\tau = a\left[u(x,0+) + \frac{\partial u}{\partial t}(x,0+)\cdot\frac{\tau}{2}\right]\tau. \tag{49}$$

Let us now remark about the corresponding two dimensional equation,

$$u_t + au_x + bu_y = 0, \tag{50}$$

where a, b are constant and (x, y) are spatial variables, the flux approximation becomes harder. Consider structural (rectangular) meshes $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [y_{\ell-\frac{1}{2}}, y_{\ell+\frac{1}{2}}]$ and approximate the flux on the interface $\{(x, y); \quad x = x_{j+\frac{1}{2}}, y_{j-\frac{1}{2}} < y < y_{j+\frac{1}{2}}\}$

$$\int_0^\tau \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} au(x_{j+\frac{1}{2}}, y, t)dydt \approx \sum_q \omega_q \int_0^\tau au(x_{j+\frac{1}{2}}, y_q, t)dt, \tag{51}$$

where $y_q \in [y_{\ell-\frac{1}{2}}, y_{\ell+\frac{1}{2}}]$ are Gaussian points and ω_q are weights. We use the Lax-Wendroff approach replacing the temporal derivatives by spatial derivatives,

$$\frac{\partial^q u}{\partial t^q}(x_{j+\frac{1}{2}}, y_q, 0+) = \left[-a\frac{\partial}{\partial x} - b\frac{\partial}{\partial y}\right]^q u(x_{j+\frac{1}{2}}, y_q, 0+), \quad 1 \le q \le k, \tag{52}$$

and then carry out the Taylor expansion at each Gaussian point in the same way as for one dimensional cases. Thus, the numerical fluxes are obtained.

Remark 5 The replacement of temporal derivatives by the corresponding spatial derivatives includes the transversal effect $\frac{\partial u}{\partial y}$ in the numerical flux, which is very crucial in multidimensional numerical schemes [22].

5.2 Euler equations of compressible inviscid flow

As the prototype of hyperbolic conservation laws, the system of compressible Euler equations

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, \\ (\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v} + p) = 0, \\ (\rho E)_t + \nabla \cdot (\mathbf{v}(\rho E + p)) = 0, \end{cases} \tag{53}$$

plays an important role in the development of theory, numerics and applications, where ρ, $\mathbf{v}$, p, $E = |\mathbf{v}|^2/2 + e$ are the density, velocity, pressure and total energy, e is the internal energy.

As $\xi(x) \in V^1$, i,e., $\xi(x)$ is piecewise linear, the generalized Riemann problem (GRP) method was developed in [1, 2] and then improved in the direct Eulerian version [3]. As $\xi(x) \in V^k$, $k \ge 2$, the GRP method was extended in [17] to achieve high order approximate fluxes in the sense of (29).

Some remarks are in order.

(i) There are acoustic versions of GRP methods provided that waves involved are weak so that the equations (53) could be linearized. The popular ADER solvers were developed along this line [27]. Hence the GRP method could be regarded as a nonlinear version of discontinuous Lax-Wendroff method.

(ii) It is amazing to find that the GRP method effectively reflects the thermodynamics of compressible flows [26].

(iii) There are extensions to various systems, e.g., the relativistic fluid dynamics [29] and the blood model [30]

6 Lax-Wendroff type Convergence

The notion of high order consistency of an approximate scheme (Definition 5) is crucial in the study of the convergence of the approximate solutions to a solution of the balance law. We discuss it in the $1-D$ case.

Applying the finite volume approximation (19), we construct the discrete sequence

$$\widetilde{\theta^{n+1}}(x) = \Phi^k(\widetilde{\theta^n}) \in V^k, \quad n = 0, 1, 2, \dots, N-1. \tag{54}$$

The initial data is given by taking the projection of the initial function $u_0 \in \mathcal{U}$ on the subspace V^k

$$\theta^0 = \widetilde{\theta}^0 = P^k u_0 \in V^k. \tag{55}$$

Observe that at each step $\widetilde{\theta^n} \in V^k$ is discontinuous at cell boundaries $x = x_{j+\frac{1}{2}}$ since the element of V^k should preserve the average over I_j.

We shall further assume that these fluxes are consistent of order $q > 0$ (Definition 5).

