GRP –A DIRECT GODUNOV EXTENSION

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Abstract. The classical paper [40] by S. K. Godunov had a revolutionary effect on the field of numerical simulations of compressible fluid flows. Yet, for some twenty years after its publication, it was mostly used by the mathematical community in the Soviet Union. The seminal paper of van Leer [85] has inaugurated the period of universal interest in high-resolution extensions of Godunov's scheme. The first step to take was to modify the (locally) self-similar solution to the Riemann Problem (at discontinuities) by allowing piecewise-polynomial (rather than piecewise-constant) initial data. The GRP (Generalized Riemann Problem) analysis [6] provided analytical solutions (for piecewise-linear data) that could be readily implemented in a high-resolution robust code. This review paper focuses on the evolution of some mathematical aspects of this method. It addresses the three concepts of *consistency, stability and convergence* in the context of compact finite difference (or finite volume) schemes for systems of nonlinear hyperbolic conservation laws. The treatment utilizes the framework of "balance laws", a common viewpoint in relevant physical conservation laws. The first significant observation is that under very mild conditions a weak solution is indeed a solution to the balance law (obtained by a formal application of the Gauss-Green formula). Since high-resolution schemes require the computation of several quantities per mesh cell (e.g., slopes), the notion of "consistency" must be extended to this framework. A combination of consistency hypothesis with stability of the scheme leads to a suitable convergence theorem, generalizing the classical convergence theorem of Lax and Wendroff [52]. Finally, the limit functions are shown to be entropy solutions by using a notion of "Godunov compatibility", which serves as a substitute to the entropy condition.

1. INTRODUCTION

The seminal paper [40] by S. K. Godunov had a revolutionary effect on the field of numerical simulations of compressible fluid flows. However the initial evolution of this effect was quite slow, when compared to other parallel advances related to the numerical simulations of fluid flows (e.g., vortex methods or finite elements). Eight years after Godunov's publication, the classical book of R.D. Richtmyer and K.W. Morton characterized his paper as follows [77, Section 12.15]:

"In 1959, Godunov described an ingenious method for one-dimensional problems with shocks." Yet, later on in the same section, they add the comment: "The method appears to have been extensively used in the Soviet Union."

We begin this review by a (brief and not comprehensive) account of the developments (spanning the 1960's and 1970's) leading from the Godunov scheme to the

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MUSCL scheme. The latter has inspired an extensive research in high-resolution finite-difference simulations of solutions to nonlinear hyperbolic conservation laws, with emphasis on compressible fluid flows.

Then we concentrate on the GRP (Generalized Riemann Problem) method, which is a direct analytical extension of the Godunov scheme. Our aim is to discuss mathematical aspects of this method. More specifically, we concentrate on three issues:

- Rigorous statement of the **Lipschitz continuity of fluxes,** for any weak solution. This supports an underlying principle of the Godunov scheme, namely, designating the flux as a main object for numerical approximation. On the way, it enables the formulation of the conservation law as a **balance law.**
- Extension of the concept of **consistency**, thus making it applicable to a wide array of schemes, allowing in particular for high order, compact schemes, and taking into account the presence of discontinuities. In particular, our analysis shows that the Godunov scheme is never first-order; it's either of "zero order" or "infinite order", depending on the space of admissible approximations.
- Linking the consistency of the discrete solution to the question of its **convergence** to the solution of the balance law. It involves a renewed consideration of the classical Lax-Wendroff theorem [52].

1.1. FROM GODUNOV TO MUSCL.

In subsequent years quite a few finite-difference schemes have been proposed for the numerical simulation of nonlinear hyperbolic conservation laws. They were all (with the exception of the Lax-Wendroff scheme [52]) basically first-order. Ten years after the publication of the book by Richtmyer and Morton, the well-known survey paper by Sod [81] addressed the most prominent and widely used schemes of the time. In order to compare their performances (including computational efficiency) he proposed a test case in the form of a "shock-tube problem". This particular test has been extensively used ever since and, almost fifty years on, is still one of the most popular test cases in current papers in this field.

The results of all the considered schemes showed a great amount of smearing of discontinuities (shocks, contact discontinuities and edges of rarefaction waves). Indeed, in order to get sharper results various "corrective" mechanisms had to be added, such as artificial viscosity, Harten's "artificial compression", a variety of "flux corrections" or hybrid schemes.

Sod's conclusion [81, Section 4] was "Of all the finite difference schemes tested, without the use of corrective procedures, Godunov's and Hyman's methods produced the best results".

Of all the schemes considered, only Godunov's scheme relied on the resolution of the Riemann problem (RP) as its basic tool. In fact its numerical flux is the exact flux as obtained from the Riemann solution. It should be noted that Glimm's scheme ("random choice method" by Chorin's implementation [24]), while relying on the Riemann problem, does not really belong to this category; it is not a finite-difference scheme in the sense that (by randomness) it yields different results when successively applied to the same initial data.

It is appropriate at this point to highlight the *fundamental principles embedded in Godunov's scheme*. For years to come they persisted as the "ideological basis" for a substantial part of the numerical simulations of hyperbolic conservation laws, most importantly *finite volume schemes* [35].

PRINCIPLES OF THE GODUNOV APPROACH

(1.1)
- Determining numerical fluxes by local solution of the exact system.
- The role of upwinding in computing the fluxes.
- Updating the numerical solution via a discretized balance law.

It is fair to say that at this point the whole field was waiting for the next meaningful developments in attempting to obtain a high-order true generalization of the Godunov scheme. This expectation was fulfilled, immediately following Sod's review, in the body of the seminal paper of van Leer [85]. He introduced a second-order (in space) discretization by allowing piecewise-linear (instead of piecewise-constant) approximations, linear in each grid cell with jumps (of function and slope) at cell boundaries. Of course such a change implies that the solution (for short time) at each cell-boundary ceases to be the classical self-similar solution to the local RP and leads to a more complex wave structure emanating from the singularity. To account for this extra complexity, van Leer proposed a modification of the solution by representing the deviation from self-similarity by two (curvilinear in space-time) shocks (thus admitting also rarefaction shocks). He then concluded that in this modified setting the values of pressure and velocity near the singularity (constant along each ray in the case of RP) evolve linearly in time. In Example 4.20 below the relevant details are given. In order to avoid oscillations (inherent to second-order schemes) van Leer introduced a "limiter" mechanism, effectively forcing dissipation near strong discontinuities.

Van Leer labeled the resulting scheme as **MUSCL=**" Monotonic Upstream-centered Scheme for Conservation Laws." The numerical results obtained by this scheme were much superior to all those obtained in Sod's survey [81]. Thus in one stroke the whole field of finite-difference approximations to hyperbolic conservation laws has experienced a definitive jump in terms of accuracy at discontinuities.

We note that in his review [86] van Leer mentions that some of the ideas of MUSCL were already proposed by Kolgan [47]. His paper was not known in the West and in fact, not even by the leading Soviet mathematicians of the time.

As mentioned earlier, in Sod's survey of the group of prominent schemes, only one (Godunov's) relied on the RP solution as a main building block. An immediate consequence of the introduction of MUSCL was the renewed interest in efficient solvers for the RP. A flurry of approaches to "simplified" Riemann solvers ensued: Roe's linearized form [78] as well as a variety of "approximate Riemann solvers" [20], [39, Ch.III.4], [83]. Also consult [82] for an extensive review and references.

The solution of the RP is a hallmark of the Godunov scheme, and consequently also of MUSCL, as well as subsequent developments along this line. This has led to a very fruitful debate, that is still active, some fifty years later. On one hand, from the mathematical aspect, one should invoke any available tool in resolving a given problem; It is no different from the application of PDE techniques in the study of a problem in differential geometry. On the other hand, it should be recognized that such schemes are primarily designed for use in engineering and physical problems. They should therefore be kept at a reasonable level of simplicity (for coding purposes) as well as being efficient in their use of computing resources.

1.2. **CONVERGENCE.**

In another seminal paper coming from the Soviet Union, Kružkov [49] introduced his "entropy criterion" for uniqueness of weak solutions to (scalar) hyperbolic balance laws. This had strong impact on the study of convergence of finite-difference schemes to the unique entropy solution. Suitable discrete analogs of entropy/entropy-flux pairs were proposed [26, 28, 29, 30, 33, 54, 73, 75, 79]. By the mid 1980's the main goal has been achieved; all monotone (necessarily first-order) "E-schemes", including that of Godunov, were shown to converge to the exact entropy solution, in the case of **scalar** hyperbolic conservation laws [39, Chapter III].

Concerning the question of first-order finite-difference schemes for **systems** of hyperbolic conservation laws (notably isentropic or non-isentropic fluid flows), there have been very few works to date. Of course this is largely due to the absence of theoretical tools (e.g.,total variation or L^∞ estimates), with the remarkable exception of "compensated compactness" (for 2×2 systems). We refer to [29, 30], and also to the comprehensive review [35] of the convergence properties of the closely related finite volume methods. Remark that concerning the Godunov scheme for the non-isentropic compressible inviscid flow (in one space dimension), the convergence to a weak solution remains an open problem (and the uniqueness of such a solution is also open).

Turning back to the scalar case, and in light of the computational success of the MUSCL scheme, the (mathematical) research interest turned to convergence properties of the MUSCL (and other second-order) finite-difference schemes. Generally speaking, the discrete solution is shown to satisfy an entropy inequality subject to an appropriate discrete entropy/entropy-flux pair. Also, some structural hypotheses are imposed (e.g., limiting the number of extrema). Then, when the discrete solution converges to a limit (using some compactness property), the limit is indeed an entropy solution. We refer to [12, 17, 22, 41, 42, 45, 54, 65, 68, 74, 87] and references therein for such treatments. It is interesting to recall here the following paragraph from DiPerna's paper [30]: "In the setting of the scalar conservation it remains an open problem to establish convergence of conservative finite difference schemes which are accurate to second order. Stability and convergence results have been obtained so far only for methods which are precisely accurate to first order." Recall that van Leer's original scheme reduces to that of Godunov's when all piecewise-linear approximate solutions are actually piecewise-constant. To the best of the author's knowledge, there is yet no proof for the convergence of the MUSCL scheme (in the scalar case).

1.3. **MUSCL AND BEYOND.**

In the ensuing years a wide panoply of generalizations (or modifications) of the MUSCL scheme (for systems) have been put forward. In all of them the discrete solution is a piecewise-polynomial function, with discontinuities at cell-boundaries.

The primary objective was to enable the numerical resolution of complex (multi-dimensional) compressible flows. This is still a very active line of research, with a common theme of evaluating the numerical fluxes by some sort of a direct or indirect approximate solution to the local generalized Riemann problem. Some of the most prominent methods are the ADER [84], DG [80], GRP [9], HLL [45], PPM [25], WENO [67, 80]. We mention that there have been various attempts to obtain second-order accuracy while completely circumventing the need for a Riemann solver. They are mathematically interesting but it seems that none of them could lead to a robust scheme for complex flows [42, 44, 50].

The review [86, Section 2.2] deals with high-resolution schemes obtained by "Flux-Corrected Transport"(FCT): a first-order, nonoscillatory scheme is used to obtain an intermediate approximation at the advanced time level; a correction step then removes the large dissipative error made in the first step, obtaining a solution of higher order accuracy. We refer the reader to this review also for an account of the attempts to incorporate the requirement of "Total-Variation-Diminishing"(TVD) into high resolution schemes. Observe that the design of such schemes involves, in addition to the flux evaluations, very elaborate polynomial reconstructions in grid cells.

