

SPECTRAL THEORY OF FIRST-ORDER SYSTEMS: FROM CRYSTALS TO DIRAC OPERATORS

MATANIA BEN-ARTZI AND TOMIO UMEDA

ABSTRACT. Let

$$L_0 = \sum_{j=1}^n M_j^0 D_j + M_0^0, \quad D_j = \frac{1}{i} \frac{\partial}{\partial x_j}, \quad x \in \mathbb{R}^n,$$

be a constant coefficient first-order partial differential system, where the matrices M_j^0 are Hermitian. It is assumed that the homogeneous part is strongly propagative. In the nonhomogeneous case it is assumed that the operator is isotropic. The spectral theory of such systems and their potential perturbations is expounded, and a Limiting Absorption Principle is obtained up to thresholds. Special attention is given to a detailed study of the Dirac and Maxwell operators.

The estimates of the spectral derivative near the thresholds are based on detailed trace estimates on the slowness surfaces. Two applications of these estimates are presented:

- Global spacetime estimates of the associated evolution unitary groups, that are also commonly viewed as decay estimates. In particular the Dirac and Maxwell systems are explicitly treated.
- The finiteness of the eigenvalues (in the spectral gap) of the perturbed Dirac operator is studied, under suitable decay assumptions on the potential perturbation.

1. I INTRODUCTION

The equations of classical physics, governing acoustic, electromagnetic and elastic waves in anisotropic media, are given as first-order hyperbolic systems. Similarly,

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the Dirac equation of relativistic quantum electrodynamics is such a system. Already in the classical treatise of Courant and Hilbert the common features of these systems are brought to the fore [17, Chapter III, §2, Chapter VI, §3a].

Motivated by this approach, emphasizing a unifying point of view, we study here basic spectral properties of a class of first-order self-adjoint operators, that includes the aforementioned systems, or, more explicitly, their spatial generators.

Thus, we consider operators acting on \mathbb{C}^K -valued functions in \mathbb{R}^n , $n \geq 2$, having the form

$$(1.1) \quad L = \sum_{j=1}^n M_j^0 D_j + M_0^0 + V(x), \quad D_j = \frac{1}{i} \frac{\partial}{\partial x_j}, \quad x \in \mathbb{R}^n,$$

where the matrices M_j^0 , $0 \leq j \leq n$, are Hermitian $K \times K$ constant matrices and the matrix $V(x)$ is Hermitian for every $x \in \mathbb{R}^n$. Thus L is symmetric (with respect to the $\mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K)$ scalar product) on $C_0^\infty(\mathbb{R}^n, \mathbb{C}^K)$.

The obvious notation

$$\mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K) = \mathcal{L}^2(\mathbb{R}^n) \otimes \mathbb{C}^K$$

(resp. $C_0^\infty(\mathbb{R}^n, \mathbb{C}^K) = C_0^\infty(\mathbb{R}^n) \otimes \mathbb{C}^K$) has been used for the space of Lebesgue square-integrable (resp. smooth and compactly supported) \mathbb{C}^K -valued functions.

Our objectives in this paper are:

- Establish a “Limiting Absorption Principle” (LAP) for the special cases of the Dirac and Maxwell operators (Section 4) up to the thresholds, with sharp resolvent estimates.
- Generalize the LAP to “strongly propagative” or “isotropic operators” (Section 5).
- Extend the LAP to the case of potential perturbation (Section 6).
- Exploit the sharp resolvent estimates of the Dirac and Maxwell operators near the thresholds in the study of two applications:
 - Obtain decay conditions on the potential perturbation of the Dirac operator which guarantee the finiteness of the isolated eigenvalues in the spectral gap (Subsubsection 6.2.1).
 - Derive global spacetime estimates of the solutions in weighted L^2 spaces (Section 7). We note that while various spacetime estimates have been derived for the Dirac equation, we could not find anything in this direction for the (non-dispersive) Maxwell equations.
- Extend the global spacetime estimates of the solutions in weighted L^2 spaces to the case of strongly propagative systems (Section 8).

Both the Dirac (Subsection 2.1) and the Maxwell (Subsection 2.2) systems are isotropic. Their prime significance in physics justifies the detailed study of their spectral structure, as is carried out in Section 4.

The methodology used in this paper is very different from the techniques used so far in the literature. In particular, it enables us to treat these two physical systems using the same tools.

The operator L is viewed as a perturbation of a constant coefficient, symmetric operator

$$(1.2) \quad L_0 = \sum_{j=1}^n M_j^0 D_j + M_0^0, \quad D_j = \frac{1}{i} \frac{\partial}{\partial x_j}, \quad x \in \mathbb{R}^n.$$

The cases where L_0 is either *strongly propagative* (Definition 5.1) or *isotropic* (Definition 5.14) are studied in Section 5.

It will be seen that the LAP is closely connected to the geometry of the level sets of the characteristic surfaces (“normal”, by the terminology of [17]) in \mathbb{R}^n . In the physical context they are referred to as “Slowness Surfaces” [49, Section 4].

The presence of the non-zero matrix M_0^0 is motivated by the “massive” Dirac operator (Subsection 2.1). The basic operator L_0 is then nonhomogeneous, meaning that the associated eigenvalues are not homogeneous functions (see Section 5 for details) and as a result the geometry of the “slowness surfaces” is more complex. As far as we know, the massive Dirac operator is the only instance of a nonhomogeneous operator that has been treated in the literature. In this paper we study the nonhomogeneous operator in the general isotropic setting, of which the massive Dirac operator serves as a special case. In particular, we admit the possibility of “crossing points” (Remark 5.25) that do not exist in the Dirac case.

A general (potential) perturbation theory is presented in Section 6. As an immediate consequence of the LAP for the perturbed operator, it follows that the spectrum is absolutely continuous, except possibly for a discrete sequence of embedded eigenvalues.

The general theory is then applied to the perturbed Dirac operator, as an important example. The threshold estimates for the perturbed Dirac operator enable us (Subsubsection 6.2.1) to give criteria for the finiteness of eigenvalues in the spectral gap.

Remark that the perturbed Maxwell system can also be reduced to the case of potential perturbation [44, Section 1.4], but we choose not to treat it here in detail, as the paper is already quite long. We refer to [11, 26] and references therein for the study of the perturbed Maxwell operator in terms of self-adjointness and absolute continuity of the spectrum.

In Section 7 we establish global spacetime estimates for the Dirac and Maxwell operators, and in Section 8 such estimates are derived for general (homogeneous) strongly propagative operators.

The literature concerning various aspects of the spectral and scattering theory of first-order systems, as well as asymptotic decay of their solutions, is very extensive, hence our reference list is far from being comprehensive. Certainly there are many papers that well deserve being included in our list and of which we are not aware. However, we have made an effort to include references to works directly related to this paper and dealing primarily with the LAP (in interior intervals of the spectrum or at thresholds). Most of these papers were concerned with the Dirac operator. We refer to [36, 38, 42, 50] and references therein for classical treatments of decay of solutions of first-order symmetric systems, and to [35] in the case of exterior domains.

The first proof of the LAP for the (massive) Dirac operator with short range potential perturbation was obtained by Yamada [52]. It was subsequently proved (imposing various hypotheses on the perturbation) by several authors (see [3, 12, 13, 37, 53] for the massive case and [18, 39] for the zero mass case), as well as the recent papers [15, 22] (and references therein) for both the massive and zero mass cases. Their treatments rely on the fact (see Equation (2.8)) that the square of this operator is the Schrödinger operator. A weak* form of the LAP, using the methodology of conjugate operators, was obtained in [14]. The LAP (for the Dirac

operator) up to the threshold was proved in [31]. Recently their method has been extended in [15] in order to obtain a global LAP for the massless Dirac Operator in all dimensions. The zero modes and zero resonances of massless Dirac operators were studied in [39] and eigenfunctions at the threshold energies were studied in [40]. We refer also to [8, 46] for the closely related “relativistic Schrödinger operator”.

For the Maxwell equations in crystal optics the LAP was established in [44, Theorem 1.2].

For a uniformly propagative system (see Definition 5.3) of the form $E(x)^{-1} \sum_{j=1}^n M_j^0 D_j$ the LAP was established in [43, 51], and also in [41]. For the more general strongly propagative systems it was proved in [48, Lemma 2.1]. See however Remark 5.13 concerning this paper. These works dealt with interior intervals of the spectrum, whereas here we obtain detailed estimates at the thresholds.

We are not aware of any work where the LAP for the nonhomogeneous operator L_0 (namely, $M_0^0 \neq 0$) has been addressed, other than the massive Dirac operator.

We refer to Remark 6.19 concerning previous studies of the finiteness of eigenvalues in the spectral gap, in the case of the perturbed (massive) Dirac operator.

2. THE CLASS OF UNPERTURBED OPERATORS—DIRAC AND MAXWELL SYSTEMS

The coefficients $M_0^0, M_1^0, \dots, M_n^0$ of the unperturbed operator L_0 (1.2) are constant Hermitian $K \times K$ matrices (over \mathbb{C}). The addition of M_0^0 will enable us in particular to include the massive Dirac operator in our treatment.

The (unitary) Fourier transform is defined by

$$(2.1) \quad (\mathcal{F}u)(\xi) = \widehat{u}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} u(x) e^{-i\langle \xi, x \rangle_{\mathbb{R}^n}} dx.$$

The constant coefficient operators are transformed into multiplication operators (by *symbols*).

The homogeneous part of L_0 is assumed to be strongly propagative [50], according to Definition 5.1 below.

We shall address this general case in Section 5. We start here with the two most famous physical examples of such operators, namely, the Dirac and Maxwell systems. As we shall see, both systems (including the nonhomogeneous Dirac system with mass) share the property of being *isotropic* (Definition 5.14 below).

2.1. EXAMPLE: THE FREE DIRAC OPERATOR. As a special case we consider the free Dirac operator. It is the self-adjoint operator H_m in $\mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^4) = \mathcal{L}^2(\mathbb{R}^3) \otimes \mathbb{C}^4$ given by

$$(2.2) \quad H_m = \alpha \cdot D + m\beta, \quad m \geq 0,$$

where

$$(2.3) \quad D = \frac{1}{i} \nabla_x, \quad x \in \mathbb{R}^3,$$

and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is the triplet of 4×4 Dirac matrices

$$(2.4) \quad \alpha_j = \begin{pmatrix} O_2 & \sigma_j \\ \sigma_j & O_2 \end{pmatrix}, \quad j = 1, 2, 3.$$

Here O_2 is the 2×2 zero matrix and the 2×2 matrices σ_j (Pauli matrices) are given by

$$(2.5) \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$(2.6) \quad \beta = \begin{pmatrix} I_2 & O_2 \\ O_2 & -I_2 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The symbol $M_m(\xi)$ (corresponding to $M_0(\xi)$ above) is a 4×4 Hermitian matrix given by

$$(2.7) \quad M_m(\xi) = \alpha \cdot \xi + m\beta.$$

It is readily verified that we have the following equality of self-adjoint operators in $L^2(\mathbb{R}^3, \mathbb{C}^4)$,

$$(2.8) \quad (H_m)^2 = (-\Delta + m^2) \otimes I_4.$$

Claim 2.1. *The eigenvalues of $M_m(\xi)$ are given by $\lambda_{\pm}(\xi) = \pm\sqrt{|\xi|^2 + m^2}$, and are both of double multiplicity (except for $m = 0$ and $\xi = 0$).*

It follows that the homogeneous operator H_0 is uniformly propagative (Definition 5.3).

For any $\xi \in \mathbb{R}^3$ (assuming either $\xi \neq 0$ or $m \neq 0$) there exists a unitary matrix $U_m(\xi)$ such that

$$(2.9) \quad U_m(\xi)^* M_m(\xi) U_m(\xi) = \begin{pmatrix} \lambda_+(\xi) I_2 & O_2 \\ O_2 & \lambda_-(\xi) I_2 \end{pmatrix}.$$

For $f \in \mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^4)$ we define the transformation

$$(2.10) \quad (\mathcal{G}_m f)(\xi) = U_m^*(\xi) \hat{f}(\xi), \quad \xi \in \mathbb{R}^3.$$

Then

$$\mathcal{G}_m : \mathcal{L}_x^2(\mathbb{R}^3, \mathbb{C}^4) \rightarrow \mathcal{L}_\xi^2(\mathbb{R}^3, \mathbb{C}^4) \quad ,$$

is unitary and diagonalizes H_m in the sense that $\mathcal{G}_m H_m \mathcal{G}_m^{-1}$ in $\mathcal{L}_\xi^2(\mathbb{R}^3, \mathbb{C}^4)$ is given by the diagonal multiplication operator

$$(2.11) \quad (\mathcal{G}_m H_m \mathcal{G}_m^{-1} \hat{f})(\xi) = \begin{pmatrix} \lambda_+(\xi) I_2 & O_2 \\ O_2 & \lambda_-(\xi) I_2 \end{pmatrix} \hat{f}(\xi).$$

In fact, in the physical literature this transformation is known as the **Foldy-Wouthuysen-Tani transformation** [27]. The transformation is explicitly presented (as can be easily verified) by [45, Section 1.4] and [2, Section 2.1]:

$$(2.12) \quad U_m(\xi) = \frac{1}{\sqrt{2}\sqrt{\lambda_+(\xi)^2 + m\lambda_+(\xi)}} \left\{ (\lambda_+(\xi) + m)I_4 + \beta(\alpha \cdot \xi) \right\} \\ = \frac{1}{\sqrt{2}\sqrt{\lambda_+(\xi)^2 + m\lambda_+(\xi)}} \begin{pmatrix} \lambda_+(\xi) + m & 0 & -\xi_3 & -\xi_1 + i\xi_2 \\ 0 & \lambda_+(\xi) + m & -\xi_1 - i\xi_2 & \xi_3 \\ \xi_3 & \xi_1 - i\xi_2 & \lambda_+(\xi) + m & 0 \\ \xi_1 + i\xi_2 & -\xi_3 & 0 & \lambda_+(\xi) + m \end{pmatrix}$$

Clearly, the columns of this matrix are the eigenvectors of the symbol matrix $M_m(\xi)$.

We shall need this transformation when studying the spectral structure of the Dirac operator in Subsection 4.1.

Remark 2.2. *Another representation of (2.12) is given by (see [3])*

$$(2.13) \quad U_m(\xi) = \exp \left\{ -\beta(\alpha \cdot \xi) \theta_m(|\xi|) \right\},$$

$$\theta_m(t) = (2t)^{-1} \arctan(m^{-1}t), \quad m, t > 0.$$

Observe that due to the double multiplicity of $\lambda_{\pm}(\xi)$ the diagonalizing matrix $U_m(\xi)$ is not unique.

2.2. EXAMPLE: THE FREE MAXWELL OPERATOR. Consider a pair of three-dimensional vector functions E (the electric field) and B (the magnetic field). We shall denote by $\begin{pmatrix} E \\ B \end{pmatrix}$ the six-component (column) vector that consists of the vertical arrangement of E, B .

The free (vacuum) Maxwell operator L_{\maxwell} is a 6×6 self-adjoint operator acting on the combined vector $\begin{pmatrix} E \\ B \end{pmatrix}$ in $\mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^6) = \mathcal{L}^2(\mathbb{R}^3) \otimes \mathbb{C}^6$,

$$(2.14) \quad L_{\maxwell} \begin{pmatrix} E \\ B \end{pmatrix} = \frac{1}{i} \begin{pmatrix} O_3 & -\text{curl} \\ \text{curl} & O_3 \end{pmatrix} \begin{pmatrix} E \\ B \end{pmatrix} = \begin{pmatrix} O_3 & -D \times \\ D \times & O_3 \end{pmatrix} \begin{pmatrix} E \\ B \end{pmatrix},$$

where as above $D = (D_1, D_2, D_3) = \frac{1}{i} \nabla_x$, $x \in \mathbb{R}^3$, and O_3 is the 3×3 zero matrix.

Remark 2.3. *Recall that the electric (E) and magnetic (B) fields in vacuum are divergence-free fields, namely, there are no electric or magnetic charges. This restriction is imposed on the initial data and then the time-dependent equations ensure that the fields evolve as divergence-free fields.*

However, here we take the point-of-view that L_{\maxwell} acts on the entire space $\mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^6)$.

For the Maxwell operator L_{\maxwell} in (2.14), the matrix symbol $M_{\maxwell}(\xi)$ is readily seen to be of the form

$$(2.15) \quad M_{\maxwell}(\xi) = \begin{pmatrix} O_3 & -\gamma(\xi) \\ \gamma(\xi) & O_3 \end{pmatrix},$$

where, for $\xi = (\xi_1, \xi_2, \xi_3)$,

$$\gamma(\xi) = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}.$$

The eigenvalues of $M_{\maxwell}(\xi)$, $\xi \neq 0$, are

$$\lambda_+(\xi) = |\xi| > \lambda_0(\xi) = 0 > \lambda_-(\xi) = -|\xi|,$$

and the multiplicity of each of them is two. Since $\gamma(\xi)\xi = 0$, it follows that the kernel of $M_{maxwell}(\xi)$ is two-dimensional, with basis vectors

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}, \quad \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus the kernel is two-dimensional, independently of $\xi \in \mathbb{R}^3 \setminus \{0\}$. It follows that this operator is uniformly propagative (Definition 5.3 below).

In order to construct the full system of eigenvectors of $M_{maxwell}(\xi)$ we introduce (for any $\xi \neq 0$) the following vectors in \mathbb{C}^3 :

$$\mathbf{a}(\xi) = \begin{cases} \begin{pmatrix} -\xi_1\xi_2 \\ \xi_1^2 + \xi_3^2 \\ -\xi_2\xi_3 \end{pmatrix} \\ \begin{pmatrix} \xi_2 \\ 0 \\ \xi_2 \end{pmatrix} \end{cases}, \quad \mathbf{b}(\xi) = \begin{cases} \begin{pmatrix} -\xi_3|\xi| \\ 0 \\ \xi_1|\xi| \end{pmatrix}, \quad \xi_1^2 + \xi_3^2 > 0, \\ \begin{pmatrix} |\xi_2| \\ 0 \\ -|\xi_2| \end{pmatrix}, \quad \xi_1 = \xi_3 = 0 \end{cases}.$$

It is readily seen that the vectors $\mathbf{a}(\xi)$, $\mathbf{b}(\xi)$, ξ are mutually orthogonal in \mathbb{C}^3 . They also form a right-hand orthogonal system in \mathbb{R}^3 , and satisfy the relations

$$\gamma(\xi)\mathbf{a}(\xi) = |\xi|\mathbf{b}(\xi), \quad \gamma(\xi)\mathbf{b}(\xi) = -|\xi|\mathbf{a}(\xi).$$

In the following corollary, we write a (column) vector in \mathbb{C}^6 as a column with two (column) \mathbb{C}^3 components.

Corollary 2.4. *Consider the following three pairs of vectors in \mathbb{C}^6*

$$\Upsilon_+ = \left\{ \frac{1}{\sqrt{|\mathbf{a}(\xi)|^2 + |\mathbf{b}(\xi)|^2}} \begin{pmatrix} \mathbf{a}(\xi) \\ \mathbf{b}(\xi) \end{pmatrix}, \frac{1}{\sqrt{|\mathbf{a}(\xi)|^2 + |\mathbf{b}(\xi)|^2}} \begin{pmatrix} -\mathbf{b}(\xi) \\ \mathbf{a}(\xi) \end{pmatrix} \right\},$$

$$(2.16) \quad \Upsilon_0 = \left\{ \frac{1}{|\xi|} \begin{pmatrix} 0 \\ \xi \end{pmatrix}, \frac{1}{|\xi|} \begin{pmatrix} \xi \\ 0 \end{pmatrix} \right\},$$

$$\Upsilon_- = \left\{ \frac{1}{\sqrt{|\mathbf{a}(\xi)|^2 + |\mathbf{b}(\xi)|^2}} \begin{pmatrix} -\mathbf{a}(\xi) \\ \mathbf{b}(\xi) \end{pmatrix}, \frac{1}{\sqrt{|\mathbf{a}(\xi)|^2 + |\mathbf{b}(\xi)|^2}} \begin{pmatrix} \mathbf{b}(\xi) \\ \mathbf{a}(\xi) \end{pmatrix} \right\}.$$

Then the set of six vectors $\Upsilon_+ \cup \Upsilon_0 \cup \Upsilon_- \subseteq \mathbb{C}^6$ constitutes an orthonormal basis of eigenvectors of $M_{maxwell}(\xi)$.

The pairs Υ_{\pm} (resp. Υ_0) are eigenvectors associated with $\lambda_{\pm}(\xi)$ (resp. $\lambda_0(\xi)$).

