EIGENFUNCTION EXPANSIONS AND SPACETIME ESTIMATES FOR GENERATORS IN DIVERGENCE-FORM

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Abstract. Let $H = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{j,k}(x) \frac{\partial}{\partial x_k}$ be a formally self-adjoint (elliptic) operator in $L^2(\mathbb{R}^n)$, $n \geq 2$. The real coefficients $a_{j,k}(x) = a_{k,j}(x)$ are assumed to be bounded and to coincide with $-\Delta$ outside of a ball. The paper deals with two topics: (i) An eigenfunction expansion theorem, proving in particular that H is unitarily equivalent to $-\Delta$, and (ii) Global spacetime estimates for the associated inhomogeneous wave equation, proved under suitable ("nontrapping") additional assumptions on the coefficients. The main tool used here is a Limiting Absorption Principle (LAP) in the framework of weighted Sobolev spaces, which holds also at the threshold.

1. Introduction

Let $H = -\sum_{j,k=1}^{n} \partial_j a_{j,k}(x) \partial_k$, where $a_{j,k}(x) = a_{k,j}(x)$, be a formally self-adjoint operator in $L^2(\mathbb{R}^n)$, $n \geq 2$. The notations $\partial_j = \frac{\partial}{\partial x_j}$ and $\partial_t = \frac{\partial}{\partial t}$ are used throughout the paper.

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We assume that the real measurable matrix function $a(x) = \{a_{j,k}(x)\}_{1\leq j,k\leq n}$ satisfies, with some positive constants $a_1 > a_0 > 0, \quad \Lambda_0 > 0$,

$$a_0 I \leq a(x) \leq a_1 I, \qquad x \in \mathbb{R}^n, \tag{1.1}$$

$$a(x) = I \qquad for \quad |x| > \Lambda_0. \tag{1.2}$$

In what follows we shall use the notation $H = -\nabla \cdot a(x)\nabla$.
We retain the notation H for the self-adjoint (Friedrichs) extension associated with the form $(a(x)\nabla\varphi, \nabla\psi)$, where $(,)$ is the scalar product in $L^2(\mathbb{R}^n)$. When $a(x) \equiv I$ we set $H = H_0 = -\Delta$.

Operators of this type appear in geometry (Laplacian on noncompact Riemannian manifolds) as well as in physics, typically when physical parameters vary in space (such as the acoustic propagator in a medium with variable speed of sound).

Under our assumptions (1.1), (1.2) it follows that $\sigma(H)$, the spectrum of H, is the half-axis $[0, \infty)$, and is entirely continuous. In particular, the equality $(Hu, u) = (a(x)\nabla u, \nabla u)$ shows that H has no eigenvalue at zero. In addition, if the coefficient matrix $a(x)$ is smooth, the absence of singular continuous spectrum follows from the classical work of Mourre [58] . However, it seems that there is no proof in the literature establishing the absolute continuity of the spectrum in our case of non-smooth (and even discontinuous) coefficients. This fact is implied by our Theorem A stated in Section 3 below.

The "threshold" $z = 0$ plays a special role in this setting, as we shall see later.

The mere fact that both H and H_0 are spectrally absolutely continuous over $[0, \infty)$ does not imply that they are "identical", namely, in the functional analytic setting, that they are "unitarily equivalent". Thus one question that arises is:

Question 1. Are the operators H and H_0 unitarily equivalent, under the above assumptions on the coefficients?

We next recall the definition of the wave operators related to H, H_0 [50, Chapter X].

Consider the family of unitary operators

$$W(t) = \exp(itH)\exp(-itH_0), \quad -\infty < t < \infty.$$

The strong limits

$$W_{\pm}(H, H_0) = s - \lim_{t\to\pm\infty} W(t), \tag{1.3}$$

if they exist, are called the *wave operators* (relating H, H_0). These operators play an important role in scattering theory. They are clearly isometries. If the range of W_+ is equal to the absolutely continuous subspace of H (which here is $L^2(\mathbb{R}^n)$ itself) , we say that it is *complete*,

with a similar definition for W_-. If either one is complete, then it is unitary (in the case at hand) and provides a unitary equivalence between H and H_0. A second question that arises therefore is:

Question 2. Do the wave operators exist and, if so, are they complete?

As noted above, a positive answer to this question entails a positive answer to the first question.

Another aspect related to the spectral theory of H is its associated *eigenfunction expansion.* When available, it serves as an analytic tool which is sharper than the abstract spectral theorem. In the case of H_0, the Fourier transform

$$\mathcal{F}g(\xi) = \widehat{g}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} g(x)e^{-i\xi x}dx, \tag{1.4}$$

serves to express $g(x)$ as

$$g(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \widehat{g}(\xi)e^{i\xi x}d\xi, \tag{1.5}$$

which can be viewed as an "expansion" of g in terms of the "generalized eigenfunctions" (or "modes") $\exp(i\xi x)$, associated with the eigenvalues $|\xi|^2$. Furthermore , the operator $\mathcal{F}$ is unitary and $\mathcal{F}H_0\mathcal{F}^{-1}$ is just multiplication by $|\xi|^2$ in Fourier space. Such ("diagonalizing") expansions have been used extensively in quantum mechanics (for example, the Airy transform associated with the Stark Hamiltonian). It is therefore natural to pose the following question:

Question 3. Can one associate a similar "eigenfunction expansion" with the operator H? More specifically, can one replace the exponentials $\exp(i\xi x)$ by some approximating generalized eigenfunctions ("distorted plane waves") so that the resulting transform remains unitary and diagonalizes the operator?

As a final topic in this paper, we turn back to the evolution (unitary) group $\exp(-itH)u_0$, which solves the Schrödinger equation

$$i\partial_t u = Hu, \quad u(0) = u_0.$$

The last thirty years have seen a very intensive research on the global (spacetime) properties of these solutions, known as "Strichartz and smoothing" estimates. Instead of treating the Schrödinger equation we choose here to address the generalized wave equation,

$$\partial_t^2 u = -Hu + f, \tag{1.6}$$

subject to initial conditions $u(0) = u_0,\ \partial_t u(0) = v_0$.

The conservation of energy for this equation (in the homogeneous case, $f = 0$) is given by

$$\int_{\mathbb{R}^n} [|H^\beta \partial_t u(x,t)|^2 + |H^{\beta+\frac{1}{2}} u(x,t)|^2] dx = \int_{\mathbb{R}^n} [|H^\beta v_0(x)|^2 + |H^{\beta+\frac{1}{2}} u_0(x)|^2] dx, \tag{1.7}$$

for any $\beta \in \mathbb{R}$, and any $t \in \mathbb{R}$.