Henceforth we concentrate on the GRP (Generalized Riemann Problem) methodology, the subject matter of this paper. The paper is organized as follows.

Section 2 introduces the GRP scheme. There have been many versions of the scheme, adapted to engineering and physical applications. A survey (necessarily not exhaustive) of some of them is given, as well as some references to comparative studies involving closely related high-resolution schemes.

Section 3 deals with the basic definition of a hyperbolic "balance law". In order to show that the classically defined weak solutions are indeed solutions to the balance law, the continuity properties of the associated fluxes need to be studied. Theorem 3.2 states that under very general conditions on weak solutions (even those not satisfying any entropy condition) the fluxes are in fact locally Lipschitz continuous.

Section 4 introduces the notion of "approximate fluxes" and their "order of consistency" (Definition 4.15). These notions rely on the order of the spaces used in the approximation (Assumption 4.7). In particular, it is shown (Corollary 4.25) that the order of consistency as introduced here conforms with the previous notion when the latter is applicable. A general definition of *finite volume schemes* (FVS) is introduced (Definition 4.22), that also depends on the order of the approximating space.

It should be emphasized that *in case of discontinuous solutions, our definition of consistency may yield different orders than commonly used.* Thus, the Godunov scheme (Example 4.18) is of infinite order when applied to piecewise-constant functions but of order zero when applied to piecewise-linear functions. Similarly, the GRP or MUSCL schemes (4.18) are only first-order consistent, while second-order consistency requires smooth solutions.

The foundational "Lax Equivalence Theorem" highlighted the close connection among the three concepts: consistency, stability and convergence. Indeed, in the context of linear evolution equations it asserts that a consistent scheme is stable if and only if it is convergent [77]. Although it was formulated for *linear* evolution equations, it played a decisive role in the development of first order finite-difference schemes, with special emphasis on hyperbolic conservation laws. The growing use of high-order schemes in this context has made it difficult to adapt the theorem in a straightforward fashion. This is particularly true for nonlinear hyperbolic conservation laws, where the presence (and formation) of discontinuities does not easily allow the examination of "consistency" by standard Taylor expansions. Furthermore, compact (high-order) schemes require the computation of several quantities per mesh cell (e.g., slopes). As we shall see, in this case many of the existing definitions of consistency are not applicable.

Section 5 deals with the convergence of the approximate solutions to the exact solution of the balance law. Recall here DiPerna's observation cited above (Subsection 1.2) stating that "it remains an open problem to establish convergence of conservative finite difference schemes which are accurate to second order. Stability and convergence results have been obtained so far only for methods which are precisely accurate to first order."

Since then, the convergence of various second-order schemes to the unique entropy solution (in the scalar case) has been established as mentioned above (Subsection 1.2). These studies treated specific schemes and employed suitable discrete entropy inequalities. In particular, they had no need to introduce a general concept of consistency as had been done in the first-order case described above.

In Theorem 5.1 the convergence of the approximate solutions to solutions of the balance law is outlined for a general class of high-order finite difference scheme, under certain boundedness conditions. The hypothesis that the scheme is consistent of order $q > 0$ plays a crucial role. The natural question to be asked is whether or not the limit functions satisfy the entropy condition. In order to provide an affirmative answer the concept of "Godunov compatibility" is introduced in Definition 6.5 and is used in Theorem 6.7. It should be noted that the compatibility condition is based on the assumption that the Godunov scheme converges to an entropy solution; this fact has actually been proved in the scalar case [39] where it is known to be unique and for a class of 2×2 systems by DiPerna [30]. We refer also to [29] in the case of isentropic gas dynamics.

2. THE GENERALIZED RIEMANN PROBLEM–GRP

The term "generalized Riemann problem" [6, 8] referred originally to the following analytical problem, concerning the initial-value problem for a system of nonlinear hyperbolic conservation laws (in one space dimension):

> *Consider initial data consisting of linear profiles for $\pm x > 0$ with a jump at $x = 0$. Determine the analytical solution for (x, t) in the neighborhood of the singularity at $(0, 0)$.*

In the case of constant profiles the problem reduces to the classical Riemann problem, and on the other hand the problem is readily extended to initial data with higher order polynomial profiles (see e.g. [25, 76, 90]).

A variety of solutions have been presented, using techniques such as asymptotic expansions in the original (x, t) coordinates, application of Riemann invariants, mapping to characteristic coordinates and more. We refer to [9, 13, 18, 19, 41, 53, 59, 71, 84] and references therein.

It should be remarked that the analytical procedure can be viewed as incorporating "physical effects" (in particular entropy variations and other thermodynamical effects) into the solution. In other words, the role of such effects is more prominent in the GRP methodology than in the majority of finite-difference schemes. We refer to [62] for a detailed discussion of this topic. Obviously, this leads to a more complex task of computer implementation of this scheme.

We shall see below (Example 4.21) that the introduction of the **acoustic GRP** provides a good remedy to this difficulty. As a matter of fact it is identical to the second-order ADER scheme.

The GRP analytic solution leads in a straightforward way to an extension of the Godunov scheme, for systems of conservation or "quasi-conservation" laws. The resulting GRP scheme is a robust "shock capturing" method, that has no need for any "adjustable" parameters, apart from enforcing some dissipation via the use of a basic "limiter". This is a mechanism that suppresses oscillatory behavior near sharp discontinuities, a behavior that is typical to any high-order scheme.

As already mentioned, when the discrete profile is piecewise-constant, the GRP scheme reduces precisely to the Godunov scheme. The main highlights of the former conform to the principles of the latter (see (1.1)):

- The numerical flux is the principal object. It is computed at all "singularities", using the analytical solution. These include, naturally, all jumps at "cell boundaries", but optionally also at selected strong shocks, material interfaces and so on.
- Use the flux in a straightforward time-marching of averages of flow variables, via a balance equation.
- Apply once again the GRP analysis to determine slopes at the next time level.

The GRP scheme has been extended and applied in many physical and engineering settings. A partial list includes reacting flows [3, 5], duct flows [7], atmospheric flows on the sphere [11], adaptive meshes [43], boundary value problems [32, 63], multi-fluid multi-dimensional flows [55], shallow water [58], Lagrangian formulation on unstructured meshes [69, 70], flows of real materials [88], relativistic hydrodynamics [91, 92], laminar two-phase two-velocity flow [93]. It was even tested against laboratory experiments [36, 46, 55].

There have been various studies comparing (or combining) the GRP scheme with other well-known schemes. We mention here:

- Comparison between the GRP and gas-kinetic schemes can be found in [61].
- Combination of GRP and DG (Discontinuous Galerkin) can be found in [89].
- Combination of the GRP solution with HLLI Riemann solver can be found in [1].
- Comparison between the GRP and RK4-WENO scheme (1-D and 2-D) can be found in the survey [57].
- Combination of GRP with WENO reconstruction can be found in [31].

3. FLUXES–THE HEART OF THE MATTER

We shall now state a very general fact about the Lipschitz continuity of the fluxes for weak solutions of systems of hyperbolic conservation laws. This is in sharp contrast to the fact that the unknowns in the system are in general discontinuous. Clearly a Lipschitz continuous function is more amenable to approximation by various continuous or discontinuous functions (e.g., polynomials or step functions).This accounts for the fundamental role of fluxes in the GRP scheme.

3.1. SYSTEMS OF CONSERVATION LAWS IN ONE SPACE DIMENSION.

We consider a system of conservation laws of D unknowns in one space dimension:

$$u_t + f(u)_x = 0, \quad u,\, f(u) \in \mathbb{R}^D, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+, \tag{3.1}$$

subject to initial data

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}. \tag{3.2}$$

In order to avoid complications caused by the presence of boundaries, we limit our considerations to the pure initial value problem set on the whole line $\mathbb{R}$.

Formally, by integration we infer that for every rectangle $Q = [x_1, x_2] \times [t_1, t_2] \subseteq \mathbb{R} \times \overline{\mathbb{R}_+}$ the following equality holds.

$$\int_{x_1}^{x_2} u(x,t_2)dx - \int_{x_1}^{x_2} u(x,t_1)dx = -\Big[\int_{t_1}^{t_2} f(u(x_2,t))dt - \int_{t_1}^{t_2} f(u(x_1,t))dt\Big]. \tag{3.3}$$

Remark 3.1. *Equation* (3.3) *can be considered as an integrated (formal) form of* (3.1)*, using the Gauss-Green theorem. However, the application of this theorem is certainly not straightforward, since the function $u(x,t)$ is not even continuous (see* [37, Section 4.5]*). We refer to* [23] *and* [27, Chapter I] *for an abstract discussion of this topic. Regarding the right-hand side of* (3.3) *one needs to keep in mind the following comment concerning the identification of the boundary flux:"the drawback of this, functional analytic, demonstration is that it does not provide any clues on how the $q_{\mathfrak{D}}$ may be computed from A"* [27, Section 1.3].

In fact, the meaning of the x and t derivatives must be clarified since the solutions generate discontinuities, such as shocks or interfaces. As is well known, the concept of a **weak solution** is introduced precisely in order to handle this difficulty [34, Chapter 11], as follows.

For every rectangle $Q = [x_1, x_2] \times [t_1, t_2] \subseteq \mathbb{R} \times \overline{\mathbb{R}_+}$, if $\phi(x,t) \in C_0^\infty(Q)$, then

$$\int_{t_1}^{t_2} \int_{x_1}^{x_2} [u(x,t)\phi_t + f(u(x,t))\phi_x] dx\, dt = 0. \tag{3.4}$$

In light of the above comments, the proof of the following theorem is not so obvious [14, Theorem 2.2]:

Theorem 3.2. *Let $u(x,t)$ be a weak solution to the system* (3.1)*, with initial function $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.*

Assume that $u(x,t)$ satisfies the following properties.

- *$u(x,t)$ is locally bounded in $\mathbb{R}\times\overline{\mathbb{R}_+}$.*
- *For every fixed interval $[x_1,x_2]\subseteq\mathbb{R}$ the mass*

$$m(t)=\int_{x_1}^{x_2}u(x,t)dx \text{ is a well-defined and continuous function of } t\in\overline{\mathbb{R}_+}. \tag{3.5}$$

Then we have:

(i) *For every fixed $[t_1,t_2]\subseteq\mathbb{R}_+$ the integral $g(x)=\int_{t_1}^{t_2}f(u(x,t))dt$ is locally Lipschitz continuous in $x\in\mathbb{R}$.*
(ii) *$u(x,t)$ satisfies the equality (3.3) in every rectangle Q.*

Remark 3.3. *[Alternative continuity hypotheses] We could replace the continuity assumption (3.5) by the stronger assumption that the map $t\to u(\cdot,t)\in L^\infty(\mathbb{R})$ weak* is continuous. This latter assumption is universally imposed when dealing with entropy solutions to nonlinear conservation laws* [27, Section 4.5]*. However the continuity condition (3.5) is valid for weak solutions that are not necessarily entropy solutions. In fact, it holds for weak solutions that have bounded (locally in time) total variation. This is expressed by Dafermos as "mechanism of regularity transfer from the spatial to the temporal variables"* [27, Theorem 4.3.1].