Remark 2.5. *The double multiplicities of the eigenvalues implies that the basis vectors in each subspace are not uniquely determined. This explains the apparent asymmetry (with respect to (ξ_1, ξ_2, ξ_3)) of the eigenvectors in each of the sets*

$\Upsilon_{\pm}, \Upsilon_0$. The operator (and its symbol $M_{\maxwell}(\xi)$) is clearly symmetric with respect to orthogonal rotations in \mathbb{R}^3 .

Remark 2.6. (propagation modes) Let $\begin{pmatrix} E_0(x) \\ B_0(x) \end{pmatrix}$ be a vector function (with values in \mathbb{C}^6) whose Fourier transform $\begin{pmatrix} \widehat{E}_0(\xi) \\ \widehat{B}_0(\xi) \end{pmatrix} \in \text{span}\{\Upsilon_{\pm}\}$, for every $0 \neq \xi \in \mathbb{R}^3$. In other words, $\begin{pmatrix} \widehat{E}_0(\xi) \\ \widehat{B}_0(\xi) \end{pmatrix}$ is orthogonal to $\ker(M_{\maxwell}(\xi))$ for all $\xi \neq 0$. Then in particular

$$\langle \widehat{E}_0(\xi), \xi \rangle_{\mathbb{C}^3} = 0, \quad \langle \widehat{B}_0(\xi), \xi \rangle_{\mathbb{C}^3} = 0, \quad \xi \in \mathbb{R}^3.$$

The propagation of the initial data $\begin{pmatrix} E_0(x) \\ B_0(x) \end{pmatrix}$ by the time-dependent Maxwell system yields the solution $\begin{pmatrix} E(x, t) \\ B(x, t) \end{pmatrix}$. Both $E(x, t)$ and $B(x, t)$ are superpositions of the plane waves $e^{i(\langle x, \xi \rangle \pm t|\xi|)} \mathbf{a}(\xi)$ and $e^{i(\langle x, \xi \rangle \pm t|\xi|)} \mathbf{b}(\xi)$, each of which is transverse (in fact orthogonal) to the propagation directions ξ or $-\xi$.

The terminology of “TE-modes” (resp. “TM-modes”), introduced by Lord Rayleigh in 1897, is used to characterize such fields.

The behavior of these waves for large time is further studied below in Subsection 7.2.

The diagonalization procedure of the Dirac operator (see (2.11)) can now be repeated, with suitable modifications, for the Maxwell operator as follows.

We define a continuous map

$$\xi \mapsto V_0(\xi), \quad \xi \in \mathbb{R}^3 \setminus \{0\},$$

where $V_0(\xi)$ is unitary for all $\xi \in \mathbb{R}^3 \setminus \{0\}$, so that

$$(2.17) \quad V_0(\xi)^* M_{\maxwell}(\xi) V_0(\xi) = \begin{pmatrix} \lambda_+(\xi) I_2 & O_2 & O_2 \\ O_2 & O_2 & O_2 \\ O_2 & O_2 & \lambda_-(\xi) I_2 \end{pmatrix}.$$

We can clearly assume that $V_0(\xi)$ is homogeneous of order zero; $V_0(\beta\xi) = V_0(\xi)$, $\beta > 0$.

Remark 2.7. Continuing Remark 2.5, the fact that the eigenvalues are “separated” and are of constant multiplicity implies [32, Chapter II.1.4] that the projections on the eigenspaces are continuous (and indeed real-analytic), hence this is true also for the map

$$\xi \mapsto V_0(\xi), \quad \xi \neq 0.$$

Remark that the choice of the unitary matrix is not unique. In particular, taking the matrix whose columns are the six vectors $\Upsilon_+ \cup \Upsilon_0 \cup \Upsilon_-$, yields a unitary matrix that diagonalizes $M_{\maxwell}(\xi)$, but is not continuous (as a function of ξ).

For $f \in \mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^6)$ we define the transformation

$$(2.18) \quad (\mathcal{T}_0 f)(\xi) = V_0^*(\xi) \widehat{f}(\xi), \quad \xi \in \mathbb{R}^3 \setminus \{0\}.$$

Then

$$\mathcal{T}_0 : \mathcal{L}_x^2(\mathbb{R}^3, \mathbb{C}^6) \rightarrow \mathcal{L}_\xi^2(\mathbb{R}^3, \mathbb{C}^6),$$

is unitary and diagonalizes L_{\maxwell} in the sense that

$$(2.19) \quad \left(\mathcal{T}_0 L_{\maxwell} \mathcal{T}_0^{-1} \widehat{f} \right)(\xi) = \begin{pmatrix} \lambda_+(\xi) I_2 & O_2 & O_2 \\ O_2 & O_2 & O_2 \\ O_2 & O_2 & \lambda_-(\xi) I_2 \end{pmatrix} \widehat{f}(\xi), \quad \xi \in \mathbb{R}^3,$$

is a multiplication by a diagonal matrix.

Remark 2.8. *Note that the diagonalization equation applies only to f in the domain of L_{\maxwell} . This will be discussed below, in Subsection 4.2.*

3. WEIGHTED SOBOLEV SPACES and BASIC NOTATION

Here we introduce the weighted Sobolev spaces which play a basic role in our treatment. We shall only need these spaces in the framework of \mathcal{L}^2 .

For $s \in \mathbb{R}$ and p a nonnegative integer we define:

$$(3.1) \quad \mathcal{L}^{2,s}(\mathbb{R}^n) := \{u(x) \mid \|u\|_{0,s}^2 = \int_{\mathbb{R}^n} (1 + |x|^2)^s |u(x)|^2 dx < \infty\},$$

$$(3.2) \quad \mathcal{H}^{p,s}(\mathbb{R}^n) := \{u(x) \mid \nabla^\alpha u \in \mathcal{L}^{2,s}, \quad |\alpha| \leq p, \quad \|u\|_{p,s}^2 = \sum_{|\alpha| \leq p} \|\nabla^\alpha u\|_{0,s}^2\}.$$

The scalar product in $\mathcal{L}^{2,s}(\mathbb{R}^n)$ is

$$(3.3) \quad (u, v)_{0,s} = \int_{\mathbb{R}^n} (1 + |x|^2)^s u(x) \overline{v(x)} dx.$$

We write \mathcal{L}^2 for $\mathcal{L}^{2,0}$ and $\|u\|_0 = \|u\|_{0,0}$. We also write \mathcal{H}^p for $\mathcal{H}^{p,0}$ and, when needed for clarity, $\|u\|_{\mathcal{H}^p} = \|u\|_{p,0}$. The scalar product is then denoted by (\cdot, \cdot) . We do not distinguish in this scalar product between scalar and vector-valued functions; that will be clear from the context.

Let $K > 0$ be an integer. For functions valued in \mathbb{C}^K , if needed for clarity, we denote, as has already been done above for \mathcal{L}^2 ,

$$\mathcal{H}^{p,s}(\mathbb{R}^n, \mathbb{C}^K) = \mathcal{H}^{p,s}(\mathbb{R}^n) \otimes \mathbb{C}^K.$$

For negative (integer) indices $-p$ we denote by $\{\mathcal{H}^{-p,s}, \|\cdot\|_{-p,s}\}$ the dual space of $\mathcal{H}^{p,-s}$. In particular, observe that any function $f \in \mathcal{H}^{-1,s}$ can be represented (not uniquely) as

$$(3.4) \quad f = f_0 + \sum_{k=1}^n i^{-1} \frac{\partial}{\partial x_k} f_k, \quad f_k \in \mathcal{L}^{2,s}, \quad 0 \leq k \leq n.$$

On the side of the (Fourier) transformed functions we shall need Sobolev spaces of any real order. The Fourier transform (2.1) carries weighted- L^2 spaces to Sobolev spaces (and vice-versa).

We let $\mathcal{H}^\theta = \mathcal{H}^\theta(\mathbb{R}^n)$, $\theta \in \mathbb{R}$ be the Sobolev space (based on $L^2(\mathbb{R}^n)$) of order θ of functions obtained as Fourier transforms, namely,

$$(3.5) \quad \mathcal{H}^\theta = \{\hat{u} \mid u \in \mathcal{L}^{2,\theta}(\mathbb{R}^n), \quad \|\hat{u}\|_{\mathcal{H}^\theta} = \|u\|_{0,\theta}\}.$$

Of course for an integer $\theta = p$ the definitions of \mathcal{H}^p and \mathcal{H}^θ are consistent, noting that the latter is used in the Fourier ξ -space.

Remark that the more general (weighted) spaces $\mathcal{H}^{\theta,s}$ can be defined, for example, by interpolating between $\mathcal{H}^{p,s}$ with integer values of p , but we shall make no use of such (weighted) spaces.

In our study we rely heavily on the trace lemma for functions in $\mathcal{H}^\theta(\mathbb{R}^n)$ [5, Proposition 6.3],

Lemma 3.1. *Let $\widehat{h} \in \mathcal{H}^\theta(\mathbb{R}^n)$, $n \geq 3$, $\frac{1}{2} < \theta < \frac{3}{2}$. Denote the sphere of radius r by*

$$S_r = \{\xi \in \mathbb{R}^n / |\xi| = r\}.$$

Then

$$(3.6) \quad \int_{S_r} |\widehat{h}|^2 d\Sigma_r \leq C \min(1, r^{2\theta-1}) \|\widehat{h}\|_{\mathcal{H}^\theta}^2, \quad r > 0,$$

where $C > 0$ is independent of \widehat{h} , r , and $d\Sigma_r$ is the Lebesgue surface measure on S_r .

Furthermore, the family of trace maps $\{\Phi_r : \mathcal{H}^\theta(\mathbb{R}^n) \rightarrow \mathcal{L}^2(S_1), r \in \mathbb{R}_+\}$ given by

$$\Phi_r \widehat{h}(\omega) = r^{\frac{n-1}{2}} \widehat{h}(r\omega), \quad \omega \in S_1, \quad r \in \mathbb{R}_+,$$

is locally Hölder continuous, in the following sense.

For a closed interval $[a, b] \subseteq \mathbb{R}_+$ there exist constants $C > 0$ and $\alpha \in (0, 1)$ such that

$$(3.7) \quad \int_{S_1} \left| \Phi_{r_2} \widehat{h}(\omega) - \Phi_{r_1} \widehat{h}(\omega) \right|^2 d\Sigma_1 \leq C \|\widehat{h}\|_{\mathcal{H}^\theta}^2 |r_2 - r_1|^\alpha, \quad r_1, r_2 \in [a, b].$$

Notation. For any two Banach spaces we use $B(X, Y)$ to denote the space of linear bounded operators from X to Y , equipped with the uniform operator topology.

4. SPECTRAL STRUCTURE of the UNPERTURBED DIRAC and MAXWELL OPERATORS

In this section, we study in detail the spectral properties of the free Dirac and Maxwell operators, as introduced in Section 2. This will provide a prelude to the general case that we defer to Section 5.

4.1. THE FREE DIRAC OPERATOR. We first consider the spectral density of the free Dirac operator, with suitable estimates in weighted Sobolev spaces. These estimates allow us to derive the Limiting Absorption Principle for this operator (Theorem 4.2 below).

Recall the definition (2.10) of the transformation \mathcal{G}_m .

Observe that in view of Equation (2.11) the operator $\mathcal{G}_m H_m \mathcal{G}_m^{-1}$ is reduced by the two (complementary) orthogonal subspaces $X_\pm \subseteq \mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^4)$ given by

$$(4.1) \quad \begin{aligned} X_+ &= \{f \in \mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^4) / \text{the two last components of } \mathcal{G}_m f(\xi) \text{ vanish}\}, \\ X_- &= \{f \in \mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^4) / \text{the two first components of } \mathcal{G}_m f(\xi) \text{ vanish}\}. \end{aligned}$$

In order to determine the domain of the self-adjoint operator H_m we repeat the discussion in [32, Section V.5.4].

From (2.9) we obtain the coercivity property (in each reducing subspace)

$$(4.2) \quad \begin{aligned} | \langle M_m(\xi) \widehat{f}(\xi), \widehat{f}(\xi) \rangle_{\mathbb{C}^4} | &= | \langle U_m^*(\xi) M_m(\xi) U_m(\xi) U_m^*(\xi) \widehat{f}(\xi), U_m^*(\xi) \widehat{f}(\xi) \rangle_{\mathbb{C}^4} | \\ &\geq |\xi| |\widehat{f}(\xi)|^2, \quad f \in X_\pm, \quad \xi \in \mathbb{R}^3. \end{aligned}$$

It follows that the domain of H_m in each reducing subspace, hence in the full space $\mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^4)$, is

$$(4.3) \quad \text{Dom}(H_m) = \mathcal{H}^1(\mathbb{R}^3, \mathbb{C}^4),$$

and its spectrum (that is absolutely continuous) is

$$(4.4) \quad \text{spec}(H_m) = \mathbb{R} \setminus (-m, m)$$

(it is of course \mathbb{R} if $m = 0$).

Recall that by Equation (2.11) the operator H_m can be diagonalized with

$$\lambda_{\pm}(r) = \pm \sqrt{r^2 + m^2}.$$

Let $E_m(\lambda)$ be the spectral family associated with H_m .

As is customary we use χ_B as the indicator function for a set $B \subseteq \mathbb{R}^3$, namely, $\chi_B(\xi) = 1$ (resp. $\chi_B(\xi) = 0$) if $\xi \in B$ (resp. $\xi \notin B$).

It is easily seen that if we confine $f \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^4)$ then for $\lambda > m$ we have

$$(4.5) \quad (E_m(\lambda)f, f) = \left(\begin{pmatrix} \chi_{\lambda_+(\xi) \leq \lambda} I_2 & O_2 \\ O_2 & I_2 \end{pmatrix} \mathcal{G}_m f, \mathcal{G}_m f \right),$$

where the right-hand side is the scalar product in $\mathcal{L}_\xi^2(\mathbb{R}^3, \mathbb{C}^4)$.

Differentiating the last equality (assuming f, g to be sufficiently regular), we get (with $d\Sigma_r$ being the Lebesgue surface measure on the sphere of radius $r > 0$),

$$(4.6) \quad \frac{d}{d\lambda}(E_m(\lambda)f, f) = \frac{\lambda}{\sqrt{\lambda^2 - m^2}} \int_{|\xi|=\sqrt{\lambda^2 - m^2}} |(\mathcal{G}_m f)_+(\xi)|^2 d\Sigma_{\sqrt{\lambda^2 - m^2}},$$

where $(\mathcal{G}_m f)_+$ is a 2-vector consisting of the first two components of $\mathcal{G}_m f$.

An analogous equation is clearly valid in the case $\lambda < -m$.

Since

$$|(\mathcal{G}_m f)_+(\xi)| \leq |(\mathcal{G}_m f)(\xi)| \leq |\widehat{f}(\xi)|,$$

we conclude from Lemma 3.1 and the definition (3.5) of the space \mathcal{H}^s that, for any $|\lambda| > m$ and $\frac{1}{2} < s < \frac{3}{2}$, there exists an operator

$$A_m(\lambda) \in B\left(\mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{L}^{2,-s}(\mathbb{R}^3, \mathbb{C}^4)\right),$$

such that

$$(4.7) \quad \begin{aligned} \langle A_m(\lambda)f, f \rangle &= \frac{d}{d\lambda}(E_m(\lambda)f, f) \\ &\leq C \min\left(\frac{|\lambda|}{\sqrt{\lambda^2 - m^2}}, |\lambda|(\lambda^2 - m^2)^{s-1}\right) \|\widehat{f}\|_{\mathcal{H}^s}^2 \\ &= C \min\left(\frac{|\lambda|}{\sqrt{\lambda^2 - m^2}}, |\lambda|(\lambda^2 - m^2)^{s-1}\right) \|\widehat{f}\|_{\mathcal{H}^s}^2 \end{aligned}$$

where \langle, \rangle is the $(\mathcal{L}^{2,-s}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^4))$ pairing.

Proposition 4.1. (1) Let $s > \frac{1}{2}$. Then the weak derivative $A_m(\lambda) = \frac{d}{d\lambda}(E_m(\lambda))$ is locally bounded and locally Hölder continuous for $|\lambda| > m$, with respect to the uniform operator topology of $B(\mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{L}^{2,-s}(\mathbb{R}^3, \mathbb{C}^4))$.

(2) Let $s > 1$. Then the weak derivative $A_m(\lambda) = \frac{d}{d\lambda}(E_m(\lambda))$ is uniformly bounded and uniformly Hölder continuous for $\lambda \in \mathbb{R}$, with respect to the uniform operator topology of $B(\mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{L}^{2,-s}(\mathbb{R}^3, \mathbb{C}^4))$.

Proof. Note that the topology of $B(\mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{L}^{2,-s}(\mathbb{R}^3, \mathbb{C}^4))$ becomes weaker as s grows, so without loss of generality we can assume $s < \frac{3}{2}$, so the estimate (4.7) can be used.

Consider the expression for the spectral derivative, Equation (4.6). Using the expression (2.10), we can rewrite it explicitly as

$$(4.8) \quad \langle A_m(\lambda)f, f \rangle = \frac{\lambda}{\sqrt{\lambda^2 - m^2}} \int_{|\xi|=\sqrt{\lambda^2 - m^2}} \left| [U_m^*(\xi)\hat{f}(\xi)]_+ \right|^2 d\Sigma_{\sqrt{\lambda^2 - m^2}}.$$

We now estimate the right-hand side of Equation (4.8) in $\mathcal{H}^{\bar{s}}$ with $\bar{s} > 2$. By the Sobolev embedding theorem,

$$(4.9) \quad |\hat{f}(\xi)| \leq C\|\hat{f}\|_{\mathcal{H}^{\bar{s}}}, \quad \xi \in \mathbb{R}^3.$$

Thus

$$\left| \int_{|\xi|=\sqrt{\lambda^2 - m^2}} \left| [U_m^*(\xi)\hat{f}(\xi)]_+ \right|^2 d\Sigma_{\sqrt{\lambda^2 - m^2}} \right| \leq C(\lambda^2 - m^2)\|f\|_{0,\bar{s}}^2, \quad \bar{s} > 2,$$

and from (4.8) we obtain in this case

$$(4.10) \quad \langle A_m(\lambda)f, f \rangle \leq C\lambda(\lambda^2 - m^2)^{\frac{1}{2}}\|f\|_{0,\bar{s}}^2, \quad \bar{s} > 2.$$

Furthermore, using the explicit form (2.12) of $U_m(\xi)$ and the Sobolev embedding theorem we have, in addition to (4.9), also

$$|\nabla_\xi [U_m^*(\xi)\hat{f}(\xi)]| \leq C\|\hat{f}\|_{\mathcal{H}^{\bar{s}}}, \quad \xi \in \mathbb{R}^3, \quad \bar{s} > 3.$$

Rewriting (4.8) in the form

$$(4.11) \quad \langle A_m(\lambda)f, f \rangle = \frac{\lambda}{\sqrt{\lambda^2 - m^2}} \int_{|\omega|=1} \left| [U_m^*(\sqrt{\lambda^2 - m^2}\omega)\hat{f}(\sqrt{\lambda^2 - m^2}\omega)]_+ \right|^2 (\lambda^2 - m^2) d\Sigma_1,$$

we can differentiate to obtain, for $|\lambda| > m$,

$$(4.12) \quad \left| \frac{d}{d\lambda} \langle A_m(\lambda)f, f \rangle \right| \leq C(\lambda^2 - m^2)^{-\frac{1}{2}}\|f\|_{0,\bar{s}}^2, \quad \bar{s} > 3.$$

Since the right-hand side is uniformly locally integrable in $\lambda \in \mathbb{R} \setminus (-m, m)$, we conclude that the operator-valued function

$$A_m(\lambda) \in B(\mathcal{L}^{2,\bar{s}}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{L}^{2,-\bar{s}}(\mathbb{R}^3, \mathbb{C}^4)), \quad |\lambda| \geq m, \quad \bar{s} > 3,$$

is uniformly Hölder continuous (and vanishes at $|\lambda| = m$).

Interpolating this estimate with the boundedness estimates (4.7) we conclude that

- For $s > \frac{1}{2}$ the operator-valued function

$$A_m(\lambda) \in B(\mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{L}^{2,-s}(\mathbb{R}^3, \mathbb{C}^4)), \quad |\lambda| > m,$$

is locally bounded and locally Hölder continuous.

- For $s > 1$ the operator-valued function

$$A_m(\lambda) \in B(\mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{L}^{2,-s}(\mathbb{R}^3, \mathbb{C}^4)), \quad |\lambda| \geq m,$$

is uniformly bounded and uniformly Hölder continuous in $\lambda \in \mathbb{R}$ (and vanishes for $\lambda \in [-m, m]$).