It follows from Definition 6 (see Equation (16)) that for all grid intervals I_j,

$$\begin{aligned} &\int_{I_j} [\widetilde{\theta^{n+1}}(x) - \widetilde{\theta^n}(x)] dx \\ &= -\int_{t_n}^{t_{n+1}} [F^{\widetilde{\theta^n}}_{j+\frac{1}{2}}(t-t_n) - F^{\widetilde{\theta^n}}_{j-\frac{1}{2}}(t-t_n)] dt, \ -\infty < j < \infty. \end{aligned} \tag{56}$$

We now construct an interpolation function (in spacetime) $\widetilde{\Upsilon}^k(x,t)$ as follows.

$$\begin{aligned} \widetilde{\Upsilon^\tau}(x,t) = \frac{1}{\tau}[(t_{n+1} - t)\widetilde{\theta^n}(x) + (t - t_n)\widetilde{\theta^{n+1}}(x)], \quad t \in [t_n, t_{n+1}], \\ n = 0, 1, \dots, N-1. \end{aligned} \tag{57}$$

Observe that $t_n = n\tau$ depends on τ.

Instead of the classical Lax-Wendroff theorem [14, Section 3.1] we get here the following theorem. We refer to [5] for the proof.

Theorem 4 *Assume that the FVS* (54) *is consistent of order* $q > 0$*. Let* $\{\tau_m \downarrow 0\}$ *be a decreasing sequence of time steps. Let* $u_0 \in \mathcal{U}$ *and let* $\{\widetilde{\Upsilon^{\tau_m}}(x,t)\}_{m=1}^{\infty}$ *be the corresponding functions defined in* (57)*.*

Suppose that

(i) The sequence $\{\widetilde{\Upsilon^{\tau_m}}(x,t)\}_{m=1}^{\infty}$ *is uniformly bounded in* $L^\infty([0,T],L^\infty(\mathbb{R}))$.

(ii) The sequence $\{\widetilde{\Upsilon^{\tau_m}}(x,t)\}_{m=1}^{\infty}$ *converges in* $C([0,T],L^1_{loc}(\mathbb{R}))$ *to a function* $v(x,t)$ *(in particular it is uniformly bounded in this space).*

Then $v(x,t)$ *is a solution of the balance law* (13) *in the sense of Theorem 2.*

Acknowledgments. Dedicated to Professor Gerald Warnecke to his 65-th birthday.

The second author (the late Professor Jiequan Li) was supported by NSFC (Nos. 11771054, 12072042,91852207), the Sino-German Research Group Project (No. GZ1465) and the National Key Project GJXM92579. It is a pleasure to thank C. Dafermos and M. Slemrod for many useful comments.

Compliance with Ethical Standards.

Ethical Statement The manuscript is not submitted to other journals for simultaneous consideration. The submitted work is original and is not published elsewhere in any form or language. The authors certify that they have NO affiliations with or involvement in any organization or entity with any financial interest (such as honorarium; educational grants; participation in speakers' bureaus; membership, employment, consultan- cies, stock ownership, or other equity interest; and expert testimony or patent-licensing arrangements), or non-financial interest (such as personal or professional relationships, affiliations, knowledge or beliefs) in the subject matter or materials discussed in this manuscript.

References

[1] M. Ben-Artzi and J. Falcovitz, A second-order Godunov-type scheme for compressible fluid dynamics. *J. Comput. Phys.*, 55 (1984), 1-32.

[2] Ben-Artzi, M., Falcovitz, J.: Generalized Riemann Problems in Computational Fluid Dynamics. Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press (2003)

[3] M. Ben-Artzi, J. Li and G. Warnecke, A direct Eulerian GRP scheme for compressible fluid flows, *J. Comput. Phys.*, 218 (2006), 19–43.

[4] M. Ben-Artzi and J. Li, Hyperbolic conservation laws: Riemann invariants and the generalized Riemann problem, *Numerische Mathematik*, 106 (2007), 369–425.