The statement of Theorem 3.2 is closely related to the more fluid dynamical viewpoint: the "conservation law", which is a **partial differential equation,** is replaced by a "balance law". This latter viewpoint plays a central role in this paper, and is introduced in the following paragraphs.

3.2. **GENERAL SETUP.** We assume the existence of two Banach spaces, $\mathfrak{U}^b$ and $\mathfrak{U}^c$, with respective norms $\|\cdot\|_b$, $\|\cdot\|_c$, and set

$$\mathfrak{U}=\mathfrak{U}^b\cap\mathfrak{U}^c. \tag{3.6}$$

Definition 3.4. *The norm in $\mathfrak{U}$ is*

$$\|w\|_{\mathfrak{U}}=\|w\|_b+\|w\|_c. \tag{3.7}$$

$\mathfrak{U}$ is assumed to be a "persistence space" for the class of solutions introduced below, in the sense that they satisfy, for every initial data $u_0\in\mathfrak{U}$,

$$t\hookrightarrow u(\cdot,t)\in C(\mathbb{R}_+,\mathfrak{U}^c)\cap L^\infty(\overline{\mathbb{R}_+},\mathfrak{U}^b). \tag{3.8}$$

Note that as in the case of weak solutions, no uniqueness assumption is imposed at this stage.

We shall also make use of the Fréchet space $L^1_{loc}(\mathbb{R})$ whose metric is given (for two vector-valued functions f, g) by

$$d(f,g)=\sum_{N=1}^{\infty}2^{-N}\,\frac{\int_{-N}^{N}|f(x)-g(x)|dx}{1+\int_{-N}^{N}|f(x)-g(x)|dx},\quad f,g\in L^1_{loc}(\mathbb{R}). \tag{3.9}$$

Definition 3.5. *Let $u_0\in\mathfrak{U}$. The function $u(\cdot,t)\in C(\overline{\mathbb{R}_+},\mathfrak{U}^c)\cap L^\infty(\mathbb{R}_+,\mathfrak{U}^b)$ is a solution to the balance law (3.3) corresponding to the partial differential equation (3.1) if the following conditions are satisfied.*

- *For every $x\in\mathbb{R}$ and interval $[t_1,t_2]\subseteq\overline{\mathbb{R}_+}$ the integral $\int_{t_1}^{t_2}f(u(x,t))dt$ is well defined, and is a continuous function of $x\in\mathbb{R}$.*

- *For every* $t \geq 0$ *and interval* $[x_1, x_2] \subseteq \mathbb{R}$ *the integral* $\int_{x_1}^{x_2} u(x,t)dx$ *is well defined and is a continuous function of* t.
- *For every rectangle* $[x_1, x_2] \times [t_1, t_2] \subseteq \mathbb{R} \times \overline{\mathbb{R}_+}$ *the balance equation* (3.3) *is satisfied.*

Our definition of a solution to the balance law conforms to that introduced in [27, Chapter I]. In fact, in Dafermos' book the balance equation is assumed to hold for any domain in spacetime. We note that other authors use various other terms, such as the "integral conservation law", and the term "balance law" is applied to a conservation law with a source term.

Definition 3.5 is closely related to the physical interpretation of systems of conservation laws, in particular the Euler system of compressible fluid flow. Furthermore, the balance law serves as the foundation of numerical finite volume schemes; in fact, every interval $[x_1, x_2]$ is considered as a "control volume" in which the balance law is satisfied between arbitrary time levels $t_1 < t_2$.

3.3. **THE SCALAR CONSERVATION LAW.** We now confine the above discussion to the scalar equation, namely $u \in \mathbb{R}$. We refer to the classical paper [49] and to the books [27, 34, 39] for the notion of the Kružkov **entropy solution.** Note that in Theorem 3.2 we do not need to assume that $u(x,t)$ is an entropy solution.

In this case the function space $\mathfrak{U}^b$ (see (3.6)) is taken as the space $L^\infty(\mathbb{R})$ while $\mathfrak{U}^c$ is the space $L^1(\mathbb{R})$.

The norm in $L^p(\mathbb{R})$ is denoted by $\|w\|_p$.

We recall the basic facts concerning this evolution semigroup [39, Chapter 2]:

Claim 3.6. *The solution semigroup* $S(t) : \mathfrak{U} \hookrightarrow C(\overline{\mathbb{R}_+}, L^1(\mathbb{R}))$ *is continuous and satisfies*

(i)

$$\|S(t)u_0\|_\infty \leq \|u_0\|_\infty, \quad t \geq 0.$$

(ii)

$$\|S(t)u_0 - S(t)v_0\|_1 \leq \|u_0 - v_0\|_1, \quad t \geq 0.$$

Remark in particular that for any fixed interval $[x_1, x_2] \subseteq \mathbb{R}$ the mass $m(t) = \int\limits_{x_1}^{x_2} u(x,t)dx$ of the entropy solution $u(x,t)$ is well-defined and, indeed, is a continuous function of $t \in \overline{\mathbb{R}_+}$.

4. CONSISTENCY–IS GODUNOV'S SCHEME FIRST-ORDER?

Theorem 3.2 implies that a weak solution satisfying certain hypotheses (in particular it need not be an entropy solution) is a solution to the balance law in the sense of Definition 3.5. It is easy to see that conversely, a solution to the balance law is a weak solution of the conservation law (3.1).

For notational simplicity we shall occasionally denote by $S(t)$ the solution operator,

$$u(\cdot, t) = S(t)u_0(\cdot), \tag{4.1}$$

even though the solution is not assumed to be unique.

To recall the meaning of "consistency", let us consider a vector function $u(x,t)$ taking values in $\mathbb{R}^D$ and satisfying an evolution equation in $\mathbb{R} \times \overline{\mathbb{R}_+}$ of the form

$$u_t = \Phi(u), \quad t > 0, \tag{4.2}$$

where $u_t = \frac{\partial}{\partial t} u$ is the partial derivative of $u(x,t)$ with respect to the time variable t, and $\Phi(u)$ is a general (not necessarily linear) operator involving spatial (with respect to $x-$variable) derivatives of u (and possibly explicit dependence on x).

Fixing a time interval $\Delta t > 0$, the discretization procedure (or "difference method") is aimed at finding a sequence of functions $\left\{\widetilde{u^n}(x)\right\}_{n=0}^{\infty}$, assumed to approximate the discrete sequence of values of the exact solution $\{u(x,t_n),\, t_n = n\Delta t\}_{n=0}^{\infty}$. The function $\widetilde{u^0}(x)$ approximates the initial data $u_0(x)$. The approximating functions are usually taken from a subspace of the "admissible" functions, those on which the operator Φ is acting (in some weak sense).

Generally speaking (for a "one step procedure"), there is a family of operators $\{\Phi_{\Delta t},\, \Delta t > 0\}$ generating the discrete (in time) sequence

$$\widetilde{u^{n+1}}(x) = \widetilde{u^n}(x) + \Phi_{\Delta t}\widetilde{u^n}(x). \tag{4.3}$$

The common definition of consistency is the following [77, Section 3.2], [72, Section 5.4], [82, Section 4.1].

Definition 4.1. *Let $q > 0$. The discrete scheme* (4.3) *is* ***consistent of order*** q *with Equation* (4.2) *in the time interval* $[0,T]$ *if the exact solution satisfies*

$$u(x,t_{n+1}) - [u(x,t_n) + \Phi_{\Delta t} u(x,t_n)] = O(\Delta t^{1+q}), \quad n\Delta t < T. \tag{4.4}$$

Observe that the equality (4.4) involves a suitable norm on functions of x, pertinent to the solution and approximating functions.

Remark 4.2. *In concrete cases, the verification of* (4.4) *relies on analytical tools, notably Taylor's theorem. This poses a difficulty, since the solution is often discontinuous, as in the case of nonlinear hyperbolic conservation laws. The goal here is to introduce a consistency condition that is* ***meaningful also in the case of discontinuous solutions.***

Our definition of consistency is given in Definition 4.15 below. In fact, it involves only **consistency of fluxes.** This is in line with our approach, as already mentioned above, regarding the *balance law* rather than the partial differential equation. Our fundamental Theorem 3.2 establishes the Lipschitz continuity of the fluxes even in the presence of discontinuities, allowing us to define the order of consistency subject to the selected discrete setting. Thus, for example, the Godunov flux is consistent to any order in the setting of piecewise-constant data (Example 4.18), but of *zero order* in the setting of piecewise-linear data (see discussion following Equation (4.16)). In both cases, the uniformity of the grid does not play any role.

We give below a brief (admittedly incomplete) account of the development of the concept of consistency (assuming a uniform grid). In Corollary 4.25 we shall see that the new definition matches the old one under some assumptions (such as uniformity of the grid).

The discrete scheme (4.3) is actually only "semi discrete", since only the (continuous) time is replaced by finite time steps. In practice, in a wide array of schemes (notably "finite volume") the spatial coordinates are also discretized. Restricting to a one-dimensional framework, a constant mesh size $\Delta x > 0$ is chosen, so that the ratio

$$\lambda = \frac{\Delta t}{\Delta x} \quad \text{is a constant.}$$

The approximating function $\widetilde{u^n}(x)$ is replaced by a discrete sequence $\widetilde{u^n_{disc}} = \left\{\widetilde{u^n_j}\right\}_{j=-\infty}^{\infty}$, that is presumed to approximate the exact values $\{u(x_j, t_n\}_{j=-\infty}^{\infty}$ at the spacetime grid points $\{x_j = j\Delta x,\, t_n = n\Delta t\}_{j=-\infty}^{\infty}$.

Accordingly, the semi discrete $\Phi_{\Delta t}$ is replaced by a fully discrete operator $\Phi_{\Delta t,\Delta x}$ and Equation (4.3) is replaced by a fully (i.e., spatial and temporal) discrete scheme

$$\widetilde{u^{n+1}_{disc}} = \widetilde{u^n_{disc}} + \Phi_{\Delta t,\Delta x}\widetilde{u^n_{disc}}. \tag{4.5}$$

It is clear how to formulate the consistency Definition 4.1 in this case:

Definition 4.3. *Let $q > 0$. Denote by $u^n_{disc} = \{u(x_j, t_n\}_{j=-\infty}^{\infty}$ the set of values of u at the spatial grid at time t_n. The discrete scheme* (4.5) *is* ***consistent of order*** *q with Equation* (4.2) *in the time interval* $[0, T]$ *if*

$$u^{n+1}_{disc} - [u^n_{disc} + \Phi_{\Delta t,\Delta x}u^n_{disc}] = O(\Delta t^{1+q}), \quad n\Delta t < T. \tag{4.6}$$

we now consider the issue of consistency of a fully discrete approximation to the system of conservation laws (3.1).