This concludes the proof of the proposition. \square

We can state a slightly more general fact by taking the norms of f and g below in different weighted spaces. In fact, suppose that f, g are smooth and compactly supported. Since the bilinear form $\langle A_m(\lambda) \cdot, \cdot \rangle$ is nonnegative, applying the Cauchy-Schwarz inequality yields, for any $|\lambda| > m$ and $s, l > \frac{1}{2}$,

$$\begin{aligned}
 (4.13) \quad \left| \frac{d}{d\lambda} (E_m(\lambda)f, g) \right| &= \left| \langle A_m(\lambda)f, g \rangle \right| \leq \langle A_m(\lambda)f, f \rangle^{\frac{1}{2}} \cdot \langle A_m(\lambda)g, g \rangle^{\frac{1}{2}} \\
 &\leq C \min \left(\frac{|\lambda|}{\sqrt{\lambda^2 - m^2}}, |\lambda|(\lambda^2 - m^2)^{\frac{s+l}{2}-1} \right) \|\widehat{f}\|_{\mathcal{H}^s} \|\widehat{g}\|_{\mathcal{H}^l} \\
 &= C \min \left(\frac{|\lambda|}{\sqrt{\lambda^2 - m^2}}, |\lambda|(\lambda^2 - m^2)^{\frac{s+l}{2}-1} \right) \|f\|_{0,s} \|g\|_{0,l}.
 \end{aligned}$$

It is obvious that in the inequality above, the first \langle, \rangle is the $(\mathcal{L}^{2,-s}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^4))$ pairing, while the second is the $(\mathcal{L}^{2,-l}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{L}^{2,l}(\mathbb{R}^3, \mathbb{C}^4))$ pairing.

The general theory [5, Section 3] now yields the **Limiting Absorption Principle** (LAP) for the unperturbed Dirac operator.

Theorem 4.2. *Let $R_m(z) = (H_m - z)^{-1}$, $\text{Im} z \neq 0$. For any $s, l > \frac{1}{2}$ the limits*

$$(4.14) \quad R_m^\pm(\mu) = \lim_{\varepsilon \downarrow 0} R_m(\mu \pm i\varepsilon), \quad \mu \in \mathbb{R} \setminus [-m, m],$$

exist in the uniform operator topology of $B(\mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{H}^{1,-l}(\mathbb{R}^3, \mathbb{C}^4))$.

If $s, l > 1$ then the limits in (4.14) exist for all $\mu \in \mathbb{R}$, or, otherwise stated, they are continuous across the thresholds at $\mu = \pm m$.

Furthermore, in both cases the limit functions $R_m^\pm(\mu)$ are locally bounded and locally Hölder continuous (in their respective domains) with respect to the uniform operator topology.

Proof. As already pointed out, the properties of $A_m(\lambda)$, as given in Proposition 4.1, enable us to invoke the general theory and obtain the result in the operator setting of $B(\mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{L}^{2,-l}(\mathbb{R}^3, \mathbb{C}^4))$.

In order to complete the proof we need to show that it is possible to replace $\mathcal{L}^{2,-l}(\mathbb{R}^3, \mathbb{C}^4)$ by $\mathcal{H}^{1,-l}(\mathbb{R}^3, \mathbb{C}^4)$.

Take $f \in \mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^4)$, so that by the already established result, the limit

$$R_m^\pm(\mu)f = \lim_{\varepsilon \downarrow 0} R_m(\mu \pm i\varepsilon)f, \quad \mu \in \mathbb{R} \setminus [-m, m],$$

exists in $\mathcal{L}^{2,-l}(\mathbb{R}^3, \mathbb{C}^4)$.

We have

$$(H_m - \mu)R_m(\mu \pm i\varepsilon)f = f \pm i\varepsilon R_m(\mu \pm i\varepsilon)f,$$

and since $\{R_m(\mu \pm i\varepsilon)f, 0 < \varepsilon < 1\}$ is uniformly bounded in $\mathcal{L}^{2,-l}(\mathbb{R}^3, \mathbb{C}^4)$, we obtain the limit (in this space)

$$\lim_{\varepsilon \downarrow 0} H_m R_m(\mu \pm i\varepsilon)f = f + \mu R_m^\pm(\mu)f.$$

Note that H_m is densely defined and closable in $\mathcal{L}^{2,-l}(\mathbb{R}^3, \mathbb{C}^4)$ and in fact, in view of the coercivity (4.2) its graph norm in this space is equivalent to the norm of $\mathcal{H}^{1,-l}(\mathbb{R}^3, \mathbb{C}^4)$.

Retaining the same notation for its closure, we get

$$(4.15) \quad H_m R_m^\pm(\mu) f = f + \mu R_m^\pm(\mu) f,$$

so that indeed

$$R_m^\pm(\mu) f \in \mathcal{H}^{1,-l}(\mathbb{R}^3, \mathbb{C}^4).$$

□

Remark 4.3. • *The first part of Theorem 4.2 (namely, $s, l > \frac{1}{2}$ and avoiding the thresholds) was obtained in [3, 53]. Both papers made use of the LAP for the Schrödinger operator, by way of formula (2.8).*

- *Note that for the second part of the theorem it suffices to assume $s, l > \frac{1}{2}$ with $s + l > 2$.*
- *In Proposition 4.5 below we give a somewhat different argument for the proof of (4.15), based on the fact that H_m is a constant coefficient operator.*

We shall now extend this theorem to more general function spaces. We take $s, l > 1$.

Let $g \in \mathcal{H}^{1,l}(\mathbb{R}^3, \mathbb{C}^4)$, $f \in \mathcal{H}^{-1,s}(\mathbb{R}^3, \mathbb{C}^4)$, where f has a representation of the form (3.4), with $f_k \in \mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^4)$, $0 \leq k \leq 3$.

Equation (4.6) can be extended (at least formally) to yield

$$(4.16) \quad \begin{aligned} & \langle A_m(\lambda)[f_0 + i^{-1} \sum_{k=1}^3 \frac{\partial}{\partial x_k} f_k], g \rangle \\ &= \frac{\lambda}{\sqrt{\lambda^2 - m^2}} \int_{|\xi|^2 = \lambda^2 - m^2} \langle (\mathcal{G}_m \widehat{f}_0)_+(\xi) + \sum_{k=1}^3 \xi_k (\mathcal{G}_m \widehat{f}_k)_+(\xi), (\mathcal{G}_m \widehat{g})_+(\xi) \rangle_{\mathbb{C}^2} d\Sigma_{\sqrt{\lambda^2 - m^2}}, \\ & \quad f \in \mathcal{H}^{-1,s}, g \in \mathcal{H}^{1,l}. \end{aligned}$$

Observe that this definition makes good sense even though the representation (3.4) is not unique, since

$$f = f_0 + \sum_{k=1}^3 i^{-1} \frac{\partial}{\partial x_k} f_k = \tilde{f}_0 + \sum_{k=1}^3 i^{-1} \frac{\partial}{\partial x_k} \tilde{f}_k,$$

implies

$$\widehat{f}_0(\xi) + \sum_{k=1}^3 \xi_k \widehat{f}_k(\xi) = \widehat{\tilde{f}}_0(\xi) + \sum_{k=1}^3 \xi_k \widehat{\tilde{f}}_k(\xi)$$

(as tempered distributions).

Proposition 4.4. *Equation (4.16) can indeed be used to define an operator (for which we retain the same notation)*

$$A_m(\lambda) \in B(\mathcal{H}^{-1,s}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{H}^{-1,-l}(\mathbb{R}^3, \mathbb{C}^4)).$$

In this setting \langle, \rangle is the $(\mathcal{H}^{-1,-l}, \mathcal{H}^{1,l})$ pairing and $|\lambda| > m$.

Proof. To estimate the operator-norm of $A_m(\lambda)$ as given in (4.16) we use, for $1 \leq k \leq 3$, the estimate (4.13), in the form

$$\begin{aligned}
| \langle A_m(\lambda) \frac{\partial}{\partial x_k} f_k, g \rangle | &= \frac{|\lambda|}{\sqrt{\lambda^2 - m^2}} \left| \int_{|\xi|^2 = \lambda^2 - m^2} \langle (\mathcal{G}_m \widehat{f_k})_+(\xi), \xi_k (\mathcal{G}_m \widehat{g})_+(\xi) \rangle_{\mathbb{C}^2} d\Sigma_{\sqrt{\lambda^2 - m^2}} \right| \\
&\leq C \min \left(\frac{|\lambda|}{\sqrt{\lambda^2 - m^2}}, |\lambda|(\lambda^2 - m^2)^{\frac{s+l}{2}-1} \right) \|\widehat{f_k}\|_{\mathcal{H}^s} \|\widehat{\xi_k g}\|_{\mathcal{H}^l},
\end{aligned}$$

so from (4.16) we obtain

$$\begin{aligned}
(4.17) \quad &| \langle A_m(\lambda) f, g \rangle | \\
&\leq C \min \left(\frac{|\lambda|}{\sqrt{\lambda^2 - m^2}}, |\lambda|(\lambda^2 - m^2)^{\frac{s+l}{2}-1} \right) \|f\|_{-1,s} \|g\|_{1,l}, \quad f \in \mathcal{H}^{-1,s}, g \in \mathcal{H}^{1,l}, s, l > 1.
\end{aligned}$$

□

Theorem 4.2 can now be enhanced to yield

Proposition 4.5. *The operator-valued function $R_m(z)$ is well-defined (and analytic) for nonreal z in the following functional setting.*

$$(4.18) \quad z \rightarrow R_m(z) \in B(\mathcal{H}^{-1,s}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{L}^{2,-l}(\mathbb{R}^3, \mathbb{C}^4)).$$

where $s, l > 1$.

Furthermore, it can be extended continuously from \mathbb{C}^\pm to $\overline{\mathbb{C}^\pm}$, in this uniform operator topology. The limiting values (denoted again by $R_m^\pm(\lambda)$) are Hölder continuous in the same topology.

The extended function satisfies

$$(4.19) \quad (H_m - z)R_m(z)f = f, \quad f \in \mathcal{H}^{-1,s}(\mathbb{R}^3, \mathbb{C}^4), \quad z \in \overline{\mathbb{C}^\pm},$$

where for $z = \lambda \in \mathbb{R}$, $R_m(z) = R_m^\pm(\lambda)$.

Proof. By the estimate (4.17), we get readily $R_m(z) \in B(\mathcal{H}^{-1,s}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{H}^{-1,-l}(\mathbb{R}^3, \mathbb{C}^4))$ if $\text{Im } z \neq 0$, as well as the analyticity of the map $z \mapsto R_m(z)$, $\text{Im } z \neq 0$. Furthermore, the extension to $\text{Im } z = 0$ is carried out as in [4, Section 4].

Equation (4.19) is obvious if $\text{Im } z \neq 0$ and $f \in \mathcal{L}^{2,s}$. By the density of $\mathcal{L}^{2,s}$ in $\mathcal{H}^{-1,s}$, the continuity of $R_m(z)$ on $\mathcal{H}^{-1,s}$ and the continuity of $H_m - z$ (in the sense of distributions), we can extend it to all $f \in \mathcal{H}^{-1,s}$.

As $z \rightarrow \lambda \pm i \cdot 0$ we have $R_m(z)f \rightarrow R_m^\pm(\lambda)f$ in $\mathcal{H}^{-1,-l}$. Applying the (constant coefficient) operator $H_m - z$ yields, in the sense of distributions, $f = (H_m - z)R_m(z)f \rightarrow (H_m - \lambda)R_m^\pm(\lambda)f$ which establishes (4.19) also for $\text{Im } z = 0$.

Finally, the established continuity of $z \mapsto R_m(z) \in B(\mathcal{H}^{-1,s}, \mathcal{H}^{-1,-l})$ up to the real boundary and Equation (4.19) imply the continuity of the map $z \mapsto H_m R_m(z) \in B(\mathcal{H}^{-1,s}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{H}^{-1,-l}(\mathbb{R}^3, \mathbb{C}^4))$.

The stronger continuity claim (4.18) follows since the norm of $\mathcal{L}^{2,-l}$ is equivalent to the graph-norm of H_m as a map of $\mathcal{H}^{-1,-l}$ to itself. □

Remark 4.6. *Note that we could actually take $s, l > \frac{1}{2}$ with $s + l > 2$. This is identical to Proposition 2.4 in [12], except that here we obtain the Hölder continuity of the limiting values.*

4.2. THE FREE MAXWELL OPERATOR. In Subsection 2.2 we introduced the Maxwell operator $L_{maxwell}$. It is a constant coefficient differential operator and its symbol $M_{maxwell}(\xi)$ is Hermitian. Thus, it can be realized as a self-adjoint operator in $\mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^6)$, for which we retain the same notation.

The spectrum of the operator is readily seen to be

$$(4.20) \quad spec(L_{maxwell}) = \mathbb{R}.$$

The spectrum is absolutely continuous except for the point $\lambda = 0$. In view of Corollary 2.4 the eigenspace associated with the zero eigenvalue is given by

$$(4.21) \quad ker(L_{maxwell}) = \left\{ f \in \mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^6) / \widehat{f}(\xi) \in span\{\Upsilon_0\}, 0 \neq \xi \in \mathbb{R}^3 \right\}.$$

We now consider the spectral density and the Limiting Absorption Principle for the Maxwell operator, in the setting of weighted Sobolev spaces. The treatment is quite analogous to that of the Dirac operator and we discuss it briefly, focusing on the aspects of difference between the two cases. As in the Dirac operator case, we obtain detailed estimates at the threshold energy $\lambda = 0$. Such estimates are needed for global spacetime estimates (Subsection 7.2). Note that the Maxwell operator is not elliptic (nor bounded from below), so suitable care is needed with respect to its domain of definition.

In addition to the kernel expressed in (4.21) we define two other (complementary) subspaces, as follows.

$$(4.22) \quad X_{\pm} = \left\{ f \in \mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^6) / \widehat{f}(\xi) \in span\{\Upsilon_{\pm}\}, 0 \neq \xi \in \mathbb{R}^3 \right\}.$$

Note (compare (4.1)) that these subspaces can also be expressed as

$$(4.23) \quad \begin{aligned} X_+ &= \{ f \in \mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^6) / \text{the four last components of } \mathcal{T}_0 f(\xi) \text{ vanish} \}, \\ X_- &= \{ f \in \mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^6) / \text{the four first components of } \mathcal{T}_0 f(\xi) \text{ vanish} \}, \end{aligned}$$

where the transformation \mathcal{T}_0 is defined in (2.18).

Observe that these subspaces are those containing the “TE, TM” modes (Remark 2.6).

These subspaces are clearly reducing for $L_{maxwell}$. From (2.17) we obtain the (partial) coercivity property

$$(4.24) \quad \begin{aligned} | \langle M_{maxwell}(\xi) \widehat{f}(\xi), \widehat{f}(\xi) \rangle_{\mathbb{C}^6} | &= | \langle M_{maxwell}(\xi) \mathcal{T}_0 f(\xi), \mathcal{T}_0 f(\xi) \rangle_{\mathbb{C}^6} | \\ &\geq |\xi| |\mathcal{T}_0 f(\xi)|^2 = |\xi| |\widehat{f}(\xi)|^2, \quad \xi \in \mathbb{R}^3, f \in X_{\pm}. \end{aligned}$$

These facts enable us to give an explicit expression for the domain of $L_{maxwell}$.

Claim 4.7. *The domain of $L_{maxwell}$, as a self-adjoint operator in $\mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^6)$, is given by*

$$(4.25) \quad Dom(L_{maxwell}) = ker(L_{maxwell}) \oplus (X_+ \cap \mathcal{H}^1(\mathbb{R}^3, \mathbb{C}^6)) \oplus (X_- \cap \mathcal{H}^1(\mathbb{R}^3, \mathbb{C}^6)).$$

The eigenvalues of the symbol $M_{maxwell}(\xi)$ are

$$\lambda_{\pm}(\xi) = \pm |\xi|.$$

Let $F(\lambda)$ be the spectral family associated with $L_{maxwell}$.

It is easily seen that if we confine $f \in C_0^\infty(\mathbb{R}^3, \mathbb{C}^6)$ then for $\lambda > 0$ we have

$$(4.26) \quad (F(\lambda)f, f) = \left(\begin{pmatrix} \chi_{\lambda_+(\xi) \leq \lambda} I_2 & O_2 & O_2 \\ O_2 & O_2 & O_2 \\ O_2 & O_2 & I_2 \end{pmatrix} \mathcal{T}_0 f, \mathcal{T}_0 f \right),$$

where the right-hand side is the scalar product in $\mathcal{L}_\xi^2(\mathbb{R}^3, \mathbb{C}^6)$.

Differentiating the last equality (assuming f, g to be sufficiently regular), we get

$$(4.27) \quad \frac{d}{d\lambda}(F(\lambda)f, f) = \int_{|\xi|=\lambda} |(\mathcal{T}_0 f)_+(\xi)|^2 d\Sigma_\lambda,$$

where $(\mathcal{T}_0 f)_+$ is a 2-vector consisting of the first two components of $\mathcal{T}_0 f$.

An analogous equation is clearly valid in the case $\lambda < 0$.

Since

$$|(\mathcal{T}_0 f)_+(\xi)| \leq |(\mathcal{T}_0 f)(\xi)| \leq |\widehat{f}(\xi)|,$$

we conclude from Lemma 3.1 that, for any $\lambda \in \mathbb{R} \setminus \{0\}$ and $s > \frac{1}{2}$,

$$(4.28) \quad \begin{aligned} \frac{d}{d\lambda}(F(\lambda)f, f) &\leq C \min(1, |\lambda|^{2s-1}) \|\widehat{f}\|_{\mathcal{H}^s}^2 \\ &= C \min(1, |\lambda|^{2s-1}) \|f\|_{0,s}^2. \end{aligned}$$

It follows that there exists a map

$$\tilde{A}(\lambda) \in B(\mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^6), \mathcal{L}^{2,-s}(\mathbb{R}^3, \mathbb{C}^6)),$$

so that

$$\frac{d}{d\lambda}(F(\lambda)f, f) = \langle \tilde{A}(\lambda)f, f \rangle,$$

where \langle, \rangle is the $(\mathcal{L}^{2,-s}(\mathbb{R}^3, \mathbb{C}^6), \mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^6))$ pairing.

Proposition 4.8. *Let $s > \frac{1}{2}$. Then the weak derivative $\tilde{A}(\lambda) = \frac{d}{d\lambda}(F(\lambda))$ is locally bounded and locally Hölder continuous for $\lambda \in \mathbb{R} \setminus \{0\}$, with respect to the uniform operator topology of $B(\mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^6), \mathcal{L}^{2,-s}(\mathbb{R}^3, \mathbb{C}^6))$.*

In fact, defining $\tilde{A}(0) = 0$, The function $\tilde{A}(\lambda)$ is uniformly bounded and uniformly Hölder continuous for $\lambda \in [-r, r]$ in the operator topology, for every $r > 0$.

Proof. The proof is quite parallel to that of Proposition 4.1, so we give a rather brief exposition here. We take $\lambda > 0$.

Consider the expression for the spectral derivative, Equation (4.27). Using (2.18), we can rewrite it explicitly as

$$(4.29) \quad \langle \tilde{A}(\lambda)f, f \rangle = \int_{|\xi|=\lambda} \left| [V_0^*(\xi)\widehat{f}(\xi)]_+ \right|^2 d\Sigma_\lambda.$$

The local boundedness of $\tilde{A}(\lambda) \in B(\mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^6), \mathcal{L}^{2,-s}(\mathbb{R}^3, \mathbb{C}^6))$, $\lambda \neq 0$, is immediate from (4.28).

In what follows $C > 0$ is a generic constant that does not depend on ξ, λ, f .