In this context, the dispersive character of the equation means that the solution "escapes" from any bounded set, as $|t| \to \infty$, in some average sense. We would like to estimate this decay in terms of the initial energy norm , namely, the right-hand side of (1.7).

We therefore ask:

Question 4. Can one establish global L^2 spacetime estimates for solutions of (1.6) in terms of the initial energy norm?

In this paper we answer affirmatively the first three questions. As for Question 4, we provide such estimates by imposing restrictive hypotheses on the coeffficient matrix.

The precise statements, as well as discussions of the relevant bibliography for each topic, are given in Section 3.

The main technical tool used here consists of a close study of the properties of the resolvent $R(z)$ as z approaches the real axis.

To be more specific, we introduce the general notion of the "continuity up to the spectrum" of the resolvent.

Definition 1.1. *Let $[\alpha, \beta] \subseteq \mathbb{R}$. We say that H satisfies the "Limiting Absorption Principle" (LAP) in $[\alpha, \beta]$ if $R(z), z \in \mathcal{C}^\pm$, can be extended continuously to $Imz = 0, Rez \in [\alpha, \beta]$, in a suitable operator topology. In this case we denote the limiting values by $R^\pm(\lambda), \quad \alpha \leq \lambda \leq \beta$.*

The precise specification of the operator topology in the above definition is left open. Typically, it will be the uniform operator topology associated with weighted-L^2 or Sobolev spaces, which are introduced in Section 2.

Note that the limiting values $R^-(\lambda)$ are, generally speaking, different from $R^+(\lambda)$. In fact, one has (formally) the "Stieltjes formula"

$$A(\lambda) = \frac{1}{2\pi i}(R^+(\lambda) - R^-(\lambda)) = \frac{d}{d\lambda} E(\lambda),$$

where $E(\lambda)$ is the spectral family associated with H.

The operator $A(\lambda)$, $\lambda \in [0, \infty)$, known in the physical literature as the "density of states" [28, Chapter XIII], plays an important role in our study.

The paper is organized as follows.

Basic functional spaces and notations are introduced in Section 2.

Our results are stated as Theorems A,B,C in Section 3 . Around each of the three theorems we discuss some background material as well as relevant references. Obviously, the large amount of existing literature excludes any possibility of compiling an exhaustive bibliography.

Section 4 is devoted to revisiting the LAP as applied to the Laplacian H_0, and in particular obtaining uniform "low energy" estimates.

In Section 5 we prove Theorem A, the LAP for H.

The eigenfunction expansion theorem, Theorem B, is proved in Section 6.

The global spacetime estimates for the generalized wave equation (1.6), as stated in Theorem C, are proved in Section 7.

Some of the results presented here were announced in [9].

2. Functional Spaces and Notation

Throughout this paper we shall make use of the following weighted-L^2 and Sobolev spaces. First, for $s \in \mathbb{R}$ and m a nonnegative integer we define.

$$L^{2,s}(\mathbb{R}^n) := \{u(x) \quad / \quad \|u\|_{0,s}^2 = \int_{\mathbb{R}^n} (1+|x|^2)^s |u(x)|^2 dx < \infty\} \tag{2.1}$$

$$H^{m,s}(\mathbb{R}^n) := \{u(x) \quad / D^\alpha u \in L^{2,s}, \quad |\alpha| \le m, \quad \|u\|_{m,s}^2 = \sum_{|\alpha \le m} \|D^\alpha u\|_{0,s}^2\} \tag{2.2}$$

(we write L^2 for $L^{2,0}$ and $\|u\|_0 = \|u\|_{0,0}$).

More generally, for any $\sigma \in \mathbb{R}$, let $H^\sigma \equiv H^{\sigma,0}$ be the Sobolev space of order σ, namely,

$$H^\sigma = \{\hat{u} \quad / u \in L^{2,\sigma}, \quad \|\hat{u}\|_{\sigma,0} = \|u\|_{0,\sigma}\} \tag{2.3}$$

where the Fourier transform is defined as in (1.4).

For negative indices we denote by $\left\{H^{-m,s}, \quad \|\cdot\|_{-m,s}\right\}$ the dual space of $H^{m,-s}$. In particular, observe that any function $f \in H^{-1,s}$ can be represented (not uniquely) as

$$f = f_0 + \sum_{k=1}^n i^{-1} \frac{\partial}{\partial x_k} f_k, \qquad f_k \in L^{2,s}, \quad 0 \le k \le n. \tag{2.4}$$

In the case $n = 2$ and $s > 1$, we define

$$L_0^{2,s}(\mathbb{R}^2) = \{u \in L^{2,s}(\mathbb{R}^2) \quad /\hat{u}(0) = 0\},$$

and set $H_0^{-1,s}(\mathbb{R}^2)$ to be the space of functions $f \in H^{-1,s}(\mathbb{R}^2)$ which have a representation (2.4) where $f_k \in L_0^{2,s}, \quad k = 0, 1, 2.$

For any two normed spaces X, Y, we denote by $B(X, Y)$ the space of bounded linear operators from X to Y, equipped with the operator-norm $\| \cdot \|_{B(X,Y)}$ topology.

3. Statement of Results and Background

3.1. **The Limiting Absorption Principle–LAP.** We note that the operator H can be extended in an obvious way (retaining the same notation) as a bounded operator $H : H^1_{loc} \hookrightarrow H^{-1}_{loc}$. In particular, $H : H^{1,-s} \hookrightarrow H^{-1,-s}$, for all $s \geq 0$. Furthermore, the graph-norm of H in $H^{-1,-s}$ is equivalent to the norm of $H^{1,-s}$.

Similarly, we can consider the resolvent $R(z)$ as defined on $L^{2,s}$, $s \geq 0$, where $L^{2,s}$ is densely and continuously embedded in $H^{-1,s}$.

The basic technical tool used in the present paper is given in the following theorem. It has its own significance, stating that the resolvent is continuous up to the spectrum, including the threshold at $\lambda = 0$.

THEOREM A. *Suppose that $a(x)$ satisfies (1.1),(1.2). Then the operator H satisfies the LAP in $\mathbb{R}$. More precisely, let $s > 1$ and consider the resolvent $R(z) = (H - z)^{-1}, \quad Im\, z \neq 0$, as a bounded operator from $L^{2,s}(\mathbb{R}^n)$ to $H^{1,-s}(\mathbb{R}^n)$.*

Then:

(a) $R(z)$ is bounded with respect to the $H^{-1,s}(\mathbb{R}^n)$ norm. Using the density of $L^{2,s}$ in $H^{-1,s}$, we can therefore view $R(z)$ as a bounded operator from $H^{-1,s}(\mathbb{R}^n)$ to $H^{1,-s}(\mathbb{R}^n)$.