The classical definition of consistency, introduced by Lax and Wendroff [52], involves a Lipschitz continuous function of $2l$ variables $g(\xi_1, \dots, \xi_{2l}) \in \mathbb{R}^D$, so that Equation (4.5) can be rewritten as

$$\widetilde{u^{n+1}_j} = \widetilde{u^n_j} - \lambda\Big[g(\widetilde{u^n_{j-l+1}}, \widetilde{u^n_{j-l+2}}, \dots, \widetilde{u^n_{j+l}}) - g(\widetilde{u^n_{j-l}}, \widetilde{u^n_{j-l+1}}, \dots, \widetilde{u^n_{j+l-1}})\Big], \quad n = 0, 1, 2\dots \quad -\infty < j < \infty. \tag{4.7}$$

Definition 4.4. [52, Lax and Wendroff] *The scheme* (4.7) *is consistent with Equation* (3.1) *if*

$$g(\xi, \dots, \xi) = f(\xi), \quad \xi \in \mathbb{R}. \tag{4.8}$$

This definition has proved to be very useful in the case of first-order schemes. The Lax-Wendroff theorem [52] ensures that the approximate solutions obtained by a consistent and conservative scheme (4.7), and subject to some boundedness and (weak) convergence hypotheses, actually converge to a weak solution of (3.1). We refer to [28, 33, 38, 48] and references therein for various extensions of this convergence theorem (assuming first-order consistency).

Remark 4.5. *The methodology of discrete approximations presented here can be viewed as being in either the category of "finite volume schemes" or "finite difference schemes". The concept of "consistency" is fundamental in both. We therefore add here a brief comment concerning the difference between the two approaches. The consistency definition* (4.8) *applies both to finite difference and finite volume schemes for uniform grids. However, in the case of discrete approximations on non-uniform grids this definition needs to be carefully considered, since a given approximation can be consistent in the "sense of finite differences" but not so in the "sense of finite volume" schemes* [35, Remark 21.1].

Practical applications, as well as mathematical interests, require "high order" accuracy, or, using conventional terminology, "high order schemes". Such a requirement can be accommodated by either one of two ways.

- Take a sufficiently large l in (4.7). In other words, extend considerably the "stencil" of dependence when evaluating $\widetilde{u_j^{n+1}}$. This in turn involves a more complicated treatment of boundary conditions.
- Instead of considering only "cell averages" (where the value $\widetilde{u_j^n}$ is viewed as an average of the approximate solution in the interval $(x_j - \frac{\Delta x}{2}, x_j + \frac{\Delta x}{2})$), at time t_n, use *more information for each interval*, such as slopes or higher moments. This leads to a more complex discrete operator $\Phi_{\Delta t,\Delta x}$ in Equation (4.5), but enables the use of a "compact scheme",where only the neighboring intervals, centered at $x_{j\pm 1}$, are involved in determining the approximate solution $\widetilde{u_j^{n+1}}$.

The second alternative above is the one that is most widely implemented in various state-of-the-art schemes, such as MUSCL [85], GRP [6, 9, 13], ADER [84], PPM [25], DG [80], WENO [67, 80]. We refer also to the survey paper [82, Section 3.3]. For all these schemes, the concept of consistency must be clearly defined and its connection to the question of convergence to the exact solution should be clarified.

Remark that in our discussion of the system (3.1) the flux function $f(u)$ depends only on the unknown $u(x,t)$. It is very natural (both mathematically and in applications) to try and extend this to more general flux functions. In this case, the issues of consistency and convergence should be addressed. While there is no general framework for such extensions, there are many studies of particular cases, for example [79].

Consider the balance law for systems (3.3). In this section we introduce approximate fluxes associated with it. These fluxes serve in the construction of **compact schemes,** designed to approximate the solution of the balance law.

Let $k = \Delta t > 0$. As is common in the literature on finite difference methods (for evolution equations) we set k as the sole parameter in the study. Thus, convergence of approximate solutions to the exact ones will be studied in terms of limits as $k \to 0$.

The spatial step is $h = \Delta x = \lambda^{-1} k$, where $\lambda > 0$ is assumed to be fixed.

Definition 4.6. *The $k-$ spatial grid is the discrete set in $\mathbb{R}$,*

$$\Gamma_k = \{x_j = jh\}_{j=-\infty}^{\infty},$$

*and the **grid intervals** (or **grid cells**) are the intervals*

$$I_j = \left(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right), \quad -\infty < j < \infty,$$

where

$$x_{j\pm\frac{1}{2}} = x_j \pm \frac{h}{2}.$$

*The **spacetime** $k-$ **grid** is the discrete set*

$$\Gamma_k^{spacetime} = \Gamma_k \times \{t_n = nk,\ n = 0, 1, 2, \ldots\}. \tag{4.9}$$

Recall the persistence space $\mathfrak{U}$ as in Equation (3.8). Given a spatial grid Γ_k we assume that:

ASSUMPTION 4.7. (i) *There exists a* ***functional subspace*** $V^k \subseteq \mathfrak{U}$ *having the following property:*

The restrictions of the elements of V^k *to any grid interval* I_j *constitute a finite dimensional subspace. The dimension of these restrictions is called the* ***order of*** V^k. *Typically, it is a space of piecewise polynomial functions of fixed degree, with possible discontinuities at the boundary points* $\left\{x_{j\pm\frac{1}{2}}\right\}_{j=-\infty}^{j=\infty}$ *of every grid interval* I_j.

(ii) *There exists a projection* $P^k : \mathfrak{U} \to V^k$, *that does not change averages in grid cells, namely,*

$$\int_{I_j} P^k v(x)dx = \int_{I_j} v(x)dx, \quad -\infty < j < \infty,\ v \in \mathfrak{U}. \tag{4.10}$$

Remark 4.8. (i) *The notation of the space* V^k *(and the projection* P^k*) refers explicitly only to the variable parameter* k *(that determines the spatial step* $h = \lambda^{-1}k$*). However, this space also depends on our choice of the dimension of its restrictions to grid intervals, such as piecewise-constant ("first order"), piecewise-linear ("second order") and so on.*

(ii) *The operators* P^k *are sometimes called "reconstruction operators". They involve suitable interpolations and "slope limiters".*

Notation. Elements of V^k will be designated by Greek letters: $\xi \in V^k$. There will be no other use of Greek letters throughout the paper (except for the fixed constant ratio $\lambda = \frac{k}{h}$).

4.1. **APPROXIMATE FLUXES.** The approximate fluxes introduced here are intended to be sufficiently general, so as to cover a wide variety of finite volume schemes of any order.

We assume that there exists $\lambda_0 > 0$ so that for every fixed $\lambda = \frac{k}{h} < \lambda_0$ and every $0 < k < \frac{1}{2}T$, $h = \lambda^{-1}k$, there exists

$$\begin{aligned} &\text{a sequence of continuous functions } \left\{F^{\xi}_{j+\frac{1}{2}}(t),\ 0 \le t < k\right\}_{j=-\infty}^{\infty}, \\ &\text{for every } \ \xi \in V = V^k. \end{aligned} \tag{4.11}$$

Definition 4.9 (**Approximate Fluxes**)**.** *We say that the functions of the family* $\left\{F^{\xi}_{j+\frac{1}{2}}(t),\ 0 \le t < k\right\}_{j=-\infty}^{\infty}$ *are* ***approximate fluxes*** *(in the time interval* $[0,k)$*) corresponding to the initial function* $\xi \in V^k$, *if the following* ***finite propagation property*** *is satisfied.*

$F^{\xi}_{j+\frac{1}{2}}(t),\ 0 \le t < k$, *depends only on the restriction of* ξ *to* $I_j \cup I_{j+1}$.

Furthermore, if $\xi \equiv c = const.$ *in* $I_j \cup I_{j+1}$ *then* $F^{\xi}_{j+\frac{1}{2}}(t) \equiv f(c)$.

Note that in this definition the grid points $\left\{x_{j+\frac{1}{2}}\right\}_{j=-\infty}^{\infty}$ satisfy $x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} = h = \lambda^{-1}k$.

Remark 4.10. *The assumption above that* λ *is sufficiently small is the "CFL condition" that enables the finite propagation property of the fluxes.*

The terminology of "approximate fluxes" is suggested by the fact that they are viewed as approximating the flux values $f(u(x_{j+\frac{1}{2}}, t))$ at the nodes $\left\{x_{j+\frac{1}{2}}\right\}_{j=-\infty}^{\infty}$ in a sense that will be made rigorous below (4.12).

Remark 4.11 (**Uniformity of the spatial grid**). *While in our treatment the time step $k > 0$ is constant over the whole mesh, the spatial grid may be non uniform. This is due to the fact that Definition 4.9 relies only on fluxes restricted to cell boundaries. We have chosen to avoid this generality since it leads to notational complications (for example, the underlying discrete spaces V^k consist of piecewise polynomial functions over cells of variable size).*

4.2. **CONSISTENCY OF APPROXIMATE FLUXES.** We now introduce the key concept of *consistency.* It is discussed within our framework of *balance laws*, and therefore it deals exclusively with fluxes. The idea of "*consistency*" involves a comparison between exact and approximate fluxes, over short time intervals. It should be emphasized that this concept depends in a substantial way on the space of (discontinuous) functions in which the balance law is considered. Preparing the definition we need to introduce suitable families of initial data, contained in the spaces V^k. For such initial data we assume the short-time existence of unique solutions to the balance law, as in Assumption 4.13 below. We take $\lambda_0 > 0$ as in Definition 4.9 and consider spacetime grids satisfying $\lambda = \frac{k}{h} < \lambda_0$.

Definition 4.12. *Let $H \subseteq \bigcup_{0<k<\frac{1}{2}T} V^k$. We say that H is an **admissible set of initial data** if for every $\lambda = \frac{k}{h} < \lambda_0$ the set $H \cap V^k$ is bounded in the $\mathfrak{U}$ topology and compact in the $L^1_{loc}(\mathbb{R})$ topology* (3.9).

As an example, we can think of H (in the scalar case) as the set of uniformly bounded functions having a finite *total variation.*

ASSUMPTION 4.13. *Let $\xi(x) \in V^k$. Then the balance law* (3.3) *admits a unique solution in the time interval $t \in [0, k]$, subject to the initial condition $\xi(x)$. The uniqueness is achieved by imposing suitable constraints, such as "entropy conditions." This solution is denoted henceforth by $u(x, t; \xi) = S(t)\,\xi \in \mathfrak{U}$, $t \in [0, k]$.*

Actually, under some additional boundedness hypotheses, Assumption 4.13 can be verified [64]:

Claim 4.14. *Let $V^k \subseteq \mathfrak{U}$ be of any (finite) order. Then for every $\xi \in V^k$ there exists a unique entropy solution $u(\cdot, t; \xi) = S(t)\,\xi(\cdot)$, $t \in [0, k]$. This solution can be obtained by a constructive procedure, using characteristic curves and generalized Riemann solvers.*

Definition 4.15. *Consider the setup as in Definition 4.9. Let $q \geq 0$. The approximate fluxes $\left\{F^{\xi}_{j+\frac{1}{2}}(t),\ 0 \leq t < k\right\}_{j=-\infty}^{\infty}$ are said to be **consistent of order** q with the balance law* (3.3) *if for every admissible set of initial data H and all $\xi \in H \cap V^k$,*

$$\begin{aligned}(4.12)\quad &\Big|\int_0^k \Big[F^{\xi}_{j+\frac{1}{2}}(t) - F^{\xi}_{j-\frac{1}{2}}(t)\Big]dt - \int_0^k \Big[f(u(x_{j+\frac{1}{2}}, t; \xi)) - f(u(x_{j-\frac{1}{2}}, t; \xi))\Big]dt\Big| \\ &\leq Ck^{2+q}, \quad -\infty < j < \infty,\end{aligned}$$

where $C > 0$ depends only on H.