Next consider the right-hand side of Equation (4.29) and estimate it in $\mathcal{H}^{\bar{s}}$, $\bar{s} > 2$. In view of the Sobolev embedding theorem (see (3.5) for notation)

$$(4.30) \quad |\widehat{f}(\xi)| \leq C \|\widehat{f}\|_{\mathcal{H}^{\bar{s}}}, \quad \xi \in \mathbb{R}^3.$$

Thus

$$\left| \int_{|\xi|=\lambda} \left| [V_0^*(\xi)\widehat{f}(\xi)]_+ \right|^2 d\Sigma_{|\lambda|} \right| \leq C\lambda^2 \|f\|_{0,\bar{s}}^2, \quad \bar{s} > 2,$$

and from (4.29) we obtain in this case

$$(4.31) \quad \langle \tilde{A}(\lambda)f, f \rangle \leq C\lambda^2 \|f\|_{0,\bar{s}}^2, \quad \bar{s} > 2.$$

Using the homogeneity of $V_0(\xi)$ (of order zero) we have

$$|\nabla_\xi[V_0^*(\xi)]| \leq \frac{C}{|\xi|}, \quad \xi \in \mathbb{R}^3 \setminus \{0\}.$$

In conjunction with the Sobolev embedding theorem we have, in addition to (4.30),

$$|\nabla_\xi[V_0^*(\xi)\widehat{f}(\xi)]| \leq C \max(1, |\xi|^{-1}) \|\widehat{f}\|_{\mathcal{H}^{\bar{s}}}, \quad \xi \in \mathbb{R}^3 \setminus \{0\}, \quad \bar{s} > 3.$$

Rewriting (4.29) in the form

$$(4.32) \quad \langle \tilde{A}(\lambda)f, f \rangle = \lambda^2 \int_{|\omega|=1} \left| [V_0^*(\lambda\omega)\widehat{f}(\lambda\omega)]_+ \right|^2 d\Sigma_1,$$

we can differentiate to obtain, for $|\lambda| > 0$,

$$(4.33) \quad \left| \frac{d}{d\lambda} \langle \tilde{A}(\lambda)f, f \rangle \right| \leq C \max(\lambda, \lambda^2) \|f\|_{0,\bar{s}}^2, \quad \bar{s} > 3.$$

The right-hand side is locally integrable in $\lambda \in \mathbb{R}$, hence the operator-valued function

$$\tilde{A}(\lambda) \in B(\mathcal{L}^{2,\bar{s}}(\mathbb{R}^3, \mathbb{C}^6), \mathcal{L}^{2,-\bar{s}}(\mathbb{R}^3, \mathbb{C}^6)), \quad |\lambda| > 0, \quad \bar{s} > 3,$$

is locally Hölder continuous (and vanishes at $|\lambda| = 0$).

Interpolating between the local boundedness (4.28) for $s > \frac{1}{2}$ and the local Hölder continuity above (for $\bar{s} > 3$) we obtain the local Hölder continuity for any $s > \frac{1}{2}$. \square

Remark 4.9. *Since $\lambda = 0$ is an eigenvalue of L_{maxwell} there is no spectral derivative there. However, as seen from the estimate (4.28), the weak derivative vanishes as $\lambda \rightarrow 0$.*

We can state a slightly more general estimate than (4.28) by taking the norms of f and g below in different weighted spaces. In fact, suppose that f, g are smooth and compactly supported. Then, as in the derivation of the estimate (4.13), for $\lambda \neq 0$, and $s, l > \frac{1}{2}$,

$$(4.34) \quad \begin{aligned} \left| \frac{d}{d\lambda} (F(\lambda)f, g) \right| &\leq \langle \tilde{A}(\lambda)f, f \rangle^{\frac{1}{2}} \cdot \langle \tilde{A}(\lambda)g, g \rangle^{\frac{1}{2}} \\ &\leq C \min(1, |\lambda|^{s+l-1}) \|\widehat{f}\|_{\mathcal{H}^s} \|\widehat{g}\|_{\mathcal{H}^l} \\ &= C \min(1, |\lambda|^{s+l-1}) \|f\|_{0,s} \|g\|_{0,l}. \end{aligned}$$

It is obvious that in the inequality above, the first \langle, \rangle is the $(\mathcal{L}^{2,-s}(\mathbb{R}^3, \mathbb{C}^6), \mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^6))$ pairing, while the second is the $(\mathcal{L}^{2,-l}(\mathbb{R}^3, \mathbb{C}^6), \mathcal{L}^{2,l}(\mathbb{R}^3, \mathbb{C}^6))$ pairing.

The general theory [5, Section 3] now yields the **Limiting Absorption Principle** (LAP) for the unperturbed Maxwell operator.

Theorem 4.10. *Let $R_{\maxwell}(z) = (L_{\maxwell} - z)^{-1}$, $\text{Im} z \neq 0$. For any $s, l > \frac{1}{2}$ the limits*

$$(4.35) \quad R_{\maxwell}^{\pm}(\mu) = \lim_{\varepsilon \downarrow 0} R_{\maxwell}(\mu \pm i\varepsilon), \quad \mu \in \mathbb{R} \setminus \{0\},$$

exist in the uniform operator topology of $B(\mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^6), \mathcal{L}^{2,-l}(\mathbb{R}^3, \mathbb{C}^6))$.

The limits in (4.35) can be extended to $\mu = 0$, or, otherwise stated, they are continuous across the eigenvalue at $\mu = 0$.

Furthermore, the limit functions $R_{\maxwell}^{\pm}(\mu)$, $\mu \in \mathbb{R}$, are locally bounded and locally Hölder continuous with respect to the uniform operator topology.

Remark 4.11. *Continuing Remark 4.9, notice that the limiting values $R_{\maxwell}^{\pm}(0)$ are not, of course, limiting values of the resolvent $(L_{\maxwell} \pm i\varepsilon)^{-1}$ as $\varepsilon \rightarrow 0$. However, on the subspace (of “TE, TM” modes) orthogonal to the kernel we have the following corollary.*

Corollary 4.12. *Let $s, l > \frac{1}{2}$. Consider the operator $\mathcal{P}^{\perp} R_{\maxwell}(z) = \mathcal{P}^{\perp} (L_{\maxwell} - z)^{-1}$, $z = \mu \pm i\varepsilon$, where \mathcal{P} is the orthogonal projection on $\ker(L_{\maxwell})$ in $\mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^6)$. Then the limits*

$$(4.36) \quad \mathcal{P}^{\perp} R_{\maxwell}^{\pm}(\mu) = \lim_{\varepsilon \downarrow 0} \mathcal{P}^{\perp} R_{\maxwell}(\mu \pm i\varepsilon), \quad \mu \in \mathbb{R},$$

exist in the uniform operator topology of $B(\mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^6), \mathcal{L}^{2,-l}(\mathbb{R}^3, \mathbb{C}^6))$.

Furthermore, these limits are Hölder continuous in the same operator topology.

Proof. Indeed, the corollary follows directly from the last part of Proposition 4.8, since the weak derivative $\tilde{A}(\lambda) = \frac{d}{d\lambda}(F(\lambda))$ (extended by $\tilde{A}(0) = 0$) is uniformly bounded and uniformly Hölder continuous for $\lambda \in [-r, r]$ in the operator topology, for every $r > 0$. \square

As in the case of the Dirac operator, we can extend the theorem to more general function spaces. We continue to assume $s, l > \frac{1}{2}$.

Let $g \in \mathcal{H}^{1,l}(\mathbb{R}^3, \mathbb{C}^6)$, $f \in \mathcal{H}^{-1,s}(\mathbb{R}^3, \mathbb{C}^6)$, where f has a representation of the form (3.4), with $f_k \in \mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^6)$, $0 \leq k \leq 3$.

Equation (4.27) can be extended to yield an operator (for which we retain the same notation)

$$(4.37) \quad \tilde{A}(\lambda) \in B(\mathcal{H}^{-1,s}(\mathbb{R}^3, \mathbb{C}^6), \mathcal{H}^{-1,-l}(\mathbb{R}^3, \mathbb{C}^6)),$$

defined by (where now \langle, \rangle is used for the $(\mathcal{H}^{-1,-l}, \mathcal{H}^{1,l})$ pairing and we assume $\lambda > 0$),

$$\begin{aligned} & \langle \tilde{A}(\lambda)[f_0 + i^{-1} \sum_{k=1}^3 \frac{\partial}{\partial x_k} f_k], g \rangle \\ &= \int_{|\xi|=\lambda} \langle (\mathcal{T}_0 f_0)_1(\xi) + \sum_{k=1}^3 \xi_k (\mathcal{T}_0 f_k)_1(\xi), (\mathcal{T}_0 g)_1(\xi) \rangle_{\mathbb{C}^2} d\Sigma_{\lambda}, \quad f \in \mathcal{H}^{-1,s}, g \in \mathcal{H}^{1,l}, \end{aligned}$$

that can be rewritten as

$$\begin{aligned}
(4.38) \quad & \langle \tilde{A}(\lambda)[f_0 + i^{-1} \sum_{k=1}^3 \frac{\partial}{\partial x_k} f_k], g \rangle = \int_{|\xi|=\lambda} \langle (\mathcal{T}_0 f_0)_1(\xi), (\mathcal{T}_0 g)_1(\xi) \rangle_{\mathbb{C}^2} d\Sigma_\lambda \\
& + \sum_{k=1}^3 \int_{|\xi|=\lambda} \langle (\mathcal{T}_0 f_k)_1(\xi) \xi_k, (\mathcal{T}_0 g)_1(\xi) \rangle_{\mathbb{C}^2} d\Sigma_\lambda, \quad f \in \mathcal{H}^{-1,s}, g \in \mathcal{H}^{1,l}.
\end{aligned}$$

To estimate the operator-norm of $\tilde{A}(\lambda)$ in this setting we use (4.38) and the considerations leading to (4.34), for $1 \leq k \leq 3$

$$\begin{aligned}
& | \langle \tilde{A}(\lambda) \frac{\partial}{\partial x_k} f_k, g \rangle | \\
& \leq C \min(1, |\lambda|^{s+l-1}) \|f\|_{-1,s} \|g\|_{1,l}, \quad f \in \mathcal{H}^{-1,s}, \quad g \in \mathcal{H}^{1,l},
\end{aligned}$$

so that, instead of (4.34), we have

$$(4.39) \quad | \langle \tilde{A}(\lambda) f, g \rangle | \leq C \min(1, |\lambda|^{s+l-1}) \|f\|_{-1,s} \|g\|_{1,l}.$$

where $s, l > \frac{1}{2}$.

The following proposition is proved in the same way as Proposition 4.8.

Proposition 4.13. *Let $s, l > \frac{1}{2}$. Then the weak derivative $\tilde{A}(\lambda) = \frac{d}{d\lambda}(F(\lambda))$ is locally bounded and locally Hölder continuous for $\lambda \in \mathbb{R} \setminus \{0\}$, with respect to the uniform operator topology of $B(\mathcal{H}^{-1,s}(\mathbb{R}^3, \mathbb{C}^6), \mathcal{H}^{-1,-l}(\mathbb{R}^3, \mathbb{C}^6))$.*

When trying to establish the regularization property of the resolvent, in analogy to the Dirac case (Proposition 4.5) we need to take into account the fact that the kernel is nontrivial, so that regularization can only take part in the subspace $\mathcal{P}^\perp \mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^6)$, where \mathcal{P} is the orthogonal projection on $\ker(L_{\maxwell})$ in $\mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^6)$, as in Corollary 4.12.

Theorem 4.10 can now be enhanced to yield

Theorem 4.14. *The operator-valued function $R_{\maxwell}(z)$ is well-defined (and analytic) for nonreal z in the following functional setting.*

$$(4.40) \quad z \rightarrow \mathcal{P}^\perp R_{\maxwell}(z) \in B(\mathcal{H}^{-1,s}(\mathbb{R}^3, \mathbb{C}^6), \mathcal{L}^{2,-l}(\mathbb{R}^3, \mathbb{C}^6)).$$

where $s, l > \frac{1}{2}$.

Furthermore, it can be extended continuously from \mathbb{C}^\pm to $\mathbb{C}^\pm \cup \mathbb{R}$, in this uniform operator topology. The limiting values (denoted again by $\mathcal{P}^\perp R_{\maxwell}^\pm(\lambda)$) are locally bounded and locally Hölder continuous in the same topology.

The extended function satisfies, for $z \in \mathbb{C}^\pm \cup \{\mathbb{R} \setminus \{0\}\}$,

$$(4.41) \quad (L_{\maxwell} - z) \mathcal{P}^\perp R_{\maxwell}(z) f = f - z \mathcal{P} R_{\maxwell}(z) f, \quad f \in \mathcal{H}^{-1,s}(\mathbb{R}^3, \mathbb{C}^6),$$

where for $z = \lambda \in \mathbb{R} \setminus \{0\}$, $\mathcal{P}^\perp R_{\maxwell}(z) = \mathcal{P}^\perp R_{\maxwell}^\pm(\lambda)$.

Remark 4.15. *Note that the operator $\mathcal{P} R_{\maxwell}(z)$ is well-defined for $\text{Im} z \neq 0$ and can therefore be extended continuously (in the sense of distributions) to the real axis, in view of the continuity of the left-hand side in Equation (4.41), as is established in the following proof.*

Proof of the Proposition. The proof runs parallel to that of Proposition 4.5. In fact, in the functional setting of $B(\mathcal{H}^{-1,s}(\mathbb{R}^3, \mathbb{C}^6), \mathcal{H}^{-1,-l}(\mathbb{R}^3, \mathbb{C}^6))$ the claims follow from the general theory, in view of Proposition 4.13. Also the proof of (4.41) is identical to that of (4.19).

Since the operator L_{\maxwell} is not elliptic, we only need to justify the stronger continuity claim (4.40), namely, the fact that $\mathcal{H}^{-1,-l}(\mathbb{R}^3, \mathbb{C}^6)$ can be replaced by $\mathcal{L}^{2,-l}(\mathbb{R}^3, \mathbb{C}^6)$ in the statement. However, the restriction of L_{\maxwell} to the subspace orthogonal to its kernel is elliptic, as seen from Equation (4.25). Therefore, for any u in this subspace, the graph-norm $\|u\|_{-1,-l} + \|L_{\maxwell}u\|_{-1,-l}$ is equivalent to the $\mathcal{L}^{2,-l}$ norm $\|u\|_{0,-l}$. \square

5. STRONGLY PROPAGATIVE OR ISOTROPIC OPERATORS

We now turn back to the study of the spectral structure of the general (constant coefficient) operator (1.2):

$$L_0 = L_{0,hom} + M_0^0 = \sum_{j=1}^n M_j^0 D_j + M_0^0.$$

Its $(K \times K)$ matrix symbol is given by

$$(5.1) \quad M_0(\xi) = M_{0,hom}(\xi) + M_0^0 = \sum_{j=1}^n M_j^0 \xi_j + M_0^0.$$

All the common physical systems (Dirac, Maxwell, wave propagation in elastic medium and others) share the basic property of being **strongly propagative**, according to the following definition.

Definition 5.1. [50]: *The homogeneous operator $L_{0,hom} = \sum_{j=1}^n M_j^0 D_j$ is said to be strongly propagative if $M_{0,hom}(\xi)$ has a kernel of fixed dimension $0 \leq d < K$, independent of $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \setminus \{0\}$.*

The nonzero eigenvalues of $M_{0,hom}$ have the following properties.

- They are positive-homogeneous of degree 1.
- Let $Q_{min}^{M_{0,hom}}(\theta; \xi)$, $\theta \in \mathbb{C}$, be the minimal polynomial of $M_{0,hom}(\xi)$. Let

$$(5.2) \quad Z = \left\{ \xi \in \mathbb{R}^n \setminus \{0\} \mid \text{the discriminant of } Q_{min}^{M_{0,hom}}(\theta; \xi) \text{ vanishes} \right\}.$$

Then $\overline{Z} = Z \cup \{0\}$ is a closed cone of Lebesgue measure zero [50].

- In $\mathbb{R}^n \setminus \overline{Z}$ every eigenvalue of $M_{0,hom}(\xi)$ has constant multiplicity.

The distinct nonzero eigenvalues can therefore be enumerated as

$$(5.3) \quad \mu_\rho(\xi) > \dots > \mu_1(\xi) > 0 > \mu_{-1}(\xi) > \dots > \mu_{-\rho}(\xi), \quad \xi \in \mathbb{R}^n \setminus \overline{Z}.$$

The basic properties of these functions can be summarized as follows.

$$(5.4) \quad \begin{cases} (i) \mu_k(\xi) \text{ is continuous on } \mathbb{R}^n, \text{ and in fact real analytic on } \mathbb{R}^n \setminus \overline{Z}, \\ (ii) \mu_k(\xi) = -\mu_{-k}(-\xi), \quad k = 1, \dots, \rho, \quad \xi \in \mathbb{R}^n \setminus \overline{Z}, \\ (iii) \mu_k(\beta\xi) = \beta\mu_k(\xi), \quad \beta > 0, \quad k = 1, \dots, \rho, \quad \xi \in \mathbb{R}^n \setminus \overline{Z}. \end{cases}$$

The corresponding projections are denoted by

$$(5.5) \quad \{P_k(\xi) / \xi \in \mathbb{R}^n \setminus \overline{Z}\}_{0 \neq |k| \leq \rho}.$$

Remark that for $\xi \in Z$ the disjointness property (5.3) is not valid, but the eigenvalues clearly retain the homogeneity property. In fact, they are all bounded on every sphere $|\xi| = r > 0$, as is stated in the following claim.

Claim 5.2. *There exist constants $c_2 > c_1 > 0$ so that all nonzero eigenvalues satisfy*

$$(5.6) \quad c_1 \leq \left| \mu_{\pm k} \left(\frac{\xi}{|\xi|} \right) \right| \leq c_2, \quad \xi \neq 0, \quad k = 1, \dots, \rho.$$

Proof. The sphere $|\xi| = 1$ is compact, so the boundedness of the nonzero eigenvalues follows from the “group continuity” [32, Section II.4] of these eigenvalues and the assumption that the operator is strongly propagative.

The inequality (5.6) follows from the homogeneity property. \square

Even though we do not treat in this paper the general nonhomogeneous strongly propagative case, we shall make here a comment concerning its possible eigenvalues. Suppose then that the constant matrix $M_0^0 \neq 0$ is not a scalar matrix.

Let us consider the possibility of having an eigenvalue η of $M_0(\xi)$ that is *independent* of ξ for ξ in some open set $O \subseteq \mathbb{R}^n$. The existence of such an eigenvalue is equivalent to the fact that η is an eigenvalue of L_0 . In the homogeneous case ($M_0^0 = 0$) we can only have $\eta = 0$.

For η to be such an eigenvalue we need

$$\det(M_0(\xi) - \eta I_K) = 0, \quad \xi \in O.$$

Since the determinant is a polynomial in ξ , it follows that it actually vanishes for all $\xi \in \mathbb{R}^n$. In particular, η is an eigenvalue of M_0^0 .

We conclude that η must be contained in the finite set (subset of the set of eigenvalues of M_0^0)

$$(5.7) \quad \Lambda = \text{the set of common eigenvalues of } M_0^0 + \sum_{j=1}^n a_j M_j^0, \quad \forall a = (a_1, \dots, a_n) \in \mathbb{R}^n.$$

Any further spectral information, in our approach, requires a detailed study of the level surfaces of the eigenvalues (in analogy to the cases of the Dirac operator and the Maxwell system). The information we need (in the general nonhomogeneous case) requires the use of delicate tools of real algebraic geometry and will not be attempted here. We remark that, to the best of our knowledge, the spectral study of this general class of operators (e.g., the Limiting Absorption Principle) has never been carried out.

A more restricted class is that of operators for which $Z = \emptyset$, as follows.

Definition 5.3. [49]: *The operator $L_{0, \text{hom}} = \sum_{j=1}^n M_j^0 D_j$ is said to be uniformly propagative if it is strongly propagative and, moreover, the eigenspace associated with every eigenvalue has a constant dimension, independent of $\xi \in \mathbb{R}^n \setminus \{0\}$.*

For simplicity in what follows we shall refer also to the associated symbols as “strongly” or “uniformly” propagative.

J. Rauch studied the asymptotic behavior of solutions of first-order hyperbolic systems, imposing the assumption of a uniformly propagative system [38, Assumption (1.3)]. Note that “the equations of electromagnetic and elastic waves in crystals are not uniformly propagative. However they are strongly propagative” [48, Introduction].

In our treatment we shall always assume that $L_{0,hom}$ is strongly propagative.

We shall restrict our considerations to two classes of operators:

- Strongly propagative *homogeneous* operators, a generalization of the Maxwell system, as well as the massless Dirac operator.
- Nonhomogeneous isotropic operators (see Definition 5.14 below), a generalization of the massive Dirac operator.

5.1. SPECTRAL STRUCTURE OF HOMOGENEOUS STRONGLY PROPAGATIVE OPERATORS. The assumption that $M_0^0 = 0$ and $L_{0,hom}$ is strongly propagative permits an explicit representation of its domain as well as “partial coercivity” characterization, in full analogy to the Maxwell operator (see (4.23)).

We define for every index $k = \pm 1, \dots, k = \pm \rho$, the subspace

$$(5.8) \quad X_k = \{f \in \mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K) / P_k(\xi)\hat{f}(\xi) = \mu_k(\xi)\hat{f}(\xi), \xi \in \mathbb{R}^n \setminus \overline{Z}\},$$

where the projections $P_k(\xi)$ are as in (5.5).