(b) The operator-valued functions, defined respectively in the lower and upper half-planes,

$$z \to R(z) \in B(H^{-1,s}(\mathbb{R}^n), H^{1,-s}(\mathbb{R}^n)), \qquad s > 1, \quad \pm Im\, z > 0, \tag{3.1}$$

can be extended continuously from $\mathcal{C}^{\pm} = \{z/ \pm Im\, z > 0\}$ to $\overline{\mathcal{C}^{\pm}} = \mathcal{C}^{\pm} \bigcup \mathbb{R}$ (with respect to the operator-norm topology of $B(H^{-1,s}(\mathbb{R}^n), H^{1,-s}(\mathbb{R}^n))$).

In the case $n = 2$ replace $H^{-1,s}$ by $H_0^{-1,s}$.

Notation: We denote the limiting values of the resolvent on the real axis by

$$R^{\pm}(\lambda) = \lim_{\epsilon \to \pm 0} R(\lambda + i\epsilon).$$

The spectrum of H is therefore entirely absolutely continuous. In particular, it follows that the limiting values $R^{\pm}(\lambda)$ are continuous at $\lambda = 0$ and H has no resonance there.

The main focus of Theorem A is the LAP for H at "low energies", i.e., in intervals $[\alpha, \beta]$ where $\alpha < 0 < \beta$.

However , to review the existing literature, we consider first the LAP in $(0, \infty)$, namely, over the *interior* of the spectrum.

Under assumptions close to ours here (but also assuming that $a(x)$ is continuously differentiable) a weaker version (roughly, "strong" instead of "uniform" convergence of the resolvents) was obtained by Eidus [34, Theorem 4 and Remark 1]. His approach relied on elliptic (kernel) estimates.

The systematic treatment of the LAP started with the work of Agmon [1]. He established it for operators of the type $H_0 + V$, where V is a short-range perturbation. To obtain the LAP for H_0 he considered the action of division by symbols with simple zeros in weighted Sobolev spaces. We therefore label this approach as the "Fourier approach" (see [41, Chapter 14]). The short-range potential was treated by perturbation methods.

Soon thereafter, two other approaches to the LAP were proposed, first the "Commutator method" (known as "Mourre's method") proposed in the classical paper [58] and then the "Spectral method" , initiated in joint works of the author with Devinatz [12, 13]. In its implementation for partial differential operators, this method relies on estimates of traces of Sobolev functions on characteristic manifolds, somewhat in analogy to the division by symbols with simple zeros in the case of the Fourier method. In fact, it implies the Hölder continuity of the limiting values $R^{\pm}(\lambda)$ in a suitable operator topology.

All three approaches yielded simple proofs for the LAP associated with $H = H_0 + V$, where V is short-range, in the *interior* $(0, \infty)$ of the spectrum.

Using one of the aforementioned approaches, the LAP for H has later been established , with V being a long-range or Stark-like potential [45, 5], a potential in $L^p(\mathbb{R}^n)$ [36, 47], a potential depending only on direction $(x/|x|)$ [38] or a perturbation of such a potential [61, 62]. In these latter cases the condition $\alpha > 0$ is replaced by $\alpha > \limsup_{|x|\to\infty} V(x)$.

The LAP for operators of the type $f(-\Delta) + V$, for a certain class of functions f, was derived in [17], using the spectral method .

A remarkable success of Mourre's method was in its application to the LAP in the case of the $N-$body Schrödinger operator (outside of thresholds) [60].

As mentioned in the Introduction, if the coefficient matrix $a(x)$ is smooth, the operator H can be viewed as the Laplace-Beltrami operator Δ_g on noncompact manifolds, where g is a smooth metric that approaches the Euclidean metric at infinity . The LAP in this case (in the interior of the spectrum) has already been established by Mourre. We refer to [65] and references therein for the case of perturbations of such operators. More recent works that employ the Mourre method for the derivation of the LAP in the interior of the spectrum, for asymptotically Euclidean spaces, are [75, Section 5] and [19, Theorem 2.2].

We now turn back to our topic here, the LAP in intervals containing the threshold at the bottom of the spectrum . The study of the resolvent near the threshold $\lambda = 0$ is sometimes referred to as "low energy estimates". The literature in this case is considerably more limited. An inspection of the aforementioned works shows that the methods they employ cannot be extended in a straightforward way to our operator H .

This case has been studied for the Laplacian H_0 in [12, Appendix A] and for H in the one-dimensional case ($n = 1$) in [8, 10, 27]. The present paper deals with the multi-dimensional case $n \geq 2$.

In recent works Bouclet [21] and Bony and Häfner [20] have applied the Mourre method in order to establish "low energy" LAP for Δ_g on noncompact manifolds of dimension $n \geq 3$, where the metric $g(x)$ is smooth but long-range.

The paper [64] deals with the two-dimensional ($n = 2$) case, but the resolvent $R(z)$ is restricted to continuous compactly supported functions f , thus enabling the use of pointwise decay estimates of $R(z)f$ at infinity.

Finally we mention the case of the closely related "acoustic propagator" , where the matrix $a(x) = b(x_1)I$ is scalar and dependent on a single coordinate , has been extensively studied [10, 22, 29, 31, 48, 49, 53], as well as the "anisotropic" case where $b(x_1)$ is a general positive matrix [11]. The LAP for the periodic case (namely, $a(x)$ is symmetric and periodic) has recently been established in [59] . Note that in this case the spectrum is absolutely continuous and consists of a union of intervals ("bands").

The proof of Theorem A , based on the spectral approach, is given in Section 5. It uses an extended version of the LAP for H_0, with the resolvent $R_0(z)$ acting on elements of $H^{-1,s}$, for suitable positive values of s (see Section 4).

Since $L^{2,s}$ (resp. $H^{1,-s}$) is densely and continuously embedded in $H^{-1,s}$ (resp. $L^{2,-s}$) , we conclude that the resolvents $R_0(z), R(z)$ can

be extended continuously to $\overline{\mathcal{C}^{\pm}}$ in the $B(L^{2,s}(\mathbb{R}^n), L^{2,-s}(\mathbb{R}^n))$ operator topology. An immediate consequence of this fact is the existence and completeness of the wave operators.

Using a well-known theorem of Kato and Kuroda [51] , we have the following immediate corollary concerning the completeness of the wave operators (see (1.3) for the definition).

Corollary 3.1. *The wave operators $W_{\pm}(H, H_0)$ exist and are complete.*

Indeed, all that is needed is that H, H_0 satisfy the LAP in $\mathbb{R}$, with respect to the same operator topologies.