Remark 4.16. *Observe that the order of consistency in Definition 4.15 depends on the choice of the space V^k. This will be illustrated in Example 4.18 below.*

Also, the exponent $2+q$ is related to the exponent $1+q$ in (4.6). This will be further discussed in Corollary 4.25 below. As already noted in Remark 4.2 the order of consistency may depend on the regularity of the solution. Refer to Subsubsection 4.2.1 below for a detailed analysis of the interplay between regularity and order of consistency.

Remark 4.17. *Note that the estimate in (4.12) is given by Ck^{2+q}, where $C>0$ depends on H but is independent of j. Typically this dependence is expressed in terms of norms of the restrictions of ξ to I_j and neighboring grid intervals.*

Of course this can be relaxed by assuming, for instance, that the constant $C>0$ is "localized", so that $C=C(A)$, for all j such that $x_{j\pm\frac{1}{2}}\in[-A,A]$.

Example 4.18 (The Godunov Approximate Flux [40]). *Let V^k be of first order , namely, the space of piecewise constant (in grid intervals) functions. Then (if λ_0 is sufficiently small by the CFL condition) by definition*

$$F^{\xi}_{j+\frac{1}{2}}(t)=f(u(x_{j+\frac{1}{2}},t;\xi)),\quad 0\le t<k,\ -\infty<j<\infty, \tag{4.13}$$

so that, for this space, the approximate flux is consistent to any order.

*Recall that in this case $u(x_{j+\frac{1}{2}},t;\xi)\equiv const$ is the solution to the **Riemann problem** subject to the two sided initial data ξ_j,ξ_{j+1} (constant values in I_j, I_{j+1}, respectively).*

However, as will be seen in Remark 5.4 below, the Godunov approximate flux is only consistent of order zero ($q=0$), when used for piecewise-linear functions. In particular, this approximate flux is never first-order!

4.2.1. ***ORDER OF CONSISTENCY AND REGULARITY.*** Suppose now that we try to implement the Godunov approximation for the case of second-order spaces (namely, V^k consists of functions that are linear in grid cells). The approximate flux is therefore

$$F^{\xi}_{j+\frac{1}{2}}(t)=f(u(x_{j+\frac{1}{2}},0+;\xi)), \tag{4.14}$$

where $u(x_{j+\frac{1}{2}},0+;\xi)$ is the "instantaneous" solution to the Riemann problem subject to the two sided initial data $\xi_{j+\frac{1}{2}-},\xi_{j+\frac{1}{2}+}$, the limiting values of the piecewise linear function $\xi(x)$ at $x_{j+\frac{1}{2}}$.

Let $u_t(x_{j+\frac{1}{2}},0+;\xi)$ be the instantaneous value of the time-derivative of the solution (this is actually the solution to the **Generalized Riemann Problem (GRP)** [9, 13]). From

$$\begin{aligned} &f(u(x_{j+\frac{1}{2}},t;\xi))\\ =\ &f(u(x_{j+\frac{1}{2}},0+;\xi))+f'(u(x_{j+\frac{1}{2}},0+;\xi))u_t(x_{j+\frac{1}{2}},0+;\xi)t+\mathcal{O}(t^2), \end{aligned} \tag{4.15}$$

it follows that

$$\begin{aligned} &\int_0^k F^{\xi}_{j+\frac{1}{2}}(t)dt-\int_0^k f(u(x_{j+\frac{1}{2}},t;\xi))dt\\ =\ &\frac{1}{2}f'(u(x_{j+\frac{1}{2}},0+;\xi))u_t(x_{j+\frac{1}{2}},0+;\xi)k^2+\mathcal{O}(k^3). \end{aligned}$$

Hence the left-hand side of (4.12) is

(4.16)

$$\int_0^k \left[F^{\xi}_{j+\frac{1}{2}}(t) - F^{\xi}_{j-\frac{1}{2}}(t)\right]dt - \int_0^k \left[f(u(x_{j+\frac{1}{2}}, t;\xi)) - f(u(x_{j+\frac{1}{2}}, t;\xi))\right]dt$$
$$= \frac{1}{2}[f'(u(x_{j+\frac{1}{2}}, 0+;\xi))u_t(x_{j+\frac{1}{2}}, 0+;\xi) - f'(u(x_{j-\frac{1}{2}}, 0+;\xi))u_t(x_{j-\frac{1}{2}}, 0+;\xi)]k^2 + \mathcal{O}(k^3).$$

If no regularity of the solution $u(x,t;\xi)$ is assumed (in particular, if it is discontinuous) then the approximate flux is only consistent of order zero ($q = 0$ in (4.12)). However, in regions where the solution is smooth the difference

(4.17)

$$f'(u(x_{j+\frac{1}{2}}, 0+;\xi))u_t(x_{j+\frac{1}{2}}, 0+;\xi) - f'(u(x_{j-\frac{1}{2}}, 0+;\xi))u_t(x_{j-\frac{1}{2}}, 0+;\xi) = \mathcal{O}(k),$$

thus raising the order of consistency to $q = 1$.

In view of (4.15) the remedy here is to upgrade the approximate flux (4.14) by adding the GRP solution, thus introducing the GRP fluxes.

Definition 4.19 (**GRP Approximate Flux**). *The GRP approximate flux is given by*

$$F^{\xi}_{j+\frac{1}{2}}(t) = f(u(x_{j+\frac{1}{2}}, 0+;\xi)) + f'(u(x_{j+\frac{1}{2}}, 0+;\xi))u_t(x_{j+\frac{1}{2}}, 0+;\xi)t. \tag{4.18}$$

Now

$$\int_0^k F^{\xi}_{j+\frac{1}{2}}(t)dt - \int_0^k f(u(x_{j+\frac{1}{2}}, t;\xi))dt = \mathcal{O}(k^3),$$

so that the order of consistency is $q = 1$ in all cases. For smooth solutions we obtain second-order consistency ($q = 2$), since in analogy with (4.18)

(4.19)

$$f''(u(x_{j+\frac{1}{2}}, 0+;\xi))u_t(x_{j+\frac{1}{2}}, 0+;\xi) - f''(u(x_{j-\frac{1}{2}}, 0+;\xi))u_t(x_{j-\frac{1}{2}}, 0+;\xi) = \mathcal{O}(k).$$

Thus, when reduced to the smooth setting, the common statement about the second order consistency of this approximate flux (as well as the MUSCL flux below) is recovered.

Example 4.20 (**MUSCL Approximate Flux**). *The derivative $f'(u(x_{j+\frac{1}{2}}, 0+;\xi))$ in* (4.18) *depends solely on the Riemann solution. On the other hand, the instantaneous time derivative $u_t(x_{j+\frac{1}{2}}, 0+;\xi)$ is obtained from the GRP solution. Suppose that we can somehow find an approximation $v(x_{j+\frac{1}{2}}, 0+;\xi)$ so that*

$$u_t(x_{j+\frac{1}{2}}, 0+;\xi) - v(x_{j+\frac{1}{2}}, 0+;\xi) = \mathcal{O}(k^{\beta}), \quad \beta \geq 0. \tag{4.20}$$

Let us define new approximate fluxes by

$$F^{\xi}_{j+\frac{1}{2}}(t) = f(u(x_{j+\frac{1}{2}}, 0+;\xi)) + f'(u(x_{j+\frac{1}{2}}, 0+;\xi))v(x_{j+\frac{1}{2}}, 0+;\xi)t,$$

so that now

$$\int_0^k F^{\xi}_{j+\frac{1}{2}}(t)dt - \int_0^k f(u(x_{j+\frac{1}{2}}, t;\xi))dt = \mathcal{O}(k^3) + \mathcal{O}(k^{2+\beta}).$$

The MUSCL scheme of van-Leer [85] *provides such an approximation with $\beta = 1$* [9, Appendix D] *and we conclude that it is consistent of order $q = 1$.*

Example 4.21 (**Acoustic GRP Approximate Flux**). *The acoustic GRP flux was introduced in* [9, Proposition 5.9] *and serves as a particularly simple extension of the Godunov flux. In fact, it also serves as the foundation of the ADER methodology* [41, p.807]. *It is only applicable if no strong discontinuities are present and in this case it has the same order of consistency as the MUSCL flux* [9, Theorem 5.36]*, namely, $\beta = 1$ in* (4.20). *However, in the presence of strong discontinuities it is consistent of order $q = 0$, hence does not offer a formal improvement of the Godunov flux. It should be noted that in simulations of problems that do not involve strong discontinuities it actually yields much better approximations than those provided by the Godunov scheme* [84].

It is now clear how to obtain still higher order of consistency ($q = 2$ in discontinuous cases): a second-order time derivative is added to the generalized Riemann solution. This has already been implemented in the case of the Euler compressible flow [76].

4.3. **CONSISTENCY–COMPARING OLD AND NEW.** The approximate fluxes introduced above lead to the construction of approximate solutions by *finite volume schemes.* This construction is introduced here, along with the order of consistency of the ensuing scheme. The compatibility of the new definition of order of consistency with the classical definition is established under suitable assumptions (Corollary 4.25).

In order to conform with the conventional treatment, we consider the general step of the scheme (namely, from t_n to t_{n+1}).

Assumption 4.13 is imposed (see also Claim 4.14), guaranteeing the existence of a unique solution $u(x,t;\xi),\ \xi \in V^k$, to the balance law, in every time step.

In the following definition we assume that $\left\{F^{\xi}_{j+\frac{1}{2}}(t),\ 0 \le t < k\right\}_{j=-\infty}^{\infty}$ are approximate fluxes consistent with the balance law (3.3) .

Definition 4.22. (i) *Suppose that there is a map $\widetilde{S(k)} : V^k \to \mathfrak{U}$ so that, for every $\xi \in V^k$,*

$$\int_{I_j} \widetilde{S(k)}\xi dx - \int_{I_j} \xi dx = -\int_0^k \left[F^{\xi}_{j+\frac{1}{2}}(t) - F^{\xi}_{j-\frac{1}{2}}(t)\right] dt, \quad -\infty < j < \infty. \tag{4.21}$$

Then $\widetilde{S(k)}$ is called ***an approximate evolution operator*** *to the balance law associated with these fluxes.*

(ii) *Let $\left\{F^{\xi}_{j+\frac{1}{2}}(t),\ 0 \le t < k\right\}_{j=-\infty}^{\infty}$ be approximate fluxes consistent with the balance law* (3.3). *We say that a family of maps $\left\{\Phi^k : V^k \to V^k\right\}_{k>0}$ is a* ***Finite Volume Scheme (FVS)*** *for the balance law* (3.3) *if*

$$\Phi^k = P^k \widetilde{S(k)}, \tag{4.22}$$

where P^k is the projection as in (4.10).

The validity of such maps and their connection to the consistency of the fluxes is addressed in Propositions 4.23 and 4.24 below.

Given a time step $k > 0$, we define a sequence $\{\theta^n\}_{n=0}^{\infty} \subseteq V^k$ as follows.

First, $\theta^0 = P^k u_0(x)$. We construct this sequence successively by letting first

$$u(x, t - t_n; \theta^n) = S(t - t_n)\theta^n, \tag{4.23}$$

and then

$$\theta^{n+1} = P^k(u(x, t_{n+1} - t_n; \theta^n)). \tag{4.24}$$

The set of cell averages of these functions is defined by

$$\left\{ \theta_j^{n+1} = h^{-1} \int_{I_j} \theta^{n+1}(x)dx \right\}_{j=-\infty}^{\infty}, \quad n = 0, 1, 2, \ldots$$

For the proof of the following proposition we refer to [14, Proposition 3.18].