These subspaces are clearly reducing for $L_{0,hom}$. In view of (5.6) we obtain the (partial) coercivity property ,

$$| \langle M_{0,hom}(\xi)\hat{f}(\xi), \hat{f}(\xi) \rangle_{\mathbb{C}^K} | \geq c_1 |\xi| |\hat{f}(\xi)|^2, \quad f \in X_k, 1 \leq |k| \leq \rho,$$

(compare the analogous fact (4.24) for the Maxwell system). We therefore conclude that

$$(5.9) \quad Dom(L_{0,hom}) = ker(L_{0,hom}) \oplus \sum_{1 \leq |k| \leq \rho} \oplus (X_k \cap \mathcal{H}^1(\mathbb{R}^n, \mathbb{C}^K)).$$

5.1.1. The Limiting Absorption Principle for homogeneous strongly propagative systems. Recall that the set Z was defined in (5.2). Let $\mu_j(\xi)$, $\xi \in \mathbb{R}^n \setminus \overline{Z}$, $0 \neq |j| \leq \rho$ be a nonzero eigenvalue. Let $\lambda \in \mathbb{R} \setminus \{0\}$ and consider the surface

$$(5.10) \quad \Gamma_j(\lambda) = \{\xi \in \mathbb{R}^n \setminus \overline{Z} / sgn(j) = sgn(\lambda), \mu_j(\xi) = \lambda\}.$$

It is an open smooth submanifold of codimension 1. It is bounded (and bounded away from the origin) in view of Claim 5.2.

The homogeneity property implies that the surfaces are homothetic in the sense that

$$(5.11) \quad \Gamma_j(\lambda) = \lambda \Gamma_j(1), \quad \lambda \neq 0.$$

The surface $\Gamma_j(1)$ plays a basic role in the wave propagation associated with the operator.

Definition 5.4. The surfaces $\Gamma_j(1) = \{\mu_j(\xi) = sgn(j)\}$ are called the **slowness surfaces** of the system (see [49, Section 4]).

The term used in [17] is **normal surfaces**. We note that treatments by means of global Fourier integral operators necessitate a very careful study of these surfaces as well as very special assumptions on the system (see e.g. [33]).

Since $\Gamma_j(1) = -\Gamma_{-j}(1)$, we shall henceforth assume $j > 0$, with $\lambda > 0$.

The homogeneity of $\mu_j(\xi)$ implies, by the Euler identity, that

$$\langle \xi, \nabla \mu_j(\xi) \rangle_{\mathbb{R}^n} = \mu_j(\xi) = \lambda, \quad \xi \in \Gamma_j(\lambda),$$

so that the Cauchy-Schwarz inequality and (5.6) yield

$$(5.12) \quad |\nabla \mu_j(\xi)| \geq c_1 > 0, \quad \xi \in \Gamma_j(\lambda), \quad \lambda > 0, \quad 1 \leq j \leq \rho.$$

Remark 5.5. *The inequality (5.12) means that the “group velocity” at the wavefront Γ_j is bounded away from zero. Compare with the analogous situation in the study of the asymptotic behavior of solutions of first-order systems [38, Equation (1.7)].*

Let $d\Sigma_{\Gamma_j(\lambda)}$ be the Lebesgue measure on $\Gamma_j(\lambda)$. The scaling property (5.11) yields

$$(5.13) \quad d\Sigma_{\Gamma_j(\lambda)} = \lambda^{n-1} d\Sigma_{\Gamma_j(1)}.$$

Let $\omega \in \Gamma_j(1)$ be a general point, with a corresponding $\lambda\omega \in \Gamma_j(\lambda)$. By (5.13) the traces of any bounded continuous function f on the two submanifolds satisfy

$$(5.14) \quad \int_{\Gamma_j(\lambda)} |f(\lambda\omega)|^2 d\Sigma_{\Gamma_j(\lambda)} = \int_{\Gamma_j(1)} \lambda^{n-1} |f(\lambda\omega)|^2 d\Sigma_{\Gamma_j(1)}.$$

Define (using appropriate scaling) the trace maps of the Sobolev space $\mathcal{H}^\theta(\mathbb{R}^n)$ into $L^2(\Gamma_j(1))$, by

$$(5.15) \quad (\Phi_\lambda^j h)(\omega) = \lambda^{\frac{n-1}{2}} h(\lambda\omega), \quad \omega \in \Gamma_j(1), \quad 1 \leq j \leq \rho.$$

To estimate these trace maps we invoke [6, Lemma A.5]. The uniform lower bound (5.12) implies that the essential condition in that lemma ($|\nabla g| \leq d$) is satisfied *uniformly* for any compact $K \Subset \Gamma_j(1)$. We conclude (by exhaustion) that the estimate can be applied to the smooth manifold $\Gamma_j(1)$, hence these maps are uniformly bounded for any $\theta > \frac{1}{2}$:

$$(5.16) \quad \sup_{\lambda > 0} \left\{ \|\Phi_\lambda^j\|_{B(\mathcal{H}^\theta(\mathbb{R}^n), \mathcal{L}^2(\Gamma_j(1)))} \right\} < \infty,$$

and the operator-valued map $\lambda \mapsto B(\mathcal{H}^\theta(\mathbb{R}^n), \mathcal{L}^2(\Gamma_j(1)))$ is locally hölder continuous in the uniform operator topology (compare Lemma 3.1).

Let $\{E_{0,hom}(\lambda), \lambda \in \mathbb{R}\}$ be the spectral family of $L_{0,hom}$. Since $E_{0,hom}(\lambda)$ commutes with $L_{0,hom}$, it also has a symbol, which we denote by $E_{0,hom}(\lambda; \xi)$. The following claim gives an explicit expression in terms of the projections on the eigenspaces (assuming $\lambda > 0$, with a similar expression for $\lambda < 0$). We use χ_B as the indicator function for a set $B \subseteq \mathbb{R}^n$, namely, $\chi_B(x) = 1$ (resp. $\chi_B(x) = 0$) if $x \in B$ (resp. $x \notin B$).

Claim 5.6. *Let $\lambda > 0$. Then,*

$$(5.17) \quad E_{0,hom}(\lambda; \xi) = \sum_{j=1}^{\rho} P_{-j}(\xi) + P_0(\xi) + \sum_{j=1}^{\rho} P_j(\xi) \chi_{\{\mu_j(\xi) \leq \lambda\}}.$$

(refer to Equations (5.4) and (5.5) for notation of eigenvalues and projections).

If $\widehat{f}, \widehat{g} \in C_0^\infty(\mathbb{R}^n \setminus \overline{Z})$ then, assuming $\lambda > 0$, we obtain by a well known formula (“coarea formula” [25, Appendix C.3]) for differentiation of volume integrals

$$(5.18) \quad \frac{d}{d\lambda}(E_{0,hom}(\lambda)f, g) = \sum_{j=1}^{\rho} \int_{\Gamma_j(\lambda)} \frac{\langle P_j(\xi)\widehat{f}(\xi), P_j(\xi)\widehat{g}(\xi) \rangle_{\mathbb{C}^K}}{|\nabla \mu_j(\xi)|} d\Sigma_{\Gamma_j(\lambda)},$$

where $d\Sigma_{\Gamma_j(\lambda)}$ is the Lebesgue surface measure (compare Equation (4.27)).

Note that the coarea formula requires global Lipschitz condition on $\mu_j(\xi)$. However, it is obtained by multiplying $\mu_j(\xi)$ by a cutoff smooth function such that $\varphi(\xi) = 1$ on the supports of $\widehat{f}(\xi)$, $\widehat{g}(\xi)$ and vanishes in a neighborhood of \overline{Z} . Then we see from (5.17) that for $\lambda > 0$

$$(5.19) \quad \frac{d}{d\lambda}(E_{0,hom}(\lambda)f, g) = \sum_{j=1}^{\rho} \frac{d}{d\lambda} \int_{\mathbb{R}^n} \langle P_j(\xi) \chi_{\{\varphi(\xi)\mu_j(\xi) \leq \lambda\}} \widehat{f}(\xi), P_j(\xi) \widehat{g}(\xi) \rangle_{\mathbb{C}^K} d\xi.$$

Since the level set $\{\xi \mid \varphi(\xi)\mu_j(\xi) = \lambda\}$ is a smooth $(n-1)$ -dimensional hypersurface for $\lambda > 0$, one can apply the coarea formula to the right hand side of (5.19), obtaining (5.18).

The real analyticity of the functions $\mu_j(\xi)$, $\xi \in \mathbb{R}^n \setminus \overline{Z}$, guarantees that Equation (5.18) can be repeatedly differentiated, using higher derivatives of $\widehat{f}(\xi)$, $\widehat{g}(\xi)$.

We introduce the subspace of functions permitting such recurrent differentiation in the following definition.

Definition 5.7. Let $\widehat{\Upsilon}_Z^s$ be the closure of $C_0^\infty(\mathbb{R}^n \setminus \overline{Z}, \mathbb{C}^K)$ in $\mathcal{H}^s(\mathbb{R}^n, \mathbb{C}^K)$, for $s > \frac{1}{2}$, and let $\Upsilon_Z^s \subseteq \mathcal{L}^{2,s}(\mathbb{R}^n, \mathbb{C}^K)$ be the subspace of its inverse Fourier transforms. It is a closed subspace, equipped with the same norm $\|\cdot\|_{0,s}$ (3.1).

Since Z has Lebesgue measure zero (in \mathbb{R}^n) it is clear that Υ_Z^s is dense in $\mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K)$. In the following claim we characterize the orthogonal complement of Υ_Z^s in $\mathcal{L}^{2,s}(\mathbb{R}^n, \mathbb{C}^K)$.

Claim 5.8. Let $(\Upsilon_Z^s)^\perp \subseteq \mathcal{L}^{2,s}(\mathbb{R}^n, \mathbb{C}^K)$, $s > \frac{1}{2}$, be the orthogonal complement to Υ_Z^s (using the scalar product associated with (3.1)). Let $h(x) \in (\Upsilon_Z^s)^\perp$. Then the Fourier transform of $(1 + |x|^2)^s h(x)$ is supported on \overline{Z} :

$$(5.20) \quad \text{supp } \mathcal{F}\{(1 + |x|^2)^s h(x)\}(\xi) \subseteq \overline{Z}.$$

Proof. Let $\psi \in \mathcal{L}^{2,s}(\mathbb{R}^n, \mathbb{C}^K)$ such that $\widehat{\psi} \in C_0^\infty(\mathbb{R}^n \setminus \overline{Z}, \mathbb{C}^K)$. The scalar product in $\mathcal{L}^{2,s}(\mathbb{R}^n, \mathbb{C}^K)$ can be expressed as

$$(5.21) \quad (h, \psi)_{0,s} = \int_{\mathbb{R}^n} \langle \widehat{h}(\xi), (I - \Delta)^s \widehat{\psi}(\xi) \rangle_{\mathbb{C}^K} d\xi = \langle (I - \Delta)^s \widehat{h}, \widehat{\psi} \rangle,$$

where \langle, \rangle in the last term stands for the $\mathcal{H}^{-s}(\mathbb{R}^n, \mathbb{C}^K)$, $\mathcal{H}^s(\mathbb{R}^n, \mathbb{C}^K)$ pairing. The assumption $h(x) \in (\Upsilon_Z^s)^\perp$ means that $(h, \psi)_{0,s} = 0$ hence

$$\text{supp } (I - \Delta)^s \widehat{h} = \text{supp } \mathcal{F}\{(1 + |x|^2)^s h(x)\} \subseteq \overline{Z}.$$

□

Remark 5.9. Note that if $h(x) \in \mathcal{L}^{2,s}(\mathbb{R}^n, \mathbb{C}^K)$ then $(1 + |x|^2)^{\frac{s}{2}} h(x) \in \mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K)$ and $(1 + |x|^2)^s h(x) \in \mathcal{L}^{2,-s}(\mathbb{R}^n, \mathbb{C}^K)$. Thus $(\Upsilon_Z^s)^\perp \subseteq \mathcal{L}^{2,s}(\mathbb{R}^n, \mathbb{C}^K)$ is non-trivial if and only if the set of functions in $\mathcal{H}^{-s}(\mathbb{R}_\xi^n, \mathbb{C}^K)$, for $s > \frac{1}{2}$, supported on \overline{Z} ,

is non-trivial. In other words Υ_Z^s is not necessarily equal to $\mathcal{L}^{2,s}(\mathbb{R}^n, \mathbb{C}^K)$ if Z is “large” in some “ $(n-1)$ – dimensional” Hausdorff measure sense. In fact, using the terminology of [30], Υ_Z^s is equal to $\mathcal{L}^{2,s}(\mathbb{R}^n, \mathbb{C}^K)$ if and only if Z is “ s –polar”, namely, if it has zero Bessel potential theoretic capacity of order s [34, Sections 10.4, 13.2]. A sufficient condition is that the α –dimensional Hausdorff measure is zero for any $\alpha > n - 2s$ [28, Theorem 5.12].

Given the special algebraic structure of Z (see (5.2)) we introduce the following conjecture.

CONJECTURE 5.10. *The set Z is s –polar for $s \in (\frac{1}{2}, \frac{3}{2})$, hence the subspace Υ_Z^s is equal to $\mathcal{L}^{2,s}(\mathbb{R}^n, \mathbb{C}^K)$.*

In conjunction with (5.12) and (5.16) we conclude from Equation (5.18) that, for any $s > \frac{1}{2}$ there exists a constant $C > 0$, depending only on s, c_1 , so that, for all $\lambda > 0$, and all $f, g \in \Upsilon_Z^s$,

$$\begin{aligned} \left| \frac{d}{d\lambda}(E_{0,hom}(\lambda)f, g) \right| &\leq \frac{1}{c_1} \sum_{j=1}^{\rho} \int_{\Gamma_j(\lambda)} |\widehat{f}(\xi)| \cdot |\widehat{g}(\xi)| d\Sigma_{\Gamma_j(\lambda)} \\ (5.22) \quad &= \frac{1}{c_1} \sum_{j=1}^{\rho} \int_{\Gamma_j(1)} |\Phi_{\lambda}^j \widehat{f}(\omega)| |\Phi_{\lambda}^j \widehat{g}(\omega)| d\Sigma_{\Gamma_j(1)} \leq C \|f\|_{0,s} \|g\|_{0,s}. \end{aligned}$$

Any continuous functional on the closed subspace Υ_Z^s can be uniquely extended to a functional on $\mathcal{L}^{2,s}(\mathbb{R}^n, \mathbb{C}^K)$ (namely, a function in $\mathcal{L}^{2,-s}(\mathbb{R}^n, \mathbb{C}^K)$) by defining it as zero on the orthogonal complement.

It follows (compare Proposition 4.8) that

Corollary 5.11. *There exists a map*

$$\widetilde{A_{0,hom}(\lambda)} \in B(\Upsilon_Z^s, \mathcal{L}^{2,-s}(\mathbb{R}^n, \mathbb{C}^K)), \quad s > \frac{1}{2},$$

so that

$$(5.23) \quad \frac{d}{d\lambda}(E_{0,hom}(\lambda)f, f) = \langle \widetilde{A_{0,hom}(\lambda)}f, f \rangle, \quad f \in \Upsilon_Z^s,$$

where \langle, \rangle is the $(\mathcal{L}^{2,-s}(\mathbb{R}^n, \mathbb{C}^K), \mathcal{L}^{2,s}(\mathbb{R}^n, \mathbb{C}^K))$ pairing.

The map $\widetilde{A_{0,hom}(\lambda)}$ is uniformly bounded

$$(5.24) \quad \|\widetilde{A_{0,hom}(\lambda)}\|_{B(\Upsilon_Z^s, \mathcal{L}^{2,-s}(\mathbb{R}^n, \mathbb{C}^K))} \leq C, \quad \lambda \neq 0,$$

and locally Hölder continuous in the uniform operator topology.

The global uniform boundedness (5.24) will play a crucial role in the spacetime estimates of Section 8.

The general theory (see Theorem 6.2 below with $\mathcal{X} = \Upsilon_Z^s$) now yields the LAP in this case as follows.

Theorem 5.12. *Let $R_{0,hom}(z) = (L_{0,hom} - z)^{-1}$, $\text{Im } z \neq 0$. For any $s > \frac{1}{2}$ the limits*

$$(5.25) \quad R_{0,hom}^{\pm}(\mu) = \lim_{\varepsilon \downarrow 0} R_{0,hom}(\mu \pm i\varepsilon), \quad \mu \in \mathbb{R} \setminus \{0\},$$

exist in the uniform operator topology of $B(\Upsilon_Z^s, \mathcal{L}^{2,-s}(\mathbb{R}^n, \mathbb{C}^K))$.

Furthermore, the limit functions $R_{0,hom}^\pm(\mu)$, $\mu \in \mathbb{R} \setminus \{0\}$, are locally bounded and locally Hölder continuous with respect to the uniform operator topology.

Remark 5.13. The LAP result of Theorem 5.12 was proved by Weder in [47, 48] for intervals interior to $\mathbb{R} \setminus \{0\}$, in the operator space $B(\mathcal{L}^{2,s}(\mathbb{R}^n, \mathbb{C}^K), \mathcal{H}^{1,-s}(\mathbb{R}^n, \mathbb{C}^K))$. We emphasize that we do not believe that the presence of the singular set Z could be entirely dismissed. The proof in [47] relies on the commutator approach, and we were not quite able to follow the details there. On the other hand the proof in [48] is essentially based on the methodology of trace maps. It seems to us to be fundamentally flawed, and this impression has not changed even after a long correspondence with him. The proof of [48, Theorem A.1] involves a deformation map of the slowness surface $\Gamma_j(1)$ onto the unit sphere followed by an application of the trace theorem on the sphere. Thus, the measure dw_j on $\Gamma_j(1)$ is defined by the radial projection on the unit sphere (see [48, Eq. (A.20)]) so as to obtain a “polar decomposition” $d^n k = \rho^{n-1} d\rho dw_j$ [48, Eq. (A.19)]. This is of course wrong, since the coarea formula is ignored. That formula introduces a denominator $|\nabla \mu_j(k)|$ (see Eq. (5.18)) in the last expression, which is singular on Z . Effectively, he argues that Z is of “measure zero” in $\Gamma_j(1)$, so our Υ_Z^s can be identified with $\mathcal{L}^{2,s}(\mathbb{R}^n, \mathbb{C}^K)$. This whole argument is applicable in obtaining a trace on every star-shaped surface, no matter how singular, and this is clearly wrong, see Theorem 2.3 in [1] and Remark 5.9.

5.2. SPECTRAL DENSITY OF ISOTROPIC OPERATORS. The examples of the Dirac and Maxwell operators motivate our next definition. In fact, like these two examples, all physical models where there is no “built in” preference for specific (spatial) directions, naturally fall into the category of **isotropic operators** [49, Section 4], that we recall next.

Definition 5.14. The operator L_0 (1.2) is said to be isotropic if the eigenvalues of its symbol (see (5.1)) $M_0(\xi)$ are functions of $|\xi|$.

We assume in addition that $L_{0,hom}$ is **strongly propagative** (Definition 5.1).

If L_0 is **homogeneous**, namely, $M_0^0 \equiv 0$, then $M_0(\xi) = M_{0,hom}(\xi)$. By definition the zero eigenvalue $\mu_0(\xi) = 0$ is of fixed dimension $d_0 \geq 0$. The singular set Z (5.2) is empty and the eigenvalues μ_k (see Equation (5.3)) satisfy

$$(5.26) \quad \mu_k(\xi) = \text{sign}(k) \mu_{|k|}^c |\xi|, \quad \xi \neq 0, \quad \pm k = 1, 2, \dots, J,$$

where μ_k^c are positive constants such that $\mu_J^c > \dots > \mu_1^c > 0$.

Each eigenvalue μ_k is of fixed dimension $d_k > 0$, $\pm k = 1, 2, \dots, J$, and $d_k = d_{-k}$. In particular, in this case the operator is **uniformly propagative** (Definition 5.3).

It is therefore a special case of the class considered above in Subsection 5.1 and does not require a further consideration here.

We now turn to the **nonhomogeneous operator**.

In this case the eigenvalues are functions of a single variable $r = |\xi|$, and we denote them by

$$(5.27) \quad \lambda_1(r) \leq \lambda_2(r) \leq \dots \leq \lambda_\rho(r).$$

However now these eigenvalues are not homogeneous functions of $r > 0$, and, unlike the homogeneous case, their multiplicity is not fixed. In other words, two (or more) different eigenvalues $\lambda_i(r) \neq \lambda_j(r)$ can “coalesce” at a point $r = r_0$. Such a point r_0 is called “a crossing point”.