We refer to the paper [46] where the existence and completeness of the wave operators $W_{\pm}(H, H_0)$ is established under suitable smoothness assumptions on $a(x)$ (however, $a(x)-I$ is not assumed to be compactly supported and H can include also magnetic and electric potentials).

3.2. **The Eigenfunction Expansion Theorem.** The spectral theorem (for self-adjoint operators) can be viewed as a "generalized eigenfunction theorem". In fact, using the result of Theorem A one can obtain a more refined version in this case as follows.

Let $\{E(\lambda),\ \lambda \in \mathbb{R}\}$ be the spectral family associated with H. Let $A(\lambda) = \frac{d}{d\lambda}E(\lambda)$ be its weak derivative. More precisely, we use the well-known formula,

$$A(\lambda) = \frac{1}{2\pi i}\lim_{\epsilon\to 0+}(R(\lambda+i\epsilon)-R(\lambda-i\epsilon)) = \frac{1}{2\pi i}(R^{+}(\lambda)-R^{-}(\lambda)). \tag{3.2}$$

By Theorem A we know that $A(\lambda) \in B(L^{2,s}(\mathbb{R}^n), L^{2,-s}(\mathbb{R}^n))$. The formal relation $(H-\lambda)A(\lambda) = 0$ can be given a rigorous meaning if, for example, we can find a bounded operator T such that $T^{*}A(\lambda)T$ is bounded in $L^2(\mathbb{R}^n)$ and has a complete set (necessarily at most countable) of eigenvectors. These will serve as "generalized eigenvectors" for H. We refer to [18, Chapters V,VI] and [23] for a development of this approach for self-adjoint elliptic operators. Note that by this approach we have at most a countable number of such generalized eigenvectors for any fixed λ. In the case of $H_0 = -\Delta$ they correspond to $|x|^{-\frac{n-3}{2}}J_{\sqrt{\kappa_j}}(\sqrt{\lambda}|x|)\psi_j(\omega)$, where $\kappa_j = \lambda_j + \frac{(n-1)(n-3)}{4}$, λ_j being the $j-$th eigenvalue of the Laplace-Beltrami operator on the unit sphere S^{n-1}, ψ_j the corresponding eigenfunction and J_ν is the Bessel function of order ν.

On the other hand, the Fourier expansion (1.5) can be viewed as expressing a function in terms of the "generalized eigenfunctions" $\exp(i\xi x)$ of H_0. Observe that now there is a continuum of such functions corresponding to $\lambda > 0$, namely, $|\xi|^2 = \lambda$.

From the physical point-of-view this expansion in terms of "plane waves" proves to be more useful for many applications. In particular, replacing $-\Delta$ by the Schrödinger operator $-\Delta + V(x)$ one can expect, under certain hypotheses on the potential V, a similar expansion in terms of "distorted plane waves". This has been accomplished, in increasing order of generality (more specifically, decay assumptions on $V(x)$ as $|x| \to \infty$) in [63, 44, 1, 68, 2]. See also [74] for an eigenfunction expansion for relativistic Schrödinger operators.

Here we use the LAP result of Theorem A in order to derive a similar expansion for the operator H. In fact, our generalized eigenfunctions are given by the following definition.

Definition 3.2. *For every $\xi \in \mathbb{R}^n$ let*

$$\begin{aligned} &\psi_\pm(x,\xi) = -R^\mp(|\xi|^2)((H-|\xi|^2)\exp(i\xi x)) = \\ &R^\mp(|\xi|^2)(\textstyle\sum_{l,j=1}^n \partial_l(a_{l,j}(x)-\delta_{l,j})\partial_j)\exp(i\xi x). \end{aligned} \tag{3.3}$$

The generalized eigenfunctions *of H are defined by*

$$\varphi_\pm(x,\xi) = \exp(i\xi x) + \psi_\pm(x,\xi). \tag{3.4}$$

We assume $n \geq 3$ in order to simplify the statement of the theorem. As we show below (see Proposition 6.1) the generalized eigenfunctions are (at least) continuous in x, so that the integral in the statement makes sense.

THEOREM B. *Suppose that $n \geq 3$ and that $a(x)$ satisfies (1.1),(1.2). For any compactly supported $f \in L^2(\mathbb{R}^n)$ define*

$$(\mathbb{F}_\pm f)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x)\overline{\varphi_\pm(x,\xi)}dx, \quad \xi \in \mathbb{R}^n. \tag{3.5}$$

Then the transformations $\mathbb{F}_\pm$ can be extended as unitary transformations (for which we retain the same notation) of $L^2(\mathbb{R}^n)$ onto itself. Furthermore, these transformations "diagonalize" H in the following sense.

$f \in L^2(\mathbb{R}^n)$ is in the domain $D(H)$ if and only if $|\xi|^2(\mathbb{F}_\pm f)(\xi) \in L^2(\mathbb{R}^n)$ and

$$H = \mathbb{F}_\pm^* M_{|\xi|^2}\mathbb{F}_\pm, \tag{3.6}$$

where $M_{|\xi|^2}$ is the multiplication operator by $|\xi|^2$.

3.3. **Spacetime Estimates for a Generalized Wave Equation.** The Strichartz estimates [72] have become a fundamental ingredient in the study of nonlinear wave equations. They are L^p spacetime estimates that are derived for operators whose leading part has constant coefficients. We refer to the books [70, 71] and [4] for detailed accounts and further references.

Here we focus exclusively on spacetime estimates pertinent to the framework of this paper, namely, weighted L^2 estimates. Indeed, once the "low energy estimates" of Theorem A are established, the method of proof here follows a standard methodology.

We recall first some results related to the Cauchy problem for the classical wave equation

$$\Box u = \frac{\partial^2 u}{\partial t^2} - \Delta u = 0, \tag{3.7}$$

subject to the initial data

$$u(x,0) = u_0(x), \quad \partial_t u(x,0) = v_0(x). \quad x \in \mathbb{R}^n. \tag{3.8}$$

The Morawetz estimate [56] yields

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} |x|^{-3} |u(x,t)|^2 dxdt \leq C(\|\nabla u_0\|_0^2 + \|v_0\|_0^2), \quad n \geq 4, \tag{3.9}$$

while in [7] we gave the estimate

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} |x|^{-2\alpha-1} |u(x,t)|^2 dxdt \leq C_\alpha(\||\nabla|^\alpha u_0\|_0^2 + \||\nabla|^{\alpha-1} v_0\|_0^2), \quad n \geq 3, \tag{3.10}$$

for every $\alpha \in (0,1)$.

Related results were obtained in [55] (allowing also dissipative terms) , [42] (with some gain in regularity), [76] (with short-range potentials) and [39] for spherically symmetric solutions .