Proposition 4.23. *Assume that the approximate fluxes* $\left\{ F^{\xi}_{j+\frac{1}{2}}(t),\ 0 \le t < k \right\}_{j=-\infty}^{\infty}$ *are consistent of order* q*, in the sense of Definition 4.15. Then the sequence of cell averages over the intervals* I_j *satisfies*

$$\theta_j^{n+1} - \theta_j^n = -\frac{\lambda}{k} \int_{t_n}^{t_{n+1}} [F^{\theta^n}_{j+\frac{1}{2}}(t - t_n) - F^{\theta^n}_{j-\frac{1}{2}}(t - t_n)]dt + \mathcal{O}(k^{1+q}), \quad -\infty < j < \infty. \tag{4.25}$$

Proposition 4.24. *Assume that the approximate fluxes* $\left\{ F^{\xi}_{j+\frac{1}{2}}(t),\ 0 \le t < k \right\}_{j=-\infty}^{\infty}$ *are consistent of order* q *and let* Φ^k *be a FVS as in Definition 4.22. Let* $\theta^n \in V^k$ *and*

$$\widetilde{\theta^{n+1}}(x) = \Phi^k(\theta^n) \in V^k. \tag{4.26}$$

Let

$$\left\{ \widetilde{\theta_j^{n+1}} = h^{-1} \int_{I_j} \widetilde{\theta^{n+1}}(x)dx \right\}_{j=-\infty}^{\infty}.$$

Then

$$|\widetilde{\theta_j^{n+1}} - \theta_j^{n+1}| = \mathcal{O}(k^{1+q}), \quad -\infty < j < \infty. \tag{4.27}$$

where the averages θ_j^{n+1} *are as in Proposition 4.23.*

Proof. By (4.21) (and the fact that P^k does not change averages) we get

$$\int_{I_j} \Phi^k(\theta^n)dx - \int_{I_j} \theta^n dx = - \int_{t_n}^{t_{n+1}} \left[F^{\theta^n}_{j+\frac{1}{2}}(t - t_n) - F^{\theta^n}_{j-\frac{1}{2}}(t - t_n) \right] dt, \quad -\infty < j < \infty. \tag{4.28}$$

Thus

$$\widetilde{\theta_j^{n+1}} - \theta_j^n = -\frac{\lambda}{k} \int_{t_n}^{t_{n+1}} \left[F^{\theta^n}_{j+\frac{1}{2}}(t - t_n) - F^{\theta^n}_{j-\frac{1}{2}}(t - t_n) \right] dt, \quad -\infty < j < \infty. \tag{4.29}$$

Comparing this equality with (4.25) we obtain (4.27). □

We can now compare the consistency result of Proposition 4.24 to the classical consistency definition as recalled in the Introduction (Definition 4.3).

Define a discrete time evolution, with $\Delta t = k$, by

$$\left[\Phi_{\Delta t, \Delta x} \widetilde{\theta^n} \right]_j = \widetilde{\theta_j^{n+1}} - \widetilde{\theta_j^n}, \quad -\infty < j < \infty, \quad n = 0, 1, 2, \ldots.$$

Corollary 4.25. *Under the assumptions of Proposition 4.24 the discrete operator $\Phi_{\Delta t,\Delta x}$ is of order q in the sense of Definition 4.3. More explicitly, when viewed as acting on the sequence of averages of the exact solution, it satisfies Equation* (4.6).

Proof. It is assumed that the discrete operator acts on the exact solution, namely, $\widetilde{\theta_j^n} = \theta_j^n$. Hence Equation (4.27) can be rewritten as

$$\left|\theta_j^{n+1} - [\theta_j^n + (\Phi_{\Delta t,\Delta x}\theta^n)_j]\right| = \mathcal{O}(k^{1+q}), \quad -\infty < j < \infty.$$

This is therefore identical to Equation (4.6). □

Thus our definition, while suitable for discontinuous solutions, is in line with the classical definition, when the latter is applicable.

At this point it should be pointed out that the Godunov scheme, as well as the GRP (and the other high-resolution schemes) are based on one-dimensional analysis. The common practice in two-dimensional settings is to determine fluxes perpendicular to a given side of a grid cell by resorting to the one-dimensional methodology in the perpendicular direction. We refer to [56] for a GRP scheme that uses the local analysis of the tangential component of the momentum. The scheme is then modified by incorporating the transversal effects in the evaluation of the fluxes.

In regular 2-D grids, the conventional approach has been the use of a nonlinear version of the "Strang spatial splitting" [9, Chapter 7]. It has led to remarkably good computational results but to the best of the author's knowledge it is still lacking a substantial analytic treatment. We refer to [10] and references therein.

5. CONVERGENCE–THE LAX-WENDROFF THEOREM REVISITED

The question of the convergence of the approximate solution to a solution of the balance law is discussed in this section.

Recall the "Lax-Wendroff Convergence Theorem" [52], in the context of approximations to conservation laws. By using a particular notion of consistency (see Definition 4.4) it yields the convergence of discrete approximate solutions to weak solutions. Introducing a different notion of consistency necessarily forces a revision of this convergence theorem.

Here we address the convergence issue in the framework of "balance laws", in one space dimension.

Our goal is to impose conditions on the FVS (Definition 4.22) that will guarantee the convergence of the approximate solutions to a solution of the balance law (Definition 3.5) at a fixed time $t = T$ as $k \to 0$. Observe that since we are dealing with *systems* and do not assume any entropy condition, we cannot infer that such a solution to the balance law is unique.

The consistency result of Proposition 4.24 does not imply such convergence. In fact, it deals with the action, over one time step, of the discrete operator on the *exact solution*. In the construction of the approximate solution at time t_{n+1}, on the other hand, the operator acts on the *approximate solution* obtained at time t_n. It is given by (see Definition 4.22)

$$\widetilde{\theta^{n+1}}(x) = \Phi^k(\widetilde{\theta^n}) \in V^k, \quad n = 0, 1, 2, \dots \tag{5.1}$$

Thus, the procedure produces errors that accrue at each time step and do not necessarily vanish at the final time $t = T$ as the time step is refined.

The above discussion can simply be summarized by saying that once *consistency* is assumed then *stability* is (in one form or another) needed in order to ensure convergence. In the *linear* case, this is precisely the claim of the celebrated "Lax equivalence theorem" [77, Section 3.5].

At this stage, it is useful to recall the two main approaches to convergence.

- "compactness"–establishing the boundedness of the discrete solutions in a stronger space that is compactly embedded in the expected convergence space. In the case of discontinuous solutions this is universally carried out in total variation spaces.
- "stability"–imposing some boundedness assumptions on the discrete solutions and using consistency in order to control the accumulation of errors.

The second approach is what can be referred to as the "Lax-Wendroff methodology". It necessarily assumes the existence of an exact solution but, on the other hand, the assumptions imposed on the discrete solutions are typically easier to verify in a concrete computation.

Our presentation here is in the framework of the second approach.

We remark that in the case of *a linear evolution equation* (even in Banach space) stability (with a suitable assumption on the action of the discrete operator on the residual terms) is sufficient to establish convergence [28]. See also Remark 4.5. However there is no similar result that is applicable to the case of interest here, namely, nonlinear hyperbolic balance laws.

The special consistency condition (4.8), used in establishing the Lax-Wendroff convergence theorem, must be replaced by the notion of consistency as in Definition 4.15.

We shall impose certain conditions that will guarantee that the approximate solution constructed in (5.1) converges to a solution of the balance law.

5.1. **THE CONVERGENCE THEOREM.** Fix $T > 0$. Recall the construction of the discrete (in time) sequence of short time exact solutions (see Claim 4.14 and (4.24))

$$\begin{aligned} &\theta^{n+1}(x) = P^k(u(x, t_{n+1} - t_n; \theta^n)) \in V^k, \\ &\qquad\qquad n = 0, 1, 2, \ldots, N-1, \;\; N = N(k) = k^{-1}T, \end{aligned} \tag{5.2}$$

and the sequence of approximate solutions (5.1)

$$\widetilde{\theta^{n+1}}(x) = \Phi^k(\widetilde{\theta^n}) \in V^k, \quad n = 0, 1, 2, \ldots, N-1. \tag{5.3}$$

For both sequences the initial data is given by taking the projection of the initial function $u_0 \in \mathfrak{U}$ on the subspace V^k

$$\theta^0 = \widetilde{\theta^0} = P^k u_0 \in V^k. \tag{5.4}$$

We assume that the conditions of Definition 4.9 (and in particular the CFL condition) are satisfied, so that approximate fluxes can be constructed. We shall further assume that these fluxes are consistent of order $q > 0$ (Definition 4.15). Since the projection P^k does not change averages in cells, it follows that for all grid intervals I_j,

$$\int_{I_j}\theta^{n+1}(x)dx-\int_{I_j}\theta^n(x)dx$$
$$=-\int_{t_n}^{t_{n+1}}\Big[f(u(x_{j+\frac12},t-t_n;\theta^n))-f(u(x_{j-\frac12},t-t_n;\theta^n))\Big]dt, \qquad -\infty<j<\infty, \tag{5.5}$$

and from (4.21) that for all grid intervals I_j,

$$\int_{I_j}[\widetilde{\theta^{n+1}}(x)-\widetilde{\theta^n}(x)]dx$$
$$=-\int_{t_n}^{t_{n+1}}[F^{\widetilde{\theta^n}}_{j+\frac12}(t-t_n)-F^{\widetilde{\theta^n}}_{j-\frac12}(t-t_n)]dt,\ -\infty<j<\infty. \tag{5.6}$$

We construct a function $\widetilde{\Upsilon^k}(x,t)$ as follows.

$$\widetilde{\Upsilon^k}(x,t)=\frac{1}{k}[(t_{n+1}-t)\widetilde{\theta^n}(x)+(t-t_n)\widetilde{\theta^{n+1}}(x)],\quad t\in[t_n,t_{n+1}], \qquad n=0,1,\dots,N-1. \tag{5.7}$$

Observe that $t_n=nk$ depends on k.

Instead of the classical Lax-Wendroff theorem we get here the following theorem. For the (rather long) proof see [14, Theorem 4.1].

Theorem 5.1. *Assume that the FVS* (5.3) *is consistent of order* $q>0$. *Let* $\{k_m\downarrow 0\}$ *be a decreasing sequence of time steps. Let* $u_0\in\mathfrak{U}$ *(see* (3.6)*) and let* $\left\{\widetilde{\Upsilon^{k_m}}(x,t)\right\}_{m=1}^{\infty}$ *be the corresponding functions defined in* (5.7).

Suppose that

(i) *The sequence* $\left\{\widetilde{\Upsilon^{k_m}}(x,t)\right\}_{m=1}^{\infty}$ *is uniformly bounded in* $L^\infty([0,T],L^\infty(\mathbb{R}))$.

(ii) *The sequence* $\left\{\widetilde{\Upsilon^{k_m}}(x,t)\right\}_{m=1}^{\infty}$ *converges in* $C([0,T],L^1_{loc}(\mathbb{R}))$ *to a function* $v(x,t)$ *(in particular it is uniformly bounded in this space).*

Then $v(x,t)$ *is a solution of the balance law* (3.3) *in* $\mathbb{R}\times[0,T]$.