As we shall see below, it will be necessary to forsake their ordering in order to maintain their analyticity.

Since L_0 is isotropic, the eigenvalues of

$$M_0(\xi) = M_{0,hom}(\xi) + M_0^0 = |\xi| \sum_{j=1}^n M_j^0 \omega_j + M_0^0 \quad (\omega = \xi/|\xi|)$$

are functions of $|\xi|$. In particular, the eigenvalues of $M_0(\xi) = M_0(|\xi|\omega)$, together with their multiplicities, are independent of $\omega \in \mathbb{S}^{n-1}$.

By virtue of this fact, we can take $\omega = e_1 = (1, 0, \dots, 0)$, and study the eigenvalues of

$$(5.28) \quad M_0(|\xi|e_1) = |\xi| M_1^0 + M_0^0.$$

Taking $r = |\xi| > 0$, the eigenvalue study is reduced to the study of the symmetric matrix, depending (linearly) on a positive parameter,

$$(5.29) \quad T(r) = r M_1^0 + M_0^0, \quad r > 0.$$

However, it is useful to regard $T(r)$ as a function of the coordinate $r \in \mathbb{R}$. Then the analytic perturbation theory of Hermitian matrices [32, Section II.6] can be invoked.

We conclude that the eigenvalues $\lambda_1(r), \dots, \lambda_\rho(r)$ of $T(r)$ are analytic functions of $r \in \mathbb{R}$. The sum of their multiplicities is K , and each of them is constant in intervals not containing crossing points, as will be explained below.

We denote by

$$(5.30) \quad P_1(\xi), \dots, P_\rho(\xi),$$

the corresponding projections.

Note that $P_j(\xi)$ cannot be assumed to depend on $|\xi|$.

Clearly, to maintain $\lambda_1(r), \dots, \lambda_\rho(r)$ as analytic functions we cannot order them, as there may be **crossing points**:

Definition 5.15. *A value $r = r_0 > 0$ is a crossing point if there are two indices $1 \leq j < k \leq \rho$ such that $\lambda_j(r_0) = \lambda_k(r_0)$.*

In every closed interval $[\alpha, \beta] \subseteq \mathbb{R}$ there is at most a finite number of crossing points. In fact, the fact that $T(r)$ depends *linearly* on r , enables us to claim even more.

Claim 5.16. *There is at most a finite number of crossing points in the whole real line $r \in \mathbb{R}$.*

Proof. The crossing points are zeros of the discriminant of the minimal polynomial of $T(r)$, as a function of $r \in \mathbb{R}$. However, clearly this discriminant is an algebraic function of r , and as such can have at most a finite number of zeros. \square

We define the **crossing values** as the finite set

$$\{\lambda \in \mathbb{R} / \lambda = \lambda_j(q), \text{ for some } j \text{ and some crossing point } q\}.$$

In preparation for the Limiting Absorption Principle in the present case, we need to determine the set Λ to be excluded from the spectrum of L_0 . As noted above, the eigenvalues remain analytic (as functions of $r > 0$) even at crossing points. Thus,

the only values of λ to be excluded from the spectrum of L_0 are the critical values, defined as follows.

Definition 5.17. *A value $\lambda \in \mathbb{R}$ is a **critical value** if $\lambda = \lambda_j(r)$ and $\lambda'_j(r) = 0$ for some $j \in \{1, 2, \dots, \rho\}$ and some $r \in \mathbb{R}$.*

Claim 5.18. *There is at most a finite number of critical values of the eigenvalues $\lambda_j(r)$, $j = 1, 2, \dots, \rho$.*

Proof. The functions $\lambda_j(r)$ are roots of the algebraic equation $\det(T(r) - \lambda I_K) = 0$ ($T(r)$ is defined in (5.29)), and are therefore algebraic functions of the real variable $r \in \mathbb{R}$. We can now apply the classical argument in [29, Section 14.3]; the set of critical values is of measure zero by Sard's theorem, hence being semi-algebraic set it must be finite. \square

Recall the definition (5.7) of the finite set Λ , that contains all possible eigenvalues of L_0 .

Definition 5.19. *The set Λ of (5.7) is extended (retaining the same notation) to include also the finitely many critical values.*

Remark 5.20. *Continuing Remark 5.5: the requirement that λ is not a critical value means that the “group velocity” at the wavefront $\Gamma_j(\lambda)$ is bounded away from zero, for all $j \in \{1, 2, \dots, \rho\}$.*

5.2.1. The Limiting Absorption Principle for isotropic operators. Let $E_0(\lambda)$ be spectral family associated with L_0 . Since $E_0(\lambda)$ commutes with L_0 , it also has a symbol, which we denote by $E_0(\lambda; \xi)$. As in the case of Claim 5.6 we have here.

The finite set $\Lambda \subseteq \mathbb{R}$ is as in Definition 5.19.

Claim 5.21. *Let $\lambda \in \mathbb{R} \setminus \Lambda$. Then, with projections P_j as in (5.30),*

$$(5.31) \quad E_0(\lambda; \xi) = \sum_{j=1}^{\rho} P_j(\xi) \chi_{\{\lambda_j(|\xi|) \leq \lambda\}}.$$

Corollary 5.22. *If $\hat{f}, \hat{g} \in C_0^\infty(\mathbb{R}^n, \mathbb{C}^K)$ and $\lambda \in \mathbb{R} \setminus \Lambda$, then*

$$(5.32) \quad \begin{aligned} \frac{d}{d\lambda} (E_0(\lambda) f, g) &= \sum_{j=1}^{\rho} \int_{\Gamma_j(\lambda)} \frac{\langle P_j(\xi) \hat{f}(\xi), \hat{g}(\xi) \rangle_{\mathbb{C}^K}}{|\nabla \lambda_j(\xi)|} \chi_{\{\lambda_j(|\xi|) = \lambda\}} d\Sigma_{r_j} \\ &= \sum_{j=1}^{\rho} |\lambda'_j(r_j)|^{-1} \int_{\Gamma_j(\lambda)} \langle P_j(\xi) \hat{f}(\xi), \hat{g}(\xi) \rangle_{\mathbb{C}^K} \chi_{\{\lambda_j(|\xi|) = \lambda\}} d\Sigma_{r_j}. \end{aligned}$$

with $d\Sigma_{r_j}$ being the Lebesgue surface measure on the sphere $\Gamma_j(\lambda)$ of radius $r_j > 0$ such that $\lambda_j(r_j) = \lambda$.

Remark 5.23. *In Equation (5.32) we know that $\lambda'_j(r_j) \neq 0$ since λ is not a critical value.*

In analogy to the case of the strongly propagative system (Theorem 5.12) we can state here the LAP as following from the general theory.

Theorem 5.24. *Let $L_0(D)$ be isotropic and $R_0(z) = (L_0 - z)^{-1}$, $\text{Im } z \neq 0$. Then for any $s > \frac{1}{2}$ the limits*

$$(5.33) \quad R_0^\pm(\mu) = \lim_{\varepsilon \downarrow 0} R_0(\mu \pm i\varepsilon), \quad \mu \in \mathbb{R} \setminus \Lambda,$$

exist in the uniform operator topology of $B(\mathcal{L}^{2,s}(\mathbb{R}^n, \mathbb{C}^K), \mathcal{L}^{2,-s}(\mathbb{R}^n, \mathbb{C}^K))$.

Furthermore, the limit functions $R_0^\pm(\mu)$, $\mu \in \mathbb{R} \setminus \Lambda$, are locally bounded and locally Hölder continuous with respect to the uniform operator topology.

Remark 5.25. Remark that the LAP is valid even at crossing values, provided they are not critical values.

6. THE CLASS OF PERTURBED OPERATORS

We now turn to the study of the spectral structure of perturbations of the operators introduced in Section 2. Our ultimate goal is the study of the operator L introduced in (1.1). This is done by regarding this operator as a perturbation of L_0 , as given in (1.2).

We take the most basic perturbation of the form

$$(6.1) \quad L = L_0 + V(x),$$

where $V(x)$ is a Hermitian $K \times K$ matrix that decays as $|x| \rightarrow \infty$. In Subsection 6.1 we expound the abstract theory of such perturbations. In particular, general conditions are given that imply the absolute continuity (and LAP) of the continuous spectrum, apart possibly from a discrete sequence of embedded eigenvalues.

In subsection 6.2 we apply the general theory to the potential perturbation of the Dirac operator. Observe that this includes the case of the magnetic Dirac operator, as the magnetic field can be merged into the potential perturbation. We choose to deal with this example in detail as it illustrates the applicability of the general theory, and also allows us to give decay conditions on the potential, to the effect that there are only *finitely many* eigenvalues in the spectral gap (of the massive Dirac operator, see Subsubsection 6.2.1).

The perturbed Maxwell system can also be reduced to the case of potential perturbation [44, Section 1.4], but we choose not to treat it here in detail, as the paper is already quite long.

6.1. GENERAL THEORY: PERTURBATION BY A POTENTIAL.

Our treatment is based on the general theory expounded in [7, Sections 3-4]. The abstract setting will allow us to consider various operator settings in a unified way. We briefly recall some definitions and statements that will be needed here.

Let H be a self-adjoint operator in $\mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K)$. The scalar product in $\mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K)$ is denoted by (\cdot, \cdot) .

Suppose that there exists a Hilbert space such that $\mathcal{X} \subseteq \mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K)$, and the embedding is dense and continuous. In other words, \mathcal{X} can be considered as a dense subspace of $\mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K)$, equipped with a stronger norm. Then, of course, $\mathcal{X} \hookrightarrow \mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K) \hookrightarrow \mathcal{X}^*$, where \mathcal{X}^* is the anti-dual of \mathcal{X} ; the continuous additive functionals l on \mathcal{X} , such that $l(\alpha v) = \bar{\alpha} l(v)$, $\alpha \in \mathbb{C}$.

We use $\|x\|_{\mathcal{X}}$, $\|x^*\|_{\mathcal{X}^*}$ for the norms in \mathcal{X} , \mathcal{X}^* , respectively, and designate by $\langle \cdot, \cdot \rangle$ the $(\mathcal{X}^*, \mathcal{X})$ pairing.

The (linear) embedding $h \in \mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K) \hookrightarrow h^* \in \mathcal{X}^*$ is obtained as usual by the scalar product (in $\mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K)$), $h^*(x) = (h, x)$.

We introduce still another Hilbert space \mathcal{X}_H^* , which is a dense subspace of \mathcal{X}^* , equipped with a stronger norm (so that the embedding $\mathcal{X}_H^* \hookrightarrow \mathcal{X}^*$ is continuous). However, we do not require that $\mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K)$ be embedded in \mathcal{X}_H^* . As indicated

by the notation, \mathcal{X}_H^* may depend on H . A typical case would be when H can be extended as a densely defined operator in \mathcal{X}^* and \mathcal{X}_H^* would be its domain there, equipped with the graph norm.

Let $\{E(\lambda)\}$ be the spectral family of H . We denote by $E(B)$ the spectral projection on any Borel set B (so that $E(\lambda) = E(-\infty, \lambda)$).

Definition 6.1. *Let $U \subseteq \mathbb{R}$ be open and let $0 < \alpha \leq 1$. Assume that U is of full spectral measure, namely, $E(\mathbb{R} \setminus U) = 0$. Then H is said to be of type $(\mathcal{X}, \mathcal{X}_H^*, \alpha, U)$ if the following conditions are satisfied:*

- (1) *The operator-valued function*

$$\lambda \rightarrow E(\lambda) \in B(\mathcal{X}, \mathcal{X}^*), \quad \lambda \in U,$$

is weakly differentiable with a locally Hölder continuous derivative in $B(\mathcal{X}, \mathcal{X}_H^)$; that is, there exists an operator-valued function*

$$\lambda \rightarrow A(\lambda) \in B(\mathcal{X}, \mathcal{X}_H^*), \quad \lambda \in U,$$

so that

$$\frac{d}{d\lambda}(E(\lambda)x, y) = \langle A(\lambda)x, y \rangle, \quad x, y \in \mathcal{X}, \lambda \in U,$$

and such that for every compact interval $K \subseteq U$, there exists an $M_K > 0$ satisfying

$$\|A(\lambda) - A(\mu)\|_{B(\mathcal{X}, \mathcal{X}_H^*)} \leq M_K |\lambda - \mu|^\alpha, \quad \lambda, \mu \in K.$$

- (2) *For every bounded open set $J \subseteq U$ and for every compact interval $K \subseteq J$, the operator-valued function (defined in the weak sense)*

$$z \rightarrow \int_{U \setminus J} \frac{A(\lambda)}{\lambda - z} d\lambda, \quad z \in \mathbb{C}, \Re z \in K, |\Im z| \leq 1,$$

takes values and is Hölder continuous in the uniform operator topology of $B(\mathcal{X}, \mathcal{X}_H^)$, with exponent α .*

We can now state the basic theorem, concerning the *Limiting Absorption Principle* (LAP) in this setting. We use the notation $\mathbb{C}^\pm = \{z \in \mathbb{C}, \pm \Im z > 0\}$, and denote by $R(z) = (H - z)^{-1}$, $z \in \mathbb{C}^\pm$, the resolvent of H .

Theorem 6.2. *Let H be of type $(\mathcal{X}, \mathcal{X}_H^*, \alpha, U)$ (where $U \subseteq \mathbb{R}$ is open and $0 < \alpha \leq 1$). Then H satisfies the LAP in U . More explicitly, the limits*

$$R^\pm(\lambda) = \lim_{\varepsilon \downarrow 0} R(\lambda \pm i\varepsilon), \quad \lambda \in U,$$

exist in the uniform operator topology of $B(\mathcal{X}, \mathcal{X}_H^)$ and the extended operator-valued function*

$$R(z) = \begin{cases} R(z), & z \in \mathbb{C}^+, \\ R^+(z), & z \in U, \end{cases}$$

is locally Hölder continuous in the same topology (with exponent α).

A similar statement applies when \mathbb{C}^+ is replaced by \mathbb{C}^- , but note that the limiting values $R^\pm(\lambda)$ are in general different.

Remark 6.3. *In view of the Stieltjes formula we have*

$$(6.2) \quad A(\lambda) = \frac{1}{2\pi i} (R^+(\lambda) - R^-(\lambda)), \quad \lambda \in U.$$

In particular, H is absolutely continuous in U .

We now consider a perturbation by a potential function $V(x)$. To deal with the requirements on V in this framework we introduce the following definition.

Definition 6.4. *An operator $V : \mathcal{X}_H^* \rightarrow \mathcal{X}$ will be called*

- (1) *Short-range with respect to H if it is compact.*
- (2) *Symmetric if $D(H) \cap \mathcal{X}_H^*$ is dense in $\mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K)$ and the restriction of V to $D(H) \cap \mathcal{X}_H^*$ is symmetric in $\mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K)$.*

The following lemma shows that (with some additional assumption) the operator $H + V$ is well defined.

Lemma 6.5. *Let H be of type $(\mathcal{X}, \mathcal{X}_H^*, \alpha, U)$ and let V be short-range and symmetric. Suppose that there exists $z \in \mathbb{C}$, $\Im z \neq 0$, and a linear subspace $D_z \subseteq D(H) \cap \mathcal{X}_H^*$ such that $(H - z)(D_z)$, the image of D_z under $H - z$, is dense in \mathcal{X} .*

Then $P = H + V$, defined on $D(H) \cap \mathcal{X}_H^$, is essentially self-adjoint.*

In what follows we always assume that H is of type $(\mathcal{X}, \mathcal{X}_H^*, \alpha, U)$ and that V is short-range and symmetric. Thus, by the lemma, $P = H + V$ can be extended as a self-adjoint operator with domain $D(P) \supseteq D(H) \cap \mathcal{X}_H^*$, and we retain the notation P for this extension.

Our aim is to study the spectral properties of P , particularly the LAP, in this abstract framework.

Denote by $S(z) = (P - z)^{-1}$, $\Im z \neq 0$, the resolvent of P . Our starting point is the resolvent equation

$$S(z)(I + VR(z)) = R(z), \quad \Im z \neq 0.$$

It can be shown that the inverse $(I + VR(z))^{-1}$ exists on \mathcal{X} if $\Im z \neq 0$. This leads to

$$(6.3) \quad S(z) = R(z)(I + VR(z))^{-1},$$

where the equality is certainly valid from $\mathcal{X} \rightarrow \mathcal{X}_H^*$.

Suppose now that $\lambda \in U$. In view of Theorem 6.2 and the assumption on V we have

$$\lim_{\varepsilon \downarrow 0} VR(\lambda \pm i\varepsilon) = VR^\pm(\lambda) \text{ in } B(\mathcal{X}).$$

Thus, if $(I + VR^\pm(\lambda))^{-1}$ exists (in $B(\mathcal{X})$), then Eq. (6.3) implies the existence of the limits

$$(6.4) \quad S^\pm(\lambda) = \lim_{\varepsilon \downarrow 0} S(\lambda \pm i\varepsilon) = R^\pm(\lambda)(I + VR^\pm(\lambda))^{-1},$$

in the uniform operator topology of $B(\mathcal{X}, \mathcal{X}_H^*)$.

Let $\lambda \in U$ be a point at which, say, $(I + VR^+(\lambda))^{-1}$ does not exist (in $B(\mathcal{X})$). Since $VR^+(\lambda)$ is compact in \mathcal{X} , there exists a non-zero $\phi \in \mathcal{X}$ so that

$$\phi = -VR^+(\lambda)\phi.$$

Let $\psi = R^+(\lambda)\phi \in \mathcal{X}_H^*$. Then

$$\langle \psi, \phi \rangle = -\lim_{\varepsilon \downarrow 0} \langle R(\lambda + i\varepsilon)\phi, VR(\lambda + i\varepsilon)\phi \rangle.$$

By the symmetry of V the right-hand side of this equality is real, so we conclude that $\Im \langle R^+(\lambda)\phi, \phi \rangle = 0$, and invoking Eq. (6.2) we conclude that

$$\langle A(\lambda)\phi, \phi \rangle = 0.$$

Now the form $\langle A(\lambda)x, y \rangle = \frac{d}{d\lambda} \langle E(\lambda)x, y \rangle$ on $\mathcal{X} \times \mathcal{X}$ is symmetric and positive semi-definite. Hence, for every $y \in \mathcal{X}$,

$$(6.5) \quad |\langle A(\lambda)\phi, y \rangle| \leq \langle A(\lambda)\phi, \phi \rangle^{\frac{1}{2}} \langle A(\lambda)y, y \rangle^{\frac{1}{2}} = 0,$$

and we conclude that

$$(6.6) \quad A(\lambda)\phi = 0.$$

In particular, $R^+(\lambda)\phi = R^-(\lambda)\phi$ and

$$\phi = -VR^\pm(\lambda)\phi.$$

Definition 6.6. We designate by Σ_P the set

$$\Sigma_P = \{\lambda \in U \mid \text{There exists a non-zero } \phi_\lambda \in \mathcal{X} \text{ such that } \phi_\lambda = -VR^\pm(\lambda)\phi_\lambda\}.$$

Remark 6.7. The set Σ_P is (relatively) closed in U . Indeed, if $(I + VR^\pm(\lambda_0))^{-1}$ exists (in $B(\mathcal{X})$), then $(I + VR^\pm(\lambda))^{-1}$ exists for λ close to λ_0 .

The discussion above leads to the following theorem.

Theorem 6.8. The operator $P = H + V$ satisfies the LAP in $U \setminus \Sigma_P$, in the uniform operator topology of $B(\mathcal{X}, \mathcal{X}_H^*)$, and the limiting values of its resolvent there are given by Eq. (6.4).

In particular, the spectrum of P in $U \setminus \Sigma_P$ is absolutely continuous. We single out this fact, stated in terms of the eigenvalues, in the following corollary.

Corollary 6.9. Let $\sigma_p(P)$ be the point spectrum of P . Then

$$\sigma_p(P) \cap U \subseteq \Sigma_P.$$

6.1.1. The exceptional set Σ_P . Our aim is to identify the set Σ_P introduced in Definition 6.6. It will turn out that (modulo one additional assumption on the smoothness of the spectral measure of H) we have equality of the sets in the last corollary. In other words, Σ_P is the set of eigenvalues of P embedded in U , and is necessarily discrete.

Let $\mu \in \Sigma_P$, so that by definition there exists a non-zero $\phi \in \mathcal{X}$ satisfying

$$(6.7) \quad \phi = -VR^\pm(\mu)\phi.$$

In view of (6.6) we have $A(\mu)\phi = 0$, and since the form $\langle A(\lambda)\phi, \phi \rangle$ is non-negative we infer that the zero at $\lambda = \mu$ is a minimum. Thus formally this minimum is a second-order zero for the form. However, our smoothness assumption on the spectral measure (Definition 6.1) does not go so far as a second-order derivative. We therefore impose the following additional hypothesis on the spectral derivative near such a minimum.