Here we consider the equation

$$\frac{\partial^2 u}{\partial t^2} + Hu = \frac{\partial^2 u}{\partial t^2} - \sum_{i,j=1}^n \partial_i a_{i,j}(x) \partial_j u = f(x,t), \tag{3.11}$$

subject to the initial data (3.8).

We first replace the assumptions (1.1),(1.2) by stronger ones as follows.

$$(H1) \qquad a(x) = g^{-1}(x) = (g^{i,j}(x))_{1 \leq i,j \leq n}. \tag{3.12}$$

where $g(x) = (g_{i,j}(x))_{1\le i,j\le n}$ is a smooth Riemannian metric on $\mathbb{R}^n$ such that

$$g(x) = I \qquad for \quad |x| > \Lambda_0. \tag{3.13}$$

$(H2)$ The Hamiltonian flow associated with $h(x,\xi) = (g(x)\xi,\xi)$ is nontrapping for any (positive) value of h.

Recall that (H2) means that the flow associated with the Hamiltonian vectorfield $\mathcal{H} = \frac{\partial h}{\partial \xi}\frac{\partial}{\partial x} - \frac{\partial h}{\partial x}\frac{\partial}{\partial \xi}$ leaves any compact set in $\mathbb{R}^n_x$.

Identical hypotheses are imposed in the study of resolvent estimates in semi-classical theory [24, 25].

In our estimates we use "homogeneous Sobolev spaces" associated with the operator H.

We note that since H has no eigenvalue at zero, the operators H^{-1} and $H^{-\frac{1}{2}}$ are well defined self-adjoint operators. Note that $\|H^{\frac{1}{2}}\theta\|_0$ is equivalent to the homogeneous Sobolev norm $\|\nabla\theta\|_0$.

THEOREM C. *Suppose that $n \ge 3$ and that $a(x)$ satisfies Hypotheses (H1)-(H2). Let $s > 1$.*

(a) (local energy decay) Let $u_0 \in D(H^{\frac{1}{2}})$ and $v_0 \in L^2(\mathbb{R}^n)$. Then there exists a constant $C_1 = C_1(s,n) > 0$ such that the solution to (3.11),(3.8) satisfies,

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n}(1+|x|^2)^{-s}[|H^{\frac{1}{2}}u(x,t)|^2 + |u_t(x,t)|^2]dxdt \le C_1\{\|H^{\frac{1}{2}}u_0\|_0^2 + \|v_0\|_0^2 + \int_{\mathbb{R}}\int_{\mathbb{R}^n}|f(x,t)|^2dxdt\}. \tag{3.14}$$

(b) (amplitude decay) Assume $f = 0$. Let $u_0 \in L^2(\mathbb{R}^n)$ and $v_0 \in D(H^{-\frac{1}{2}})$. There exists a constant $C_2 = C_2(s,n) > 0$ such that the solution to (3.11),(3.8) satisfies,

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n}(1+|x|^2)^{-s}|u(x,t)|^2dxdt \le C_2[\|u_0\|_0^2 + \|H^{-\frac{1}{2}}v_0\|_0^2]. \tag{3.15}$$

These estimates generalize similar estimates obtained for the classical ($g = I$) wave equation [7, 55].

Remark 3.3. *The estimate* (3.14) *is an "energy decay estimate" for the wave equation* (3.11). *A localized (in space) version of the estimate*

has served to obtain global (small amplitude) existence theorems for the corresponding nonlinear equation [25, 40].

Remark 3.4. *The referee has pointed out to the author the recent preprint* [19, Theorem 1.3]*, where a more general result is obtained, with the metric being long-range.*

The weighted-L^2 spacetime estimates for the *dispersive* equation

$$i^{-1}\frac{\partial}{\partial t}u = Lu,$$

have been extensively treated in recent years. In general, in this case there is also a gain of derivatives (so called "smoothing") in addition to the energy decay. For the Schrödinger operator $L = -\Delta + V(x)$, with various assumptions on the potential V, we refer to [3, 6, 7, 15, 16, 42, 52, 67, 69, 77] and references therein. Smoothing estimates in the presence of magnetic potentials are considered in [30]. The Schrödinger operator on a Riemannian manifold is considered in [24, 33] . For more general operators see [14, 17, 26, 43, 57, 66, 73] and references therein.

4. The Operator $H_0 = -\Delta$

Let $\{E_0(\lambda)\}$ be the spectral family associated with H_0, so that

$$(E_0(\lambda)h, h) = \int\limits_{|\xi|^2\le\lambda} |\hat{h}|^2 d\xi, \qquad \lambda \ge 0, \quad h \in L^2(\mathbb{R}^n). \tag{4.1}$$

Following the methodology of [13, 32] we see that the weak derivative $A_0(\lambda) = \frac{d}{d\lambda}E_0(\lambda)$ exists in $B(L^{2,s}, L^{2,-s})$ for any $s > \frac{1}{2}$ and $\lambda > 0$. (Here and below we write $L^{2,s}$ for $L^{2,s}(\mathbb{R}^n)$). Furthermore,

$$< A_0(\lambda)h, h >= (2\sqrt{\lambda})^{-1} \int\limits_{|\xi|^2=\lambda} |\hat{h}|^2 d\tau, \tag{4.2}$$

where $<,>$ is the $(L^{2,-s}, L^{2,s})$ pairing (conjugate linear with respect to the second term) and $d\tau$ is the Lebesgue surface measure. Recall that by the standard trace lemma we have

$$\int\limits_{|\xi|^2=\lambda} |\hat{h}|^2 d\tau \le C\|\hat{h}\|^2_{H^s}, \quad s > \frac{1}{2}. \tag{4.3}$$

However, we can refine this estimate near $\lambda = 0$ as follows.

Proposition 4.1. *Let* $\frac{1}{2} < s < \frac{3}{2}$, $h \in L^{2,s}$. *For* $n = 2$ *assume further that* $s > 1$ *and* $h \in L_0^{2,s}$. *Then*

$$\int\limits_{|\xi|^2=\lambda} |\hat{h}|^2 d\tau \leq C \min(\lambda^\gamma, 1) \|\hat{h}\|_{H^s}^2, \tag{4.4}$$

where

$$\begin{aligned} &0 < \gamma = s - \frac{1}{2}, \quad n \geq 3, \\ &0 < \gamma < s - \frac{1}{2}, \quad n = 2, \end{aligned} \tag{4.5}$$

and $C = C(s, \gamma, n)$.

Proof. If $n \geq 3$, the proof follows as in [16, Appendix], using the "generalized Hardy inequality" due to Herbst [37], namely, that multiplication by $|\xi|^{-s}$ is bounded from H^s into L^2 (see also [54, Section 9.4]).