Remark 5.2. *The boundedness and convergence hypotheses in the theorem can be formulated in terms of the discrete solutions* $\widetilde{\theta^n}(x)$ *as follows, where* $N_m=k_m^{-1}T$.

- *The set* $\left\{\left\{\widetilde{\theta^n}(x)\right\}_{n=1}^{N_m}\subseteq V^{k_m}\right\}_{m=1}^{\infty}$ *is uniformly bounded in* $\mathfrak{U}$ *(in the topology* (3.7)*).*
- *There exists a function* $v(\cdot,t)\in C([0,T],L^1_{loc}(\mathbb{R}))$ *so that*

$$\lim_{m\to\infty}\sup_{1\le n\le N_m}d(\widetilde{\theta^n}(x),v(x,t_n))=0, \tag{5.8}$$

where the metric $d(y,z)$ *is given in* (3.9).

It was shown (Example 4.18) that the Godunov approximate fluxes on piecewise-constant functions are consistent of any order while the GRP upgrading (Definition 4.19) is consistent of (at least) first order, hence the following corollary holds.

Corollary 5.3 (**Godunov, GRP and MUSCL convergence**). (i) *Let the FVS* (5.3) *be given by the approximate Godunov fluxes (Example 4.18), confined to piecewise-constant functions. Then the limit of any convergent sequence, subject to the hypotheses (i)-(ii) of Theorem 5.1, is a solution to the balance law.*

(ii) *Let the FVS* (5.3) *be given by the approximate GRP fluxes* (4.18), *where the space V^k of approximating functions can be of any finite order. Then the limit of any convergent sequence, subject to the hypotheses (i)-(ii) of Theorem 5.1, is a solution to the balance law.*

(iii) *Let the FVS* (5.3) *be given by the approximate MUSCL fluxes (Example 4.20), where the space V^k of approximating functions can be of any finite order. Then the limit of any convergent sequence, subject to the hypotheses (i)-(ii) of Theorem 5.1, is a solution to the balance law.*

Remark 5.4. *A fundamental assumption in Theorem 5.1 is that the approximate fluxes are consistent of order $q > 0$. Recalling the discussion in Subsection 4.2.1 it follows that when the Godunov approximate fluxes are implemented for piecewise-constant functions,any limit function is a solution to the balance law, as stipulated by Corollary 5.3. On the other hand, taking a (spatially) second-order approximation (namely, piecewise-linear functions), and still using the Godunov approximate fluxes* (4.14), *the order of consistency is $q = 0$. The convergence theorem is not applicable and convergence may fail. On the other hand, as is stated in Corollary 5.3, implementing the GRP or MUSCL fluxes raises the order to $q = 1$ and ensures that any limit function is a solution of the balance law.*

These considerations are convincingly demonstrated in the numerical examples worked out in [62], *where the aforementioned two possibilities for approximate fluxes (with piecewise-linear data) were tested (see Figs. 6.2 and 6.5 there).*

5.2. **A MEASURE THEORY LEMMA.** Throughout the rest of this section we fix a $T > 0$. The time steps to be considered will be of size $k = \frac{1}{N}T$ for an integer $N > 1$.

Our final goal (Corollary 5.8) is to prove that the *grid averages* of the approximate solutions converge to the solution of the balance law obtained in Theorem 5.1. In proving this, some basic measure-theoretic facts are established.

Definition 5.5. *Given the spacetime grid $\Gamma_k^{spacetime}$* (4.9) *and a function $Y(x,t) \in L^1_{loc}(\mathbb{R} \times [0,T])$, we denote by $Y^{k,av}(x,t)$ the space-averaged function that consists of the averages in grid intervals,*

$$Y^{k,av}(x,t) = \frac{1}{h}\int\limits_{I_j} Y(z,t)dz, \quad (x,t) \in I_j \times [0,T], \tag{5.9}$$

$$I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \quad -\infty < j < \infty.$$

We have

$$\int\limits_{I_j} Y^{k,av}(x,t)dx = \int\limits_{I_j} Y(x,t)dx$$

and

$$\Big|\int\limits_{I_j} Y^{k,av}(x,t)dx\Big| = \int\limits_{I_j} |Y^{k,av}(x,t)|dx.$$

So integrating over any $[a,b] \subseteq [0,T]$ yields

$$\int\limits_{I_j \times [a,b]} |Y^{k,av}(x,t)| dxdt \leq \int\limits_{I_j \times [a,b]} |Y(x,t)| dxdt. \tag{5.10}$$

It follows that if the bounded set $K \subseteq \mathbb{R} \times [0,T]$ is a union of such rectangles then

$$\int\limits_K |Y^{k,av}(x,t)| dxdt \leq \int\limits_K |Y(x,t)| dxdt. \tag{5.11}$$

Using a density argument we now obtain:

Claim 5.6. *Let $Y(x,t) \in L^1_{loc}(\mathbb{R} \times [0,T])$, then for every bounded $K \subseteq \mathbb{R} \times [0,T]$,*

$$\lim_{k \to 0} \int\limits_K |Y^{k,av}(x,t) - Y(x,t)| dxdt = 0. \tag{5.12}$$

Claim 5.6 entails the following lemma.

Lemma 5.7. *Let $\{w_m(x,t)\}_{m=1}^{\infty} \subseteq C([0,T], L^1_{loc}(\mathbb{R}))$ be a sequence of functions that converges to a function $w(x,t)$ in the sense that*

$$\lim_{m \to \infty} w_m(\cdot,t) = w(\cdot,t), \quad \textit{in } C([0,T], L^1_{loc}(\mathbb{R})).$$

Let $\{k_m \downarrow 0\}$ be a decreasing sequence. Then the sequence of the corresponding average functions $\left\{w_m^{k_m,av}(x,t)\right\}_{m=1}^{\infty}$ converges to $w(x,t)$ in $L^1_{loc}(\mathbb{R} \times [0,T])$.

Proof. Let $K \subseteq \mathbb{R} \times [0,T]$ be bounded. Then in view of (5.11)

$$\lim_{m \to \infty} \int\limits_K |w_m^{k_m,av}(x,t) - w^{k_m,av}(x,t)| dxdt = 0,$$

and in view of (5.12)

$$\lim_{m \to \infty} \int\limits_K |w^{k_m,av}(x,t) - w(x,t)| dxdt = 0.$$

□

Corollary 5.8. *Assume the conditions of Theorem 5.1 and define the sequence of piecewise-constant functions*

$$\widetilde{\Upsilon^{k_m,av}}(x,t) = \frac{1}{h} \int\limits_{I_j} \widetilde{\Upsilon^{k_m}}(z,t) dz, \quad (x,t) \in I_j \times [0,T], \tag{5.13}$$

$$I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \quad -\infty < j < \infty.$$

Then the sequence $\left\{\Upsilon_m^{k_m,av}(x,t)\right\}_{m=1}^{\infty}$ converges to $v(x,t)$ in $L^1_{loc}(\mathbb{R} \times [0,T])$.

This corollary is of great practical significance, as it states that the solution to the balance law can be recovered from the cell averages of the approximate solutions. Obviously, these averages are easier to obtain, in the computational procedure, than the full (piecewise-polynomial) approximate solutions.

6. GODUNOV COMPATIBILITY AND ENTROPY

A major difficulty in the theory of nonlinear balance laws is that the limiting solutions obtained in Theorem 5.1 need not be unique. Recall that a solution to the balance law (Definition 3.5) is necessarily a weak solution to the corresponding conservation law (3.1). The entropy condition (see Subsection 3.3 above), essentially the only tool available for establishing uniqueness, has been applicable only in the scalar case [27, 39] (and some 2×2 systems [66]).

It is well-known that the Godunov FVS $\Phi^{k,G}$ provides a "reference frame" to full classes of (first order, scalar) approximate solutions, for example to all "$E-$schemes" [39, Chapter 3, Lemma 4.1]. Also, for a class of 2×2 systems it was shown in [30] that all limits of approximate solutions obtained by the Godunov scheme are entropy solutions. These observations seem to justify the introduction of the concept of "Godunov-compatible" schemes (Definition 6.5). It is based on the Assumption 6.4 that the Godunov FVS converges to a unique entropy solution. it is then shown that the approximate solutions produced by Godunov-compatible schemes converge to the same solution.

The treatment here may be compared to that of the Glimm scheme: Under suitable conditions all weak solutions obtained as limits satisfy the entropy condition [51, Theorem 2.2] and the solution is unique in the class of approximate solutions obtained by the front tracking method [21] (see also [27, Chapter XIV]).

Let us first consider the Godunov FVS as introduced in Example 4.18 (for general systems, not only scalar). The scheme is used for first order (namely, piecewise constant) spaces V^k. The notation $\Phi^{k,G}$ is used for the Godunov FVS.

In this case the FVS yields a discrete sequence $\left\{\widetilde{\theta^{n,G}}(x)\right\}_{n=0}^{N}$ of piecewise constant functions as in (5.1). Thus

$$\widetilde{\theta^{n,G}}(x) = \widetilde{\theta_j^{n,G}},\ x \in I_j,\ -\infty < j < \infty. \tag{6.1}$$

The values $\left\{\widetilde{\theta_j^{n,G}}\right\}_{j=-\infty}^{\infty}$ satisfy

$$\begin{aligned} &\widetilde{\theta_j^{n+1,G}} - \widetilde{\theta_j^{n,G}} \\ &= -\lambda\Big[f(u^G(x_{j+\frac{1}{2}}, t-t_n; \widetilde{\theta^{n,G}})) - f(u^G(x_{j-\frac{1}{2}}, t-t_n; \widetilde{\theta^{n,G}}))\Big], -\infty < j < \infty, \end{aligned} \tag{6.2}$$

where $u^G(x_{j+\frac{1}{2}}, t-t_n; \widetilde{\theta^n}) \equiv const$ is the solution to the Riemann problem at $x = x_{j+\frac{1}{2}}$.

As in Equation (5.7) we can now use the set $\left\{\widetilde{\theta^{n,G}}(x)\right\}_{n=0}^{N}$ in order to define the function $\widetilde{\Upsilon^{k,G}}(x,t)$ for the Godunov scheme.

6.1. THE SCALAR GODUNOV SCHEME.

It is well-known that **in the scalar case** the Godunov scheme possesses the same boundedness and contraction properties as the exact solution to the balance law (Claim 3.6):

Claim 6.1 ([39, Section 3.3]). *The sequence of solutions to the Godunov scheme satisfies*

- $$\|\widetilde{\theta^{n+1,G}}\|_\infty \le \|\widetilde{\theta^{n,G}}\|_\infty, \quad n = 0,1,2,\dots,N-1.$$
- *If* $\widetilde{\theta^{0,G}}$ *and* $\widetilde{\chi^{0,G}}$ *are two piecewise constant functions and* $\left\{\widetilde{\theta^{n,G}}(x)\right\}_{n=0}^{N}$, $\left\{\widetilde{\chi^{n,G}}(x)\right\}_{n=0}^{N}$ *are the corresponding solutions by the Godunov scheme, then*
$$\|\widetilde{\theta^{n+1,G}} - \widetilde{\chi^{n+1,G}}\|_1 \le \|\widetilde{\theta^{n,G}} - \widetilde{\chi^{n,G}}\|_1, \quad n = 0,1,2,\dots,N-1.$$

The fact that the approximate solutions derived by the Godunov scheme converge to the unique entropy solution is a fundamental fact of the theory of discretization of (scalar) conservation laws:

Claim 6.2 ([39, Chapter 3, Theorem 4.1])**.** *Assume that the CFL condition is satisfied and also that the initial function* u_0 *has finite total variation. Then the limit*
$$v(x,t) = \lim_{k\to 0} \widetilde{\Upsilon^{k,G}}(x,t)$$
exists in $C([0,T], L^1_{loc}(\mathbb{R}))$ *and is the unique entropy solution of the balance law.*

Remark 6.3. *Note that the condition that* u_0 *has finite total variation is not really needed* [35, Theorem 29.2].