[5] Ben-Artzi, M., Li, J.Q.: Consistency of finite volume approximations to nonlinear hyperbolic balance laws. Math.Comp. **90**, 141–169 (2021)

[6] Ben-Artzi, M., Li, J.Q.: Regularity of fluxes in nonlinear hyperbolic balance laws, Communications on Applied Mathematics and Computation, https://doi.org/10.1007/s42967-022-00224-y, (2023)

[7] Chen, G.Q., Comi, G.E., Torres, M.: Cauchy fluxes and Gauss-Green formulas for divergence measure fields over general open sets. Arch. Rat. Mech.Anal. **233**, 87–166 (2019)

[8] Chen, G.Q., Frid, H.: Divergence-measure fields and hyperbolic conservation laws. Arch. Rat.Mech. Anal. **147**, 89–118 (1999)

[9] Chen, G.Q., Torres, M., Ziemer, W.: Gauss-Green theorem for weakly differentiable vector fields, sets of finite perimeter, and balance laws. Comm. Pure Appl. Math. **62**, 242–304 (2009)

[10] Dafermos, C.M.: Hyperbolic Conservation Laws in Continuum Physics, Fourth Edition. Grundlehren der Mathematischen Wissenschaften, Springer, Heidelberg (2016)

[11] Evans, L.C.: Partial Differential Equations. American Mathematical Society (1998)

[12] Eymard. R., Gallouët, T. Herbin, R.: Finite Volume Methods. In Ciarlet, P.G., Lions, J.L. (eds.) Handbook of Numerical Analysis, Vol. VII, pp. 713–1020. North-Holland (2000)

[13] Federer, H.: Geometric Measure Theory. Springer-Verlag, Heidelberg (1969)

[14] Godlewski, E., Raviart, P.A.: Hyperbolic Systems of Conservation Laws. Ellipses (1991)

[15] S. K. Godunov, Finite difference methods for numerical computations of discontinuous solutions of equations of fluid dynamics, Mat. Sb., **47** (1959), 271–295.

[16] Gurtin, M.E., Martins, L.C.: Cauchy's theorem in classical physics, Arch. Rat. Mech. Anal. **60**, 305–324 (1975/76).

[17] J.Qian, J. Li and S. Wang, The generalized Riemann problems for compressible fluid flows: Towards high order, *J. Comput. Phys.*, 259 (2014) 358-389.

[18] Šilhavý, M.: The existence of the flux vector and the divergence theorem for general Cauchy fluxes. Arch. Rat. Mech.Anal. **90**, 195–212 (1985)

[19] Šilhavý, M.: Divergence-measure fields and Cauchy's stress theorem. Rend. Sem. Mat. Padova **113**, 15–45 (2005)

[20] Spivak, M.: A Comprehensaive Introduction to Differential Geometry, Vol. I. Publish or Perish (1979)

[21] P. Lax and B. Wendroff, Systems of conservation laws, Comm. Pure Appl. Math., **13** (1960), 217–237.

[22] X. Lei and J. Li, Transversal effects of high order numerical schemes for compressible fluid flows, *Applied Mathematics and Mechanics (English Edition)*, 40(2019), 343-354.

[23] J. Li and Z. Du, A two-stage fourth order time-accurate discretization for Lax-Wendroff type flow solvers, I. Hyperbolic conservation laws, *SIAM J. Sci. Comput.,* 38 (2016), 3045-3069.

[24] van Leer, B.: On the relation between the upwind-differencing schemes of Godunov, Engquist-Osher and Roe. SIAM J. Sci. Stat. Comput. **5**, 1–20 (1984).

[25] J. Li and Z. Du, A two-stage fourth order time-accurate discretization for Lax-Wendroff type flow solvers, I. Hyperbolic conservation laws, *SIAM J. Sci. Comput.,* 38 (2016), 3045-3069.

[26] J. Li and Y. Wang, Thermodynamical effects and high resolution methods for compressible fluid flows, *J. Comput. Phys.*, 343 (2017), 340-354.

[27] E. Toro, The ADER path to high-order Godunov methods. *Continuum mechanics, applied mathematics and scientific computing: Godunov's legacy, 359-366, Springer, Cham, [2020], ©2020.*

[28] E. Toro, Riemann solvers and numerical methods for fluid dynamics: a practical introduction. Third edition, *Springer-Verlag*, 2009.

[29] K. Wu and H. Tang, A direct Eulerian GRP scheme for spherically symmetric general relativistic hydrodynamics. *SIAM J. Sci. Comput.*, 38 (2016), no. 3, B458-B489.

[30] W. Sheng, Q. Zhang and Y. Zheng, A direct Eulerian GRP scheme for a blood flow model in arteries. *SIAM J. Sci. Comput.*, 43 (2021), no. 3, A1975–A1996.

We have added the reference to Warnecke's 65-th birthday, as indicated by the reviewer.