If $n = 2$ and $1 < s < \frac{3}{2}$ we have, for $h \in L_0^{2,s}$,

$$|\hat{h}(\xi)| = |\hat{h}(\xi) - \hat{h}(0)| \leq C_{s,\delta} |\xi|^\delta \|\hat{h}\|_{H^s},$$

for any $0 < \delta < \min(1, s-1)$. Using this estimate in the integral in the right-hand side of (4.4) the claim follows also in this case. □

Combining Equations (4.2),(4.3) and (4.4) we conclude that,

$$\begin{aligned} &| < A_0(\lambda) f, g > | \leq < A_0(\lambda) f, f >^{\frac{1}{2}} < A_0(\lambda) g, g >^{\frac{1}{2}} \\ &\leq C \min(\lambda^{-\frac{1}{2}}, \lambda^\eta) \|f\|_{0,s} \|g\|_{0,\sigma}, \quad f \in L^{2,s}, \quad g \in L^{2,\sigma}, \end{aligned} \tag{4.6}$$

where either

$$\begin{aligned} &(i) \quad n \geq 3, \quad \frac{1}{2} < s, \sigma < \frac{3}{2}, \quad s + \sigma > 2 \quad \text{and} \quad 0 < 2\eta = s + \sigma - 2, \\ &\qquad\qquad\qquad\qquad \text{or} \\ &(ii) \quad n = 2, 1 < s < \frac{3}{2}, \quad \frac{1}{2} < \sigma < \frac{3}{2}, \quad s + \sigma > 2, \quad 0 < 2\eta < s + \sigma - 2 \\ &\qquad\qquad\qquad\qquad \text{and} \quad \hat{f}(0) = 0. \end{aligned} \tag{4.7}$$

In both cases, $A_0(\lambda)$ is Hölder continuous and vanishes at $0, \infty$, so as in [13] we obtain

Proposition 4.2. *The operator-valued function*

$$z \to R_0(z) \in \begin{cases} B(L^{2,s}, L^{2,-\sigma}), & n \geq 3, \\ B(L_0^{2,s}, L^{2,-\sigma}), & n = 2, \end{cases} \tag{4.8}$$

where s, σ *satisfy (4.7), can be extended continuously from* $\mathcal{C}^\pm$ *to* $\overline{\mathcal{C}^\pm}$, *in the respective uniform operator topologies.*

Remark 4.3. *We note that the conditions* (4.7) *yield the continuity of* $A_0(\lambda)$ *across the threshold* $\lambda = 0$ *and hence the continuity property of the resolvent as in Proposition 4.2. However, for the local continuity at any* $\lambda_0 > 0$, *it suffices to take* $s, \sigma > \frac{1}{2}$, *as in* [1].

This remark applies equally to the statements below, where the resolvent is considered in other functional settings.

We shall now extend this proposition to more general function spaces. Let $g \in H^{1,\sigma}$, where s, σ satisfy (4.7). Let $f \in H^{-1,s}$ have a representation of the form (2.4). Equation (4.2) can be extended to yield an operator (for which we retain the same notation)

$$A_0(\lambda) \in B(H^{-1,s}, H^{-1,-\sigma}),$$

defined by (where now $<,>$ is used for the $(H^{-1,s}, H^{1,\sigma})$ pairing),

(4.9)
$$< A_0(\lambda)[f_0 + i^{-1}\sum_{k=1}^{n}\frac{\partial}{\partial x_k}f_k],\, g >$$
$$= (2\sqrt{\lambda})^{-1}\int\limits_{|\xi|^2=\lambda}[\hat{f}_0(\xi) + \sum_{k=1}^{n}\xi_k\hat{f}_k(\xi)]\overline{\hat{g}(\xi)}d\tau, \quad f \in H^{-1,s},\, g \in H^{1,\sigma}.$$

(replace $H^{-1,s}$ by $H_0^{-1,s}$ if $n = 2$).

Observe that this definition makes good sense even though the representation (2.4) is not unique, since

$$f = f_0 + \sum_{k=1}^{n} i^{-1}\frac{\partial}{\partial x_k}f_k = \tilde{f}_0 + \sum_{k=1}^{n} i^{-1}\frac{\partial}{\partial x_k}\tilde{f}_k,$$

implies

$$\hat{f}_0(\xi) + \sum_{k=1}^{n}\xi_k\hat{f}_k(\xi) = \hat{\tilde{f}}_0(\xi) + \sum_{k=1}^{n}\xi_k\hat{\tilde{f}}_k(\xi)$$

(as tempered distributions).

To estimate the operator-norm of $A_0(\lambda)$ in this setting we use (4.9) and the considerations preceding Proposition 4.2 , to obtain ,instead of (4.6), for $k = 1, 2, ..., n$,

(4.10)
$$\begin{aligned} &| < A_0(\lambda)\frac{\partial}{\partial x_k}f_k, g > | \\ &\leq C\min(\lambda^{-\frac{1}{2}}, \lambda^{\eta})\|f\|_{-1,s}\|g\|_{1,\sigma} \quad , \quad f \in H^{-1,s}, \quad g \in H^{1,\sigma}, \end{aligned}$$

where s, σ satisfy (4.7) (replace $H^{-1,s}$ by $H_0^{-1,s}$ if $n = 2$).

We now define the extension of the resolvent operator by

$$R_0(z) = \int_0^\infty \frac{A_0(\lambda)}{\lambda - z} d\lambda, \quad Im\, z \neq 0. \tag{4.11}$$

The convergence of the integral (in operator-norm) follows from the estimate (4.10).

The LAP in this case is given in the following proposition.

Proposition 4.4. *The operator-valued function $R_0(z)$ is well-defined (and analytic) for nonreal z in the following functional setting.*

$$z \to R_0(z) \in \begin{cases} B(H^{-1,s}, H^{1,-\sigma}), & n \geq 3, \\ B(H_0^{-1,s}, H^{1,-\sigma}), & n = 2, \end{cases} \tag{4.12}$$

where s, σ satisfy (4.7). Furthermore, it can be extended continuously from $\mathcal{C}^\pm$ to $\overline{\mathcal{C}^\pm}$, in the respective uniform operator topologies. The limiting values are denoted by $R_0^\pm(\lambda)$.

The extended function satisfies

$$(H_0 - z)R_0(z)f = f, \quad f \in H^{-1,s}, \quad z \in \overline{\mathcal{C}^\pm}, \tag{4.13}$$

where for $z = \lambda \in \mathbb{R}$, $\quad R_0(z) = R_0^\pm(\lambda)$.

Proof. We assume for simplicity $n \geq 3$. By Definition (4.11) and the estimate (4.10) , we get readily $R_0(z) \in B(H^{-1,s}, H^{-1,-\sigma})$ if $Im\, z \neq 0$, as well as the analyticity of the map $z \hookrightarrow R_0(z)$, $Im\, z \neq 0$. Furthermore, the extension to $Im\, z = 0$ is carried out as in [13].