REGULARITY ASSUMPTION ON SPECTRAL DENSITY. Let $K \subseteq U$ be compact and $\phi \in \mathcal{X}$ a solution to (6.7), where $\mu \in K$. Then there exist constants $C, \varepsilon > 0$, depending only on K , so that

$$(6.8) \quad \langle A(\lambda)\phi, \phi \rangle \leq C |\lambda - \mu|^{1+\varepsilon} \|\phi\|_{\mathcal{X}}^2, \quad \lambda \in K.$$

Remark 6.10. This assumption is satisfied if the operator-valued function $\lambda \rightarrow A(\lambda) \in B(\mathcal{X}, \mathcal{X}_H^*)$ has a Hölder continuous (in the uniform operator topology) Fréchet derivative in a neighborhood of μ . Indeed, in this case we have

$$\langle A(\lambda)\phi, \phi \rangle = \langle (A(\lambda) - A(\mu))\phi, \phi \rangle = \frac{d}{d\theta} \langle A(\theta)\phi, \phi \rangle_{\theta \in [\mu, \lambda]} (\lambda - \mu).$$

Theorem 6.11. *Let V be symmetric and short-range, and assume that the condition of Lemma 6.5 is satisfied, so that $P = H + V$ is a self-adjoint operator. Assume, in addition, that the assumption above is satisfied and (6.8) holds. Then*

$$\Sigma_P = \sigma_p(P) \cap U.$$

Furthermore, every eigenvalue is of finite multiplicity and the set of eigenvalues $\sigma_p(P)$ has no accumulation point in U .

6.2. PERTURBATION OF THE DIRAC OPERATOR. .

We now consider the operator

$$(6.9) \quad H_m^V = H_m + V(x),$$

where H_m is the free Dirac operator given in (2.2) and $V(x)$ is a Hermitian 4×4 matrix that decays as $|x| \rightarrow \infty$.

We assume $m > 0$. In this case the spectrum $\text{spec}(H_m)$ has a “gap” $(-m, m)$ (see Equation (4.4)) and our focus is on the *finiteness* of the eigenvalues therein. We refer to [9, 10, 21] for general discussion of the eigenvalues in the gap. A variational characterization of the eigenvalues is given in [20].

The spectral structure of H_m was studied in Subsection 4.1. Recall that by (4.3) the domain of H_m (as a self adjoint operator in $\mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^4)$) is $\mathcal{H}^1(\mathbb{R}^3, \mathbb{C}^4)$. Using the terminology introduced in Subsection 6.1 it follows that , in view of Theorem 4.2, we can take

$$(6.10) \quad \mathcal{X} = \mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^4), \quad \mathcal{X}^* = \mathcal{L}^{2,-s}(\mathbb{R}^3, \mathbb{C}^4), \quad s > \frac{1}{2}.$$

Also, the space \mathcal{X}_H^* can then be taken as the domain of (the closure of) H_m in $\mathcal{L}^{2,-s}(\mathbb{R}^3, \mathbb{C}^4)$, so that

$$(6.11) \quad \mathcal{X}_H^* = \mathcal{H}^{1,-s}(\mathbb{R}^3, \mathbb{C}^4).$$

For the selfadjointness of H_m^V the following proposition suffices for our study here.

Proposition 6.12. *Assume that V is bounded and decays at infinity. Then the operator H_m^V is self-adjoint with domain $\mathcal{H}^1(\mathbb{R}^3, \mathbb{C}^4) \subseteq \mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^4)$. Its essential spectrum is $\mathbb{R} \setminus (-m, m)$.*

Proof. By the coercivity inequality (4.2) it follows that V is compact with respect to H_m , which establishes the claim by the general theory of self-adjoint perturbations. \square

Remark 6.13. *Indeed, the self-adjointness of H_m^V holds for a much wider class of potentials, see e.g. [2, Chapter 2], [24]. As in the case of the classical Schrödinger operators , the stronger decay assumptions are needed when studying the LAP and the spectral derivative associated with the operator, as we proceed to do next.*

In order to derive the Limiting Absorption Principle (LAP) of the perturbed operator H_m^V from the general theory presented in Section 6.1, we need to assume that, for some constant $C > 0$,

$$(6.12) \quad |V(x)| \leq C(1 + |x|)^{-(1+\varepsilon)}, \quad \varepsilon > 0, \quad x \in \mathbb{R}^3.$$

It is readily verified that V satisfies the conditions of Definition 6.4, so that it is short-range and symmetric.

Now let

$$U = \mathbb{R} \setminus [-m, m].$$

In view of Theorem 4.2 and Theorem 6.8 we obtain the LAP for H_m^V as follows.

Theorem 6.14. *Assume that V satisfies the decay assumption (6.12). Let $R_m^V(z) = (H_m^V - z)^{-1}$, $\Im z \neq 0$. Then:*

- *For any $\frac{1}{2} < s < \frac{1+\varepsilon}{2}$ the limits*

$$(6.13) \quad R_m^{V,\pm}(\mu) = \lim_{\varepsilon \downarrow 0} R_m^V(\mu \pm i\varepsilon), \quad \mu \in U \setminus \Sigma_{H_m}^V,$$

exist in the uniform operator topology of $B(\mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{L}^{2,-s}(\mathbb{R}^3, \mathbb{C}^4))$, where the (relatively) closed set $\Sigma_{H_m}^V$ is given by

$$\Sigma_{H_m}^V = \{\lambda \in U \mid \text{There exists a non-zero } \phi_\lambda \in \mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^4) \text{ such that } \phi_\lambda = -VR_m^\pm(\lambda)\phi_\lambda\}.$$

- *Assume that the decay assumption (6.12) is replaced by the stronger one*

$$(6.14) \quad |V(x)| \leq C(1 + |x|)^{-(2+\varepsilon)}, \quad \varepsilon > 0, \quad x \in \mathbb{R}^3,$$

then for any $1 < s < \frac{2+\varepsilon}{2}$ the limits in (6.13) exist for all $\mu \in \mathbb{R} \setminus \widetilde{\Sigma_{H_m}^V}$, where now

$$\widetilde{\Sigma_{H_m}^V} = \{\lambda \in \mathbb{R} \mid \text{There exists a non-zero } \phi_\lambda \in \mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^4) \text{ such that } \phi_\lambda = -VR_m^\pm(\lambda)\phi_\lambda\}.$$

Proof. The existence of the limits in (6.13) in $U \setminus \Sigma_{H_m}^V$ (resp. in $\mathbb{R} \setminus \widetilde{\Sigma_{H_m}^V}$) for the decay rate (6.12) (resp. (6.14)) follows from the general theory (see Theorem 6.8) and the estimates implied by Theorem 4.2. \square

Remark 6.15. *Note that the operator H_m^V is certainly defined on a dense subspace of $\mathcal{L}^{2,-s}(\mathbb{R}^3, \mathbb{C}^4)$. If it is closable and the graph-norm of its closure is equivalent to the norm $\mathcal{H}^{1,-s}(\mathbb{R}^3, \mathbb{C}^4)$, then the general theory (as in the case of the free operator, see Equation (6.11)) implies that the limits (6.13) are actually obtained in the uniform operator topology of $B(\mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{H}^{1,-s}(\mathbb{R}^3, \mathbb{C}^4))$.*

6.2.1. The discreteness and finiteness of eigenvalues. Our next goal is to identify the set $\Sigma_{H_m}^V$ with the (discrete) set of embedded eigenvalues. The abstract framework that enables us to do that was introduced in Subsubsection 6.1.1. We shall need to replace the short-range condition (6.12) by the stronger decay condition (6.14) on the potential V , even when dealing with spectral intervals that do not include the thresholds at $\pm m$.

Theorem 6.16. *Let*

$$U = \mathbb{R} \setminus [-m, m].$$

Suppose that the potential V satisfies the following decay condition:

$$(6.15) \quad |V(x)| \leq C(1 + |x|)^{-(2+\varepsilon)}, \quad \varepsilon > 0, \quad x \in \mathbb{R}^3.$$

Let $\Sigma_{H_m}^V$ be as in Theorem 6.14. Then

$$(6.16) \quad \Sigma_{H_m}^V = \sigma_p(H_m^V) = \text{the set of eigenvalues in } U,$$

and this set is discrete, with all eigenvalues of finite multiplicity.

Proof. In view of Theorem 6.11, we need to verify that the regularity condition (6.8) is satisfied under the decay assumption (6.15).

The spectral derivative is given by Equation (4.6) (assuming $\lambda > m$):

$$(6.17) \quad \langle A_m(\lambda)f, f \rangle = \frac{\lambda}{\sqrt{\lambda^2 - m^2}} \int_{|\xi|=\sqrt{\lambda^2 - m^2}} |(\mathcal{G}_m f)_+(\xi)|^2 d\Sigma_{\sqrt{\lambda^2 - m^2}}.$$

Let $\frac{1}{2} < s < \frac{1+\varepsilon}{2}$.

Suppose that $\phi_\mu \in \mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^4)$ is such that $\phi_\mu = -VR_m^\pm(\mu)\phi_\mu$, $\mu > m$.

Thus $\langle A_m(\mu)\phi_\mu, \phi_\mu \rangle = 0$, so that $(\mathcal{G}_m \phi_\mu)_+(\xi) = 0$ for all ξ on the sphere $|\xi| = \sqrt{\mu^2 - m^2}$. We need to show that

$$(6.18) \quad \langle A_m(\lambda)\phi_\mu, \phi_\mu \rangle \leq C|\lambda - \mu|^{1+\varepsilon}.$$

We note that by Theorem 4.2 we have $R_m^\pm(\mu)\phi_\mu \in \mathcal{H}^{1,-s}(\mathbb{R}^3, \mathbb{C}^4)$. Therefore (6.15) implies that $\phi_\mu = -VR_m^\pm(\mu)\phi_\mu \in \mathcal{L}^{2,-s+2+\varepsilon}(\mathbb{R}^3, \mathbb{C}^4)$. It follows by (2.10) that $\mathcal{G}_m \phi_\mu \in \mathcal{H}^\theta$, where $\theta > 2 + \varepsilon - \frac{1+\varepsilon}{2} = \frac{3}{2} + \frac{1}{2}\varepsilon$.

In particular, the trace of $\nabla_\xi \mathcal{G}_m \phi_\mu(\xi)$ on the sphere $|\xi| = \sqrt{\lambda^2 - m^2}$ is Hölder continuous (for λ near μ). From Equation (6.17) we infer that $\frac{d}{d\lambda} \langle A_m(\lambda)\phi_\mu, \phi_\mu \rangle$ is Hölder continuous in a neighborhood of μ . Since the nonnegative function $\langle A_m(\lambda)\phi_\mu, \phi_\mu \rangle$ has a minimum at $\lambda = \mu$, its derivative vanishes there, which yields (6.18) in view of the Hölder continuity. \square

Finally we treat the case of the full line (and in particular the thresholds at $\lambda = \pm m$ are included). This is related to the second part of Theorem 6.14. In fact, the following theorem can be viewed as the “perturbed” version of the second part of Theorem 4.2.

Theorem 6.17. *Suppose that the potential V satisfies the following decay condition:*

$$(6.19) \quad |V(x)| \leq C(1 + |x|)^{-(3+\varepsilon)}, \quad \varepsilon > 0, \quad x \in \mathbb{R}^3.$$

Let $\widetilde{\Sigma_{H_m}^V}$ be as in Theorem 6.14. Then

$$(6.20) \quad \widetilde{\Sigma_{H_m}^V} = \sigma_p(H_m^V) = \text{the set of eigenvalues in } \mathbb{R},$$

and this set is discrete, with all eigenvalues of finite multiplicity.

Proof. We follow the line of proof of Theorem 6.16, subject to modifications needed due to the fact that we are now looking at a full neighborhood of the thresholds.

Without loss of generality let us consider a neighborhood of $\lambda = m$.

Let $1 < s < \frac{2+\varepsilon}{2}$.

According to the second part of Theorem 6.14 the limits (6.13) exist in $\mathbb{R} \setminus \Sigma_{H_m}^V$.

Suppose that $\phi_\mu \in \mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^4)$ is such that $\phi_\mu = -VR_m^\pm(\mu)\phi_\mu$, where $\mu \in (m - \eta, m + \eta)$, for some small $\eta > 0$.

Thus $\langle A_m(\mu)\phi_\mu, \phi_\mu \rangle = 0$. For $\mu \geq m$ this means that $(\mathcal{G}_m \phi_\mu)_+(\xi) = 0$ for all ξ on the sphere $|\xi| = \sqrt{\mu^2 - m^2}$.

In view of Theorem 6.11, we need to verify that the regularity condition (6.8) is satisfied under the decay assumption (6.19).

We need to show that, for some $C > 0$ depending only on η ,

$$(6.21) \quad \langle A_m(\lambda)\phi_\mu, \phi_\mu \rangle \leq C|\lambda - \mu|^{1+\varepsilon}, \quad \lambda, \mu \in (m - \eta, m + \eta).$$

Remark that $\langle A_m(\lambda)\phi_\mu, \phi_\mu \rangle = 0$ for $\lambda \in [-m, m]$.

We note that by Theorem 4.2 we have $R_m^\pm(\mu)\phi_\mu \in \mathcal{H}^{1,-s}(\mathbb{R}^3, \mathbb{C}^4)$. Therefore (6.19) implies that $\phi_\mu = -VR_m^\pm(\mu)\phi_\mu \in \mathcal{L}^{2,-s+3+\varepsilon}(\mathbb{R}^3, \mathbb{C}^4)$. It follows by (2.10) that $\mathcal{G}_m\phi_\mu \in \mathcal{H}^\theta$, where $\theta > 3 + \varepsilon - \frac{2+\varepsilon}{2} = 2 + \frac{1}{2}\varepsilon$.

In particular, the trace of $\nabla_\xi \mathcal{G}_m\phi_\mu(\xi)$ on the sphere $|\xi| = \sqrt{\lambda^2 - m^2}$ can be estimated (for λ near μ). In view of the trace Lemma 3.1 and Equation (6.17) we infer that, for $\lambda \in [m, m + \eta)$, the derivative $\frac{d}{d\lambda} \langle A_m(\lambda)\phi_\mu, \phi_\mu \rangle$ is Hölder continuous and

$$(6.22) \quad \left| \frac{d}{d\lambda} \langle A_m(\lambda)\phi_\mu, \phi_\mu \rangle \right| \leq C \left(\sqrt{\lambda^2 - m^2} \right)^{2+\varepsilon-1-1} = C(\lambda^2 - m^2)^{\frac{\varepsilon}{2}}.$$

Now

$$\langle A_m(\mu)\phi_\mu, \phi_\mu \rangle = \frac{d}{d\lambda} \langle A_m(\lambda)\phi_\mu, \phi_\mu \rangle \Big|_{\lambda=\mu} = 0,$$

so that (6.21) follows readily, as in the conclusion of the proof of Theorem 6.16. \square

Corollary 6.18.

$$(6.23) \quad H_m^V = H_m + V(x),$$

where H_m is the free Dirac operator given in (2.2) and the potential $V(x)$ is a Hermitian 4×4 matrix satisfying the decay condition (6.19).

Then H_m^V has at most a discrete sequence of eigenvalues in \mathbb{R} . In particular, it has at most a finite number of eigenvalues in the “gap” $(-m, m)$.

Remark 6.19. Using a stronger decay hypothesis (typically $|V(x)| \leq C(1+|x|)^{-5-\varepsilon}$) it was shown in [12, Proposition 2.3] that there are no eigenvalues of H_m^V near the thresholds at $\pm m$.

More generally, Cojuhari [16] established the finiteness of the number of eigenvalues in the gap under the weaker condition (6.15) (which in our study served only for the discreteness of embedded eigenvalues).

Remark 6.20. For a class of potentials, including the Coulomb potential, it was shown [45, Theorem 4.21] that there are actually no embedded eigenvalues in the essential spectrum $\mathbb{R} \setminus (-m, m)$.

On the other hand, for the special potential $V(x) = \gamma/|x|$, $|\gamma| \leq \frac{\sqrt{3}}{2}$, it is known ([2, Section 3.1], [21]) that there is an infinite sequence of eigenvalues in the gap $(-m, m)$.

7. GLOBAL SPACETIME ESTIMATES –FREE DIRAC and MAXWELL OPERATORS

As in the case of the spectral study above, we deal first with spacetime estimates for the free (“unperturbed”) Dirac and Maxwell operators. In the case of the latter we derive spacetime estimates for the “TE-modes” and “TM-modes”. We are not aware of any analogous spacetime estimates for the Maxwell equations. We mention also the case of spacetime estimates for generalized wave equations in [4, 5].

7.1. SPACETIME ESTIMATES FOR THE FREE DIRAC OPERATOR.

The free Dirac operator was defined in (2.2). We consider the unitary group associated with it

$$(7.1) \quad iu_t = H_m u, \quad t \in \mathbb{R},$$

subject to the initial condition

$$(7.2) \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}^3,$$

where u, u_0 are \mathbb{C}^4 -valued functions.

We refer to [19, 23] for Strichartz and decay estimates for the Dirac operator with magnetic potentials. Here we formulate a global spacetime estimate in the weighted- L^2 framework.

Theorem 7.1. *Let $m > 0, s > 1$, and let $u(x, t)$ be the solution to (7.1)-(7.2). There exists a constant $C = C_{s,m} > 0$, such that*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^3} (1 + |x|^2)^{-s} |u(x, t)|^2 dx dt \leq C \|u_0\|_0^2.$$

Proof. The solution $u(x, t)$ can be represented as

$$u = \exp(-itH_m)u_0.$$

In the proof we find it clearer to use the separate notations $(\cdot, \cdot), [\cdot, \cdot]$ for the scalar products in $\mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^4), \mathcal{L}^2(\mathbb{R}^4, \mathbb{C}^4)$, respectively.

We denote by

$$\tilde{v}(x, \tau) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} v(x, t) e^{-it\tau} dt$$

the partial Fourier transform with respect to t .

The spectral derivative A_m was defined in equation (4.7).

To estimate $u(x, t)$ we use a duality argument. Take $w(x, t) \in C_0^\infty(\mathbb{R}^4, \mathbb{C}^4)$. Then,

$$\begin{aligned} [u, w] &= \int_{-\infty}^{\infty} dt \int_{\mathbb{R}^3} \langle (e^{-itH_m} u_0)(x, t), w(x, t) \rangle_{\mathbb{C}^4} dx \\ &= \int_{-\infty}^{\infty} \langle A_m(\lambda) u_0, \int_{-\infty}^{\infty} e^{-it\lambda} w(\cdot, t) dt \rangle d\lambda, \\ &= (2\pi)^{1/2} \int_{-\infty}^{\infty} \langle A_m(\lambda) u_0, \tilde{w}(\cdot, \lambda) \rangle d\lambda. \end{aligned}$$

where in the last two terms \langle, \rangle is the $(\mathcal{L}^{2,-s}(\mathbb{R}^3, \mathbb{C}^4), \mathcal{L}^{2,s}(\mathbb{R}^3, \mathbb{C}^4))$ pairing.

We note that by the spectral theorem

$$(7.3) \quad \int_{-\infty}^{\infty} \langle A_m(\lambda) f, f \rangle d\lambda = \|f\|_0^2.$$

The Cauchy-Schwarz inequality yields

$$\begin{aligned}
& |[u, w]| \\
& \leq (2\pi)^{1/2} \left(\int_{-\infty}^{\infty} \langle A_m(\lambda) u_0, u_0 \rangle d\lambda \right)^{1/2} \cdot \left(\int_{-\infty}^{\infty} \langle A_m(\lambda) \tilde{w}(\cdot, \lambda), \tilde{w}(\cdot, \lambda) \rangle d\lambda \right)^{1/2} \\
& = (2\pi)^{1/2} \|u_0\|_2 \cdot \left(\int_{-\infty}^{\infty} \langle A_m(\lambda) \tilde{w}(\cdot, \lambda), \tilde{w}(\cdot, \lambda) \rangle d\lambda \right)^{1/2}.
\end{aligned}$$

and recalling (4.7) we obtain

$$\begin{aligned}
& |[u, w]| \\
& \leq (2\pi)^{1/2} C \|u_0\|_0 \cdot \left(\int_{|\lambda| > m} \min\left(\frac{|\lambda|}{\sqrt{\lambda^2 - m^2}}, |\lambda|(\lambda^2 - m^2)^{s-1}\right) \cdot \|\tilde{w}(\cdot, \lambda)\|_{0,s}^2 d\lambda \right)^{\frac{1}{2}}.
\end{aligned}$$

By the Plancherel theorem, using $s > 1$,

$$|[u, w]| \leq (2\pi)^{1/2} C \|u_0\|_0 \left(\int_{\mathbb{R}} \|w(\cdot, t)\|_{0,s}^2 dt \right)^{1/2}.$$

Let $f(x, t) \in C_0^\infty(\mathbb{R}^4, \mathbb{C}^4)$, and take $w(x, t) = (1 + |x|^2)^{-\frac{s}{2}} f(x, t)$, so that

$$\int_{\mathbb{R}} \|w(\cdot, t)\|_{0,s}^2 dt = \|f\|_0^2.$$

We infer that

$$|[(1 + |x|^2)^{-\frac{s}{2}} u, f]| \leq (2\pi)^{1/2} C \cdot \|u_0\|_0 \cdot \|f\|_0,$$

which concludes the proof of the theorem. \square

Remark 7.2. Note that we need only (for the unperturbed case) $s > 1$, including the propagation near the threshold. This is to be compared with Theorem 1.1 and Proposition 2.1 in [12], where $s > \frac{5}{2}$, is assumed (including a potential).