6.2. **THE CASE OF SYSTEMS.** Recall (5.9)) that the projection of $\xi \in V^k$ on the space of piecewise constant functions, namely, the set of averages in the grid intervals, is designated as

$$\xi^{k,av}(x) = h^{-1}\int_{I_j} \xi(z)dz, \quad x \in I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \quad -\infty < j < \infty, \tag{6.3}$$

and satisfies (compare (5.10))

$$\int\limits_{\mathbb{R}} |\xi^{k,av}(x)|dx \le \int\limits_{\mathbb{R}} |\xi(x)|dx. \tag{6.4}$$

For general systems consider the Godunov scheme and recall (Corollary 5.3) that under suitable hypotheses the approximate solutions converge to a solution of the balance law. We now impose the following **fundamental hypothesis on the Godunov scheme** regarding the uniqueness of these solutions.

ASSUMPTION 6.4. *[**Godunov Scheme**] Let* $\mathfrak{B}_K$ *be the ball of radius* $K > 0$ *in* $\mathfrak{U}$ *(see* (3.6)*) and let* $u_0 \in \mathfrak{B}_K$*. Let* $\theta^{0,G} = u_0^{k,av}$*. The Godunov scheme* $\Phi^{k,G}$*, applied to* $\theta^{0,G} = u_0^{k,av}$ *converges to a unique solution of the balance law. More precisely, if* $\Phi^{k,G}$ *is the FVS in Theorem 5.1 then, under the hypotheses of the theorem, all limits of subsequences* $\widetilde{\Upsilon^{k_m,G}}(x,t)$ *obtained in the theorem are identical.*

Furthermore, if $v_0 \in \mathfrak{B}_K$ *is another initial function and* $\psi^{0,G} = v_0^{k,av}$ *, then*

$$\|\Phi^{k,G}\theta^{0,G} - \Phi^{k,G}\psi^{0,G}\|_1 \le (1 + Ck)\|\theta^{0,G} - \psi^{0,G}\|_1, \tag{6.5}$$

where $C > 0$ *depends only on* K.

Consider a space V^k (of any order) and an FVS as in (5.3). In view of (4.10) and (4.21) the map Φ^k is conservative:

$$\int_{\mathbb{R}} \Phi^k\xi(x)dx = \int_{\mathbb{R}} \xi(x)dx, \quad \xi \in V^k. \tag{6.6}$$

Let $\left\{\widetilde{\theta^n} \in V^k\right\}_{n=0}^{N}$ be the discrete set of approximate solutions, as constructed in (5.3). From them we obtain the function $\widetilde{\Upsilon^k}(x,t)$ defined in (5.7) and the "average function" $\widetilde{\Upsilon^{k,av}(x,t)}$ as in (5.9). The set of cell averages of $\widetilde{\theta^n}$ is denoted by $\left\{\widetilde{\theta^n_j} = h^{-1}\int_{I_j}\widetilde{\theta^n}(x)dx\right\}_{j=-\infty}^{\infty}$.

The following definition encapsulates the meaning of the FVS Φ^k as being compatible with the Godunov scheme.

Definition 6.5. *[**Godunov Compatibility**] The FVS Φ^k (consistent of order $q > 0$) is compatible with the Godunov scheme if the following conditions hold.*

(i) *The FVS Φ^k coincides with the Godunov scheme on piecewise constant functions; if $\xi \in V^k$ is piecewise constant then*

$$\Phi^k\xi = \Phi^{k,G}\xi. \tag{6.7}$$

(ii) *Let H be an admissible set (Definition 4.12). Then*

$$\int\limits_{\mathbb{R}} |\Phi^k\xi(x) - \Phi^{k,G}\xi^{k,av}(x)|dx = o(k), \quad \xi \in H, \tag{6.8}$$

where $o(k)$ is uniform for all $\xi \in H$.

Remark 6.6. *[**Explaining** (6.8)] This remark is intended as a* motivation*, certainly not a proof, for* (6.8)*.*

Let $\eta(x) = \Phi^k\xi(x)$ and $\chi(x) = \Phi^{k,G}\xi^{k,av}(x)$. Observe that in light of (4.12) *and* (4.25)*, for a fixed index $j \in \mathbb{Z}$,*

$$\begin{aligned}\int_{I_j}[\eta(x) - \xi(x)]dx &= -\int\limits_0^k [F^{\xi}_{j+\frac{1}{2}} - F^{\xi}_{j-\frac{1}{2}}]dt \\ &= -\int_0^k \Big[f(u(x_{j+\frac{1}{2}},t;\xi)) - f(u(x_{j-\frac{1}{2}},t;\xi))\Big]dt + \mathcal{O}(k^{2+q}) \\ &= \int_{I_j}[u(x,k;\xi)(x) - \xi(x)]dx + \mathcal{O}(k^{2+q}).\end{aligned} \tag{6.9}$$

(See Assumption 4.13 for the definition of $u(x,t;\xi)$).

Since the Godunov scheme yields the exact mean value,

$$\int_{I_j}[\chi(x) - \xi^{k,av}(x)]dx = \int_{I_j}[u(x,k;\xi^{k,av})(x) - \xi(x)]dx. \tag{6.10}$$

Subtracting (6.10) *from* (6.9) *yields*

$$\int_{I_j}(\eta(x) - \chi(x))dx = \int_{I_j}[u(x,k;\xi)(x) - u(x,k;\xi^{k,av})(x)]dx + \mathcal{O}(k^{2+q}). \tag{6.11}$$

Assuming that the exact solution u is a contraction in L^1, in conjunction with the finite propagation speed property leads to

$$\int_{I_j}|\eta(x) - \chi(x)|dx \le C_1 \sum_{m=j-l}^{j+l}\int_{I_m}|\xi(x) - \xi^{k,av}(x)|dx + \mathcal{O}(k^{2+q}), \tag{6.12}$$

where $C_1 > 0$ and the integer $l \ge 1$ are independent of j.

If the projection P^k (see (4.10)) involves a slope limiter, then the right-hand side of (6.12) is estimated by $C_2k^2(1+\mathcal{O}(k^q))$, and summation over j leads to

$$\int_{\mathbb{R}} |\eta(x)-\xi(x)|dx \leq C_2k(1+\mathcal{O}(k^q)). \tag{6.13}$$

Thus, (6.8) can be understood as assuming that in at most $o(k^{-1})$ grid cells there is a significant discrepancy between $\Phi^k\xi$ and the piecewise-constant function $\Phi^{k,G}\xi^{k,av}$ obtained by application of the Godunov scheme.

For the proof of the following theorem we refer to [14, Theorem 5.7].

Theorem 6.7. *Assume the validity of Assumption 6.4 and that the FVS Φ^k is consistent of order $q>0$ and compatible with the Godunov scheme. Let $\left\{\widetilde{\theta^n}\in V^k\right\}_{n=0}^N$ ($N=k^{-1}T$) be the discrete set of approximate solutions, as constructed in (5.3). Then the limit function obtained in Theorem 5.1 is unique, namely, under the hypotheses of the theorem there is a unique limit function for all converging subsequences.*

Remark 6.8. *[**Godunov Scheme and Entropy**] Systems that allow for entropy/entropy-flux formulations play a special role in the study of balance laws. This is true in particular in various (hyperbolic) models of fluid dynamics. In such cases, Assumption 6.4 can be relaxed, requiring only that all possible limit functions are entropy solutions. As is well-known, this requirement is not sufficient to ensure uniqueness. However, in this case Theorem 6.7 can be modified (under the same hypotheses) to state that all possible limits obtained by the FVS Φ^k are entropy solutions.*

Example 6.9. *[**Isentropic Gas Dynamics**] Consider the Euler system of compressible, isentropic flow in one space dimension:*

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x &= 0, \end{aligned} \tag{6.14}$$

subject to initial conditions

$$\rho(x,0)=\rho_0(x)\geq 0, \quad u(x,0)=u_0(x), \quad x\in\mathbb{R}.$$

Here ρ is the density, u is the velocity and the gas is polytropic: $p=k\rho^\gamma$ with $1<\gamma\leq\frac{5}{3}$. Then we have the following corollary to Theorem 6.7.

Corollary 6.10. *Suppose that the FVS Φ^k is consistent of order $q>0$ and compatible with the Godunov scheme. Let $\left\{\widetilde{\theta^n}\in V^k\right\}_{n=0}^N$ ($N=k^{-1}T$) be the discrete set of approximate solutions, obtained by applying Φ^k to the system (6.14). Then all limit functions obtained in Theorem 5.1 are entropy solutions of the system.*

Proof. It is shown in [29] that, under some additional conditions on the initial data, the approximate solutions obtained by the Godunov scheme converge to entropy solutions of the system. □

7. SUMMARY

A main ingredient in the Godunov legacy is the construction of the <u>numerical flux</u> by invoking a local analytic solution to the nonlinear hyperbolic conservation law, subject to discretized initial data. The fact that this flux is Lipschitz continuous (Theorem 3.2) implies that it is the proper object to be approximated, even in the presence of various discontinuities. In addition, it enables the "balance law" formulation, merging the finite volume and finite difference viewpoints.

The cosistency of the original Godunov scheme was an automatic (albeit somewhat implicit) consequence in the case of a piecewise-constant discretization. However, if higher order resolution is sought, this concept must be carefully studied. Here we offer a systematic study of consistency (and its order) for high order schemes (Section 4). The point to be emphasized is that it solely relates to the numerical flux and depends on the order of the underlying discretization space (Remark 4.16). Thus the two fundamental concepts of *numerical flux* and *consistency* are profoundly intertwined. Once they are established, we can proceed to state a generalization (to high order schemes) of the classical Lax-Wendroff convergence theorem.

7.1. **FUTURE DIRECTIONS.** We conclude by suggesting some research directions that represent natural extensions to the subject matter of this review.

- In the two-dimensional case, the Strang spatial splitting seems to be very effective when used on a regular grid. Beyond that, the focus should be turned to unstructured meshes:
  (a) Establish suitable discretization spaces (piecewise constant, multilinear etc). In the piecewise-constant case, give a rigorous Godunov scheme.
  (b) Determine a "Godunov-type", as well as higher order numerical fluxes, along each side of a mesh cell. In particular, such fluxes should account for transversal effects.
  (c) Define a suitable consistency concept for such fluxes.
- Based on such fluxes, establish a generalization of the Lax-Wendroff convergence theorem. In particular, it is essential that the assumptions imposed in such theorems are amenable to verification in actual computations.
- Extend the treatment to systems of conservation laws with source terms (e.g. reacting flows or topography in shallow-water systems). The source terms play an important role in the derivation of local analytical solutions.

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