Equation (4.13) is obvious if $Im\, z \neq 0$ and $f \in L^{2,s}$. By the density of $L^{2,s}$ in $H^{-1,s}$, the continuity of $R_0(z)$ on $H^{-1,s}$ and the continuity of $H_0 - z$ (in the sense of distributions) , we can extend it to all $f \in H^{-1,s}$.

As $z \to \lambda \pm i \cdot 0$ we have $R_0(z)f \to R_0^\pm(\lambda)f$ in $H^{-1,-\sigma}$. Applying the (constant coefficient) operator $H_0 - z$ yields, in the sense of distributions, $f = (H_0 - z)R_0(z)f \to (H_0 - \lambda)R_0^\pm(\lambda)f$ which establishes (4.13) also for $Im\, z = 0$.

Finally, the established continuity of $z \hookrightarrow R_0(z) \in B(H^{-1,s}, H^{-1,-\sigma})$ (up to the real boundary) and Equation (4.13) imply the continuity of the map $z \hookrightarrow H_0 R_0(z) \in B(H^{-1,s}, H^{-1,-\sigma})$.

The stronger continuity claim (4.12) follows since the norm of $H^{1,-\sigma}$ is equivalent to the graph-norm of H_0 as a map of $H^{-1,-\sigma}$ to itself. □

Remark 4.5. *The main point here is the fact that the limiting values can be extended continuously to the threshold at $\lambda = 0$.*

In the neighborhood of any $\lambda > 0$ this proposition follows from [68, Theorem 2.3]*, where a very different proof is used. In fact, using the terminology there, the limit functions $R_0^\pm(\lambda)f$ are the unique (on either*

side of the positive real axis) radiative functions and they satisfy a suitable "Sommerfeld radiation condition". We recall it here for the sake of completeness, since we will need it in the next section.

Let $z = k^2 \in \mathcal{C} \setminus \{0\}$, $Im\, k \geq 0$. For $f \in H^{-1,s}$ let $u = R_0(z)f \in H^{1,-\sigma}$ be as defined above. Then

$$\mathcal{R}u = \int\limits_{|x|>\Lambda_0} |r^{-\frac{n-1}{2}} \frac{\partial}{\partial r}(r^{\frac{n-1}{2}} u) - iku|^2 dx < \infty, \tag{4.14}$$

where $r = |x|$.

We shall refer to $\mathcal{R}u$ as the radiative norm *of u.*

Furthermore, we can take $\frac{1}{2} < s, \sigma$, as in Remark 4.3.

5. The Operator H

Fix $[\alpha, \beta] \in \mathbb{R}$ and let

$$\Omega = \big\{ z \in \mathcal{C}^+ \quad / \ \alpha < Re\, z < \beta, \quad 0 < Im\, z < 1 \big\}. \tag{5.1}$$

Let $z = \mu + i\varepsilon \in \Omega$ and consider the equation

$$(H - z)u = f \in H^{-1,s}, \quad u \in H^{1,-\sigma}, \quad (f \in H_0^{-1,s} \quad \text{if} \quad n = 2). \tag{5.2}$$

(Observe that in the case $n = 2$ also $u \in L_0^{2,\sigma}$).
With Λ_0 as in (1.2), let $\chi(x) \in C^\infty(\mathbb{R}^n)$ be such that

$$\chi(x) = \begin{cases} 0, & |x| < \Lambda_0 + 1, \\ 1, & |x| > \Lambda_0 + 2. \end{cases} \tag{5.3}$$

Equation (5.2) can be written as

$$(H_0 - z)(\chi u) = \chi f - 2\nabla\chi \cdot \nabla u - u\Delta\chi. \tag{5.4}$$

Letting $\psi(x) = 1 - \chi(\frac{x}{2}) \in C_0^\infty(\mathbb{R}^n)$ and using Proposition 4.4 and standard elliptic estimates, we obtain from (5.4)

$$\|u\|_{1,-\sigma} \leq C[\|f\|_{-1,s} + \|\psi u\|_{0,-s}], \tag{5.5}$$

where s, σ satisfy (4.7), and $C > 0$ depends only on Λ_0, σ, s, n.

We note that since ψ is compactly supported, the term $\|\psi u\|_{0,-s}$ can be replaced by $\|\psi u\|_{0,-s'}$ for any real s'.

In fact, the second term in the right-hand side can be dispensed with, as is demonstrated in the following proposition.

Proposition 5.1. *The solution to (5.2) satisfies,*

$$\|u\|_{1,-\sigma} \leq C\|f\|_{-1,s}, \tag{5.6}$$

where s, σ satisfy (4.7) and $C > 0$ depends only on σ, s, n, Λ_0.

Proof. In view of (5.5) we only need to show that

$$\|\psi u\|_{0,-s} \leq C\|f\|_{-1,s}. \tag{5.7}$$

Since $L^{2,s}(\mathbb{R}^n)$ is dense in $H^{-1,s}(\mathbb{R}^n)$ it suffices to prove this inequality for $f \in L^{2,s}(\mathbb{R}^n) \cap H^{-1,s}(\mathbb{R}^n)$ (using the norm of $H^{-1,s}$).

We argue by contradiction. Let

$$\{z_k\}_{k=1}^{\infty} \subseteq \Omega, \quad \{f_k\}_{k=1}^{\infty} \subseteq L^{2,s}(\mathbb{R}^n) \cap H^{-1,s}(\mathbb{R}^n)$$

(with $\hat{f}_k(0) = 0$ if $n = 2$) and

$$\{u_k = R(z_k)f_k\}_{k=1}^{\infty} \subseteq H^{1,-\sigma}(\mathbb{R}^n)$$

be such that,

$$\begin{aligned} \|\psi u_k\|_{0,-s} = 1, \quad \|f_k\|_{-1,s} \leq k^{-1}, \quad k = 1, 2, ... \\ z_k \to z_0 \in \overline{\Omega} \quad \text{as} \quad k \to \infty. \end{aligned} \tag{5.8}$$

By (5.5) $\{u_k\}_{k=1}^{\infty}$ is bounded in $H^{1,-\sigma}$. Replacing the sequence by a suitable subsequence (without changing notation) and using the Rellich compactness theorem we may assume that there exists a function $u \in L^{2,-\sigma'}$, $\sigma' > \sigma$, such that,

$$u_k \to u \quad \text{in} \quad L^{2,-\sigma'} \quad \text{as} \quad k \to \infty. \tag{5.9}$$

Furthermore, by weak compactness we actually have (restricting again to a subsequence if needed)

$$u_k \xrightarrow{w} u \quad \text{in} \quad H^{1,-\sigma} \quad \text{as} \quad k \to \infty. \tag{5.10}$$

Since H maps continuously $H^{1,-\sigma}$ into $H^{-1,-\sigma}$ we have

$$Hu_k \xrightarrow{w} Hu \quad \text{in} \quad H^{-1,-\sigma} \quad \text{as} \quad k \to \infty,$$

so that from $(H - z_k)u_k = f_k$ we infer that

$$(H - z_0)u = 0. \tag{5.11}$$

In view of (5.4) and Remark 4.5 the functions χu_k are "radiative functions" . Since they are uniformly bounded in $H^{1,-\sigma}$ their "radiative norms" (4.14) are uniformly bounded.