7.2. SPACETIME ESTIMATES FOR THE FREE MAXWELL OPERATOR.

The free Maxwell operator L_{\maxwell} was introduced in (2.14).

We consider the unitary group associated with the operator,

$$(7.4) \quad iu_t = L_{\maxwell} u, \quad t \in \mathbb{R},$$

subject to the initial condition

$$(7.5) \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}^3,$$

where u, u_0 are \mathbb{C}^6 -valued functions.

We further assume that

$$(7.6) \quad u_0 \in (I - \mathcal{P})\mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^6),$$

where \mathcal{P} is the orthogonal projection on $\ker(L_{\maxwell})$ (see Equation (4.25)).

In view of Remark 2.6 the orthogonality condition means that if $u_0(x) = \begin{pmatrix} E_0(x) \\ B_0(x) \end{pmatrix}$ (with values in \mathbb{C}^6) then Fourier transform $\widehat{\begin{pmatrix} E_0(\xi) \\ B_0(\xi) \end{pmatrix}} \in \text{span}\{\Upsilon_{\pm}\}$, for every $0 \neq \xi \in \mathbb{R}^3$. In particular

$$\langle \widehat{E_0}(\xi), \xi \rangle_{\mathbb{C}^3} = 0, \quad \langle \widehat{B_0}(\xi), \xi \rangle_{\mathbb{C}^3} = 0, \quad \xi \in \mathbb{R}^3 \setminus \{0\},$$

so that E_0 (resp. B_0) is a “TE-mode” (resp. “TM-mode”).

Theorem 7.3. *Let $s > \frac{1}{2}$, and let $u(x, t)$ be the solution to (7.4)-(7.5)-(7.6). There exists a constant $C = C_s > 0$, such that*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^3} (1 + |x|^2)^{-s} |u(x, t)|^2 dx dt \leq C \|u_0\|_0^2.$$

Proof. The solution $u(x, t)$ can be represented as

$$u = \exp(-itL_{\text{maxwell}})u_0.$$

As in the proof of Theorem 7.1, we find it clearer to use the separate notations $(\cdot, \cdot), [\cdot, \cdot]$ for the scalar products in $\mathcal{L}^2(\mathbb{R}^3, \mathbb{C}^6), \mathcal{L}^2(\mathbb{R}^4, \mathbb{C}^6)$, respectively.

We denote by

$$\tilde{v}(x, \tau) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} v(x, t) e^{-it\tau} dt$$

the partial Fourier transform with respect to t .

The spectral derivative \tilde{A} was defined in equation (4.28).

To estimate $u(x, t)$ we use a duality argument. Take $w(x, t) \in C_0^\infty(\mathbb{R}^4, \mathbb{C}^6)$. Then,

$$\begin{aligned} [u, w] &= \int_{-\infty}^{\infty} dt \int_{\mathbb{R}^3} \langle (e^{-itL_{\text{maxwell}}} u_0)(x, t), w(x, t) \rangle_{\mathbb{C}^6} dx \\ &= \int_{-\infty}^{\infty} \langle \tilde{A}(\lambda) u_0, \int_{-\infty}^{\infty} e^{-it\lambda} w(\cdot, t) dt \rangle d\lambda \\ &= (2\pi)^{1/2} \int_{-\infty}^{\infty} \langle \tilde{A}(\lambda) u_0, \tilde{w}(\cdot, \lambda) \rangle d\lambda. \end{aligned}$$

We note that by the spectral theorem

$$(7.7) \quad \int_{-\infty}^{\infty} \langle \tilde{A}(\lambda) f, f \rangle d\lambda = \|f\|_0^2.$$

The Cauchy-Schwarz inequality yields

$$\begin{aligned}
& |[u, w]| \\
& \leq (2\pi)^{1/2} \left(\int_{-\infty}^{\infty} \langle \tilde{A}(\lambda) u_0, u_0 \rangle d\lambda \right)^{1/2} \cdot \left(\int_{-\infty}^{\infty} \langle \tilde{A}(\lambda) \tilde{w}(\cdot, \lambda), \tilde{w}(\cdot, \lambda) \rangle d\lambda \right)^{1/2} \\
& = (2\pi)^{1/2} \|u_0\|_2 \cdot \left(\int_{-\infty}^{\infty} \langle \tilde{A}(\lambda) \tilde{w}(\cdot, \lambda), \tilde{w}(\cdot, \lambda) \rangle d\lambda \right)^{1/2}.
\end{aligned}$$

and recalling (4.28) we obtain

$$\begin{aligned}
& |[u, w]| \\
& \leq (2\pi)^{1/2} C \|u_0\|_0 \cdot \left(\int_{-\infty}^{\infty} \min(1, |\lambda|^{2s-1}) \cdot \|\tilde{w}(\cdot, \lambda)\|_{0,s}^2 d\lambda \right)^{\frac{1}{2}}.
\end{aligned}$$

By the Plancherel theorem, using $s > \frac{1}{2}$,

$$|[u, w]| \leq (2\pi)^{1/2} C \|u_0\|_0 \left(\int_{\mathbb{R}} \|w(\cdot, t)\|_{0,s}^2 dt \right)^{1/2}.$$

Let $f(x, t) \in C_0^\infty(\mathbb{R}^4, \mathbb{C}^6)$, and take $w(x, t) = (1 + |x|^2)^{-\frac{s}{2}} f(x, t)$, so that

$$\int_{\mathbb{R}} \|w(\cdot, t)\|_{0,s}^2 dt = \|f\|_0^2.$$

We infer that

$$|[(1 + |x|^2)^{-\frac{s}{2}} u, f]| \leq (2\pi)^{1/2} C \cdot \|u_0\|_0 \cdot \|f\|_0,$$

which concludes the proof of the theorem. \square

8. GLOBAL SPACETIME ESTIMATES –STRONGLY PROPAGATIVE HOMOGENEOUS SYSTEMS

We now address the spacetime decay estimates for *homogeneous* strongly propagative operators $L_0 = \sum_{j=1}^n M_j^0 D_j$ (see (1.2)), for which the LAP was stated in Theorem 5.12. Recall that both the Dirac (zero mass) and Maxwell systems belong to this class, but the singular set Z (defined in (5.2)) is empty in both cases.

We consider the associated unitary group

$$(8.1) \quad iu_t = L_{0,hom} u, \quad t \in \mathbb{R},$$

subject to the initial condition

$$(8.2) \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n,$$

where u, u_0 are \mathbb{C}^K -valued functions.

We further assume that the initial function is orthogonal to the “stationary waves” of the system, namely,

$$(8.3) \quad u_0 \in (I - \mathcal{P}) \mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K),$$

where \mathcal{P} is the orthogonal projection on $\ker(L_{0,hom})$.

Clearly the solution $u(x, t) = e^{-itL_{0, hom}}u_0$ satisfies $u(\cdot, t) \in (I - \mathcal{P})\mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K)$, $t \in \mathbb{R}$.

It is represented by the Fourier integral operator

$$(8.4) \quad u(x, t) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-it \sum_{j=1}^n M_j^0 \xi_j} e^{-i\langle \xi, x \rangle} \widehat{u_0}(\xi) d\xi.$$

However, this explicit expression does not easily lend itself to asymptotic analysis by classical methods of geometric optics; the non-commutativity of the matrices $\{M_j^0\}_{j=1}^n$ requires a very detailed study of the algebraic structure of the eigenvalue manifolds and their intersections. Hence various restrictive hypotheses need to be imposed, even in the case of constant coefficients.

There is extensive literature concerning the (large-time) asymptotic behavior of solutions of such systems and their perturbations, using primarily the geometric optics approach. It is beyond the scope of the present paper to present an exhaustive account of these works, and we refer to [38] and references therein. In particular it is assumed there that the system is uniformly propagative [38, Assumption (1.3)]. The rate of decay in time of the L^2 norm of scattered solutions in balls is shown [38, Theorem 1] to be $O((\log t)^{-1})$. Confining to the constant coefficient case, the global estimate (8.13) obtained below yields a faster rate of decay (in integral sense).

The presence of the singular set Z makes our spacetime estimate somewhat more delicate. Recall Definition 5.7 of the closed subspace $\Upsilon_Z^s \subseteq \mathcal{L}^{2,s}(\mathbb{R}^n, \mathbb{C}^K)$, and note also Remark 5.9. We now define the closed subspace $\Upsilon_Z \subseteq \mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K)$ as the inverse Fourier transform of $(I - \Delta)^{\frac{s}{2}} \widehat{\Upsilon}_Z^s$, namely

$$(8.5) \quad \Upsilon_Z = (1 + |x|^2)^{\frac{s}{2}} \Upsilon_Z^s = \{g / g(x) = (1 + |x|^2)^{\frac{s}{2}} h(x) \text{ for some } h \in \Upsilon_Z^s\}.$$

Let $\mathcal{E} : \mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K) \hookrightarrow \Upsilon_Z$ be the orthogonal projection unto this subspace.

Theorem 8.1. *Assume that $L_{0, hom}$ is strongly propagative. Let $s > \frac{1}{2}$, and let $u(x, t)$ be the solution to (8.1)-(8.2)-(8.3).*

Define $u_Z(x, t)$, $t \in \mathbb{R}$, by

$$u_Z(x, t) = \mathcal{E}[(1 + |x|^2)^{-\frac{s}{2}} u(x, t)].$$

Then there exists a constant $C = C_{s,n} > 0$, such that

$$(8.6) \quad \int_{\mathbb{R}} \int_{\mathbb{R}^n} |u_Z(x, t)|^2 dx dt \leq C \|u_0\|_0^2.$$

Remark 8.2. *Note that $u_Z(x, t) \in \Upsilon_Z$ implies $(1 + |x|^2)^{-\frac{s}{2}} u_Z(x, t) \in \Upsilon_Z^s$ for every $t \in \mathbb{R}$. Fix $t = t_0 \in \mathbb{R}$. In light of Definition 5.7, given $\eta > 0$ there exists $\psi_\eta \in \mathcal{L}^{2,s}(\mathbb{R}^n, \mathbb{C}^K)$ so that $\widehat{\psi}_\eta \in C_0^\infty(\mathbb{R}^n \setminus \overline{Z}, \mathbb{C}^K)$ and*

$$\int_{\mathbb{R}^n} (1 + |x|^2)^s |(1 + |x|^2)^{-\frac{s}{2}} u_Z(x, t_0) - \psi_\eta(x)|^2 dx \leq \eta^2.$$

Thus

$$\int_{\mathbb{R}^n} |u_Z(x, t_0) - (1 + |x|^2)^{\frac{s}{2}} \psi_\eta(x)|^2 dx \leq \eta^2,$$

and by the Plancherel theorem

$$\int_{\mathbb{R}^n} |\widehat{u_Z}(\xi, t_0) - (I - \Delta)^{\frac{s}{2}} \widehat{\psi}_\eta(\xi)|^2 d\xi \leq \eta^2.$$

The function $u_Z(x, t_0)$ is therefore the part of $u(x, t_0)$ whose Fourier transform is in the closure (in $\mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K)$) of the range of $(I - \Delta)^{\frac{s}{2}}$ acting on smooth functions “supported away” from the singular set \overline{Z} .

Proof of theorem: By density (in $\mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K)$) we may assume

$$(8.7) \quad \widehat{u_0}(\xi) \in C_0^\infty(\mathbb{R}^n \setminus \overline{Z}, \mathbb{C}^K).$$

The solution $u(x, t)$ can be represented as

$$u = \exp(-itL_{0,hom})u_0.$$

In particular, $\widehat{u}(\cdot, t) \in C_0^\infty(\mathbb{R}^n \setminus \overline{Z}, \mathbb{C}^K)$ for all $t \in \mathbb{R}$.

As in the proofs of Theorems 7.1 and 7.3, we find it clearer to use the separate notations $(\cdot, \cdot), [\cdot, \cdot]$ for the scalar products in $\mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K), \mathcal{L}^2(\mathbb{R}^{n+1}, \mathbb{C}^K)$, respectively.

We denote by

$$\tilde{\gamma}(x, \tau) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \gamma(x, t) e^{-it\tau} dt$$

the partial Fourier transform with respect to t .

The spectral derivative $\widetilde{A_{0,hom}}(\lambda) \in B(\Upsilon_Z^s, \mathcal{L}^{2,-s}(\mathbb{R}^n, \mathbb{C}^K))$, $s > \frac{1}{2}$, was defined in Equation (5.23).

To estimate $u(x, t)$ we use a duality argument, as in the cases of the Dirac and Maxwell systems. Take $w(x, t) \in C^\infty(\mathbb{R}^{n+1}, \mathbb{C}^K)$, so that

$$(8.8) \quad \widehat{w}(\xi, t) \in C_0^\infty((\mathbb{R}^n \setminus \overline{Z}) \times \mathbb{R}, \mathbb{C}^K).$$

Then,

$$\begin{aligned} [u, w] &= \int_{-\infty}^{\infty} dt \int_{\mathbb{R}^3} \langle (e^{-itL_{0,hom}} u_0)(x, t), w(x, t) \rangle_{\mathbb{C}^K} dx \\ (8.9) \quad &= \int_{-\infty}^{\infty} \langle \widetilde{A_{0,hom}}(\lambda) u_0, \int_{-\infty}^{\infty} e^{-it\lambda} w(\cdot, t) dt \rangle d\lambda \\ &= (2\pi)^{1/2} \int_{-\infty}^{\infty} \langle \widetilde{A_{0,hom}}(\lambda) u_0, \tilde{w}(\cdot, \lambda) \rangle d\lambda. \end{aligned}$$

We note that by the spectral theorem

$$(8.10) \quad \int_{-\infty}^{\infty} \langle \widetilde{A_{0,hom}}(\lambda) u_0, u_0 \rangle d\lambda = \|u_0\|_0^2.$$

The Cauchy-Schwarz inequality yields

$$\begin{aligned}
& |[u, w]| \\
& \leq (2\pi)^{1/2} \left(\int_{-\infty}^{\infty} \langle \widetilde{A_{0,hom}}(\lambda) u_0, u_0 \rangle d\lambda \right)^{1/2} \cdot \left(\int_{-\infty}^{\infty} \langle \widetilde{A_{0,hom}}(\lambda) \tilde{w}(\cdot, \lambda), \tilde{w}(\cdot, \lambda) \rangle d\lambda \right)^{1/2} \\
& = (2\pi)^{1/2} \|u_0\|_2 \cdot \left(\int_{-\infty}^{\infty} \langle \widetilde{A_{0,hom}}(\lambda) \tilde{w}(\cdot, \lambda), \tilde{w}(\cdot, \lambda) \rangle d\lambda \right)^{1/2}.
\end{aligned}$$

and recalling (5.24) and the fact that the norm of Υ_Z^s is the $\mathcal{L}^{2,s}$ norm, it follows that

$$|[u, w]| \leq (2\pi)^{1/2} C \|u_0\|_0 \cdot \int_{-\infty}^{\infty} \|\tilde{w}(\cdot, \lambda)\|_{0,s}^2 d\lambda.$$

Invoking the Plancherel theorem, this estimate leads to

$$(8.11) \quad |[u, w]| \leq (2\pi)^{1/2} C \|u_0\|_0 \left(\int_{\mathbb{R}} \|w(\cdot, t)\|_{0,s}^2 dt \right)^{1/2}.$$

The estimate (8.11) was obtained under the condition (8.8), and by closure it holds for all $w(x, t) \in \mathcal{L}^2(\mathbb{R}_t, \Upsilon_Z^s)$.

Now note that $v(x, t) = (1 + |x|^2)^{\frac{s}{2}} w(x, t) \in \mathcal{L}^2(\mathbb{R}_t, \Upsilon_Z)$.

The estimate (8.11) can be rewritten as

$$(8.12) \quad |[(1 + |x|^2)^{-\frac{s}{2}} u, v]| \leq (2\pi)^{1/2} C \|u_0\|_0 \left(\int_{\mathbb{R}} \|v(\cdot, t)\|_0^2 dt \right)^{1/2}, \quad \forall v(\cdot, t) \in \mathcal{L}^2(\mathbb{R}_t, \Upsilon_Z),$$

which clearly entails (8.6). □

Corollary 8.3. *Assume that $L_{0,hom}$ is uniformly propagative (Definition 5.3). Let $s > \frac{1}{2}$, and let $u(x, t)$ be the solution to (8.1)-(8.2)-(8.3).*

Then there exists a constant $C = C_{s,n} > 0$, such that

$$(8.13) \quad \int_{\mathbb{R}} \int_{\mathbb{R}^n} (1 + |x|^2)^{-s} |u(x, t)|^2 dx dt \leq C \|u_0\|_0^2.$$

Proof. In this case by definition $Z = \emptyset$ so that $u_Z(x, t) = (1 + |x|^2)^{-\frac{s}{2}} u(x, t)$ in Theorem 8.1. □

Remark 8.4. *Note that if Conjecture 5.10 is shown to be true, then $\Upsilon_Z = \mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K)$ and $u_Z = (1 + |x|^2)^{-\frac{s}{2}} u(x, t)$ for any strongly propagative system.*

The result in Theorem 8.1 involved the “projected” function u_Z because we let the initial function u_0 be any function in $\mathcal{L}^2(\mathbb{R}^n, \mathbb{C}^K)$. However, if we restrict the support of u_0 away from \bar{Z} we can get an improved estimate, but with a constant that depends on the support of u_0 as follows.

Theorem 8.5. *Assume that $L_{0,hom}$ is strongly propagative. Let $s > \frac{1}{2}$, and let $u(x, t)$ be the solution to (8.1)-(8.2)-(8.3). Let $\mathfrak{K} \Subset \mathbb{R}^n \setminus \overline{Z}$ be a compact set and assume further that $\text{supp}[\widehat{u}_0] \subseteq \mathfrak{K}$.*

Then there exists a constant $C = C_{s,n,\mathfrak{K}} > 0$ such that

$$(8.14) \quad \int_{\mathbb{R}} \int_{\mathbb{R}^n} (1 + |x|^2)^{-s} |u(x, t)|^2 dx dt \leq C \|u_0\|_0^2.$$

Proof. We repeat the proof of Theorem 8.1. However instead of (8.8) we take any $\widehat{w}(\xi, t) \in C_0^\infty((\mathbb{R}^{n+1}, \mathbb{C}^K))$. Now let $\chi(\xi) \in C_0^\infty(\mathbb{R}^n \setminus \overline{Z})$ so that $\chi \equiv 1$ on a compact neighborhood of \mathfrak{K} . Define $\widehat{w}_1(\xi, t) = \chi(\xi) \widehat{w}(\xi, t)$ and let $w_1(x, t)$ be the inverse Fourier transform. Clearly $w_1(x, t) \in \mathcal{L}^2(\mathbb{R}_t, \Upsilon_Z^s)$ and with the notation of (8.9) we have

$$[u, w_1] = [u, w].$$

The proof now proceeds as before and is completed by noting that with a constant $C = C_{s,n,\mathfrak{K}} > 0$

$$\|w_1(\cdot, t)\|_{0,s} \leq C \|w(\cdot, t)\|_{0,s}, \quad t \in \mathbb{R}.$$

□

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MATANIA BEN-ARTZI: INSTITUTE OF MATHEMATICS, THE HEBREW UNIVERSITY, JERUSALEM 91904, ISRAEL

Email address: `mbartzi@math.huji.ac.il`

TOMIO UMEDA: DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF HYOGO, HIMEJI 671-2201, JAPAN

Email address: `umeda@sci.u-hyogo.ac.jp`