Suppose first that $z_0 \neq 0$. In view of Remark 4.5 we can take $s, \sigma > \frac{1}{2}$. Then the limit function u is a radiative solution to $(H_0 - z_0)u = 0$ in $|x| > \Lambda_0 + 2$ and hence must vanish there (see [68]). By the unique continuation property of solutions to (5.11) we conclude that $u \equiv 0$. Thus by (5.9) we get $\|\psi u_k\|_{0,-\sigma'} \to 0$ as $k \to \infty$, which contradicts (5.8).

We are therefore left with the case $z_0 = 0$. In this case $u \in H^{1,-\sigma}$ satisfies the equation

$$\nabla \cdot (a(x)\nabla u) = 0. \tag{5.12}$$

In particular, $\Delta u = 0$ in $|x| > \Lambda_0$ and

$$\int_{\Lambda_0}^{\infty} \int_{|x|=r} r^{-2\sigma} \big(|u|^2 + |\frac{\partial u}{\partial r}|^2\big) d\tau dr < \infty. \tag{5.13}$$

Consider first the case $n \geq 3$. We may then use the representation of u by spherical harmonics, so that, with $x = r\omega, \quad \omega \in S^{n-1}$,

$$u(x) = r^{-\frac{n-1}{2}} \Big\{ \sum_{j=0}^{\infty} b_j r^{\mu_j} h_j(\omega) + \sum_{j=0}^{\infty} c_j r^{-\nu_j} h_j(\omega) \Big\}, \quad r > \Lambda_0, \tag{5.14}$$

where,

$$\mu_j(\mu_j - 1) = \nu_j(\nu_j + 1) = \lambda_j + \frac{(n-1)(n-3)}{4}, \tag{5.15}$$

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ...$$

being the eigenvalues of the Laplace-Beltrami operator on S^{n-1}, and $h_j(\omega)$ the corresponding spherical harmonics. Since $\lambda_1 = n - 1$, it follows that

$$\mu_0 = \frac{n-1}{2}, \quad \mu_0 + 1 \leq \mu_1 \leq \mu_2 ... \quad \frac{n-3}{2} = \nu_0 < \nu_1 \leq \nu_2 ... \tag{5.16}$$

We now observe that (5.13) forces

$$b_0 = b_1 = \ldots = 0.$$

Also, by (5.14)

$$\int_{|x|=r} \frac{\partial u}{\partial r} d\tau = -(n-2)|S^{n-1}|c_0, \qquad r > \Lambda_0, \tag{5.17}$$

($|S^{n-1}|$ is the surface measure of S^{n-1}),
while integrating (5.12) we get

$$\int_{|x|=r} \frac{\partial u}{\partial r} d\tau = 0, \qquad r > \Lambda_0. \tag{5.18}$$

Thus $c_0 = 0$. It now follows from (5.14) that, for $r > \Lambda_0$,

$$\int_{|x|=r} \big(|u|^2 + |\frac{\partial u}{\partial r}|^2\big) d\tau \leq \big(\frac{r}{\Lambda_0}\big)^{-2\nu_1} \int_{|x|=\Lambda_0} \big(|u|^2 + |\frac{\partial u}{\partial r}|^2\big) d\tau. \tag{5.19}$$

Multiplying (5.12) by $\overline{u}$ and integrating by parts over the ball $|x| \leq r$, we infer from (5.19) that the boundary term vanishes as $r \to \infty$. Thus $\nabla u \equiv 0$, in contradiction to (5.8)-(5.9).

It remains to deal with the case $n = 2$. Instead of (5.14) we now have

(5.20)

$$u(x) = r^{-\frac{1}{2}}\Big\{\widetilde{b_0} r^{\frac{1}{2}} \log r + \sum_{j=0}^{\infty} b_j r^{\mu_j} h_j(\omega) + \sum_{j=1}^{\infty} c_j r^{-\nu_j} h_j(\omega)\Big\}, \quad r > \Lambda_0,$$

where $\mu_0 = \frac{1}{2}, \quad \mu_1 = \frac{3}{2}, \quad \nu_1 = \frac{1}{2}$.
As in the derivation above, the condition (5.13) yields $b_0 = b_1 = \ldots = 0$. Also, we get $\widetilde{b_0} = 0$ in view of (5.18). It now follows that

$$\int\limits_{|x|=r} \overline{u}\frac{\partial u}{\partial r} d\tau = -2\pi \sum_{j=1}^{\infty} (\nu_j + \frac{1}{2})|c_j|^2 r^{-2\nu_j - 1}, \qquad r \geq \Lambda_0, \tag{5.21}$$

from which, as in the argument following (5.19), we deduce that $u \equiv 0$, again in contradiction to (5.8)-(5.9). □

Proof of Theorem A. Part (a) of the theorem is actually covered by Proposition 5.1. Moreover, the proposition implies that the operator-valued function

$$z \to R(z) \in B(H^{-1,s}(\mathbb{R}^n), H^{1,-\sigma}(\mathbb{R}^n)), \qquad s > 1, \quad z \in \Omega,$$

is uniformly bounded, where s, σ satisfy (4.7). Here and below replace $H^{-1,s}$ by $H_0^{-1,s}$ if $n = 2$.

We next show that the function $z \to R(z)$ can be continuously extended to $\overline{\Omega}$ in the *weak toplogoy* of $B(H^{-1,s}(\mathbb{R}^n), H^{1,-\sigma}(\mathbb{R}^n))$. To this end, we take $f \in H^{-1,s}(\mathbb{R}^n)$ and $g \in H^{-1,\sigma}(\mathbb{R}^n)$ and consider the function

$$z \to < g, R(z)f >, \quad z \in \Omega,$$

where $<,>$ is the $(H^{-1,\sigma}, H^{1,-\sigma})$ pairing. We need to show that it can be extended continuously to $\overline{\Omega}$.

In view of the uniform boundedness established in Proposition 5.1, we can take f, g in dense sets (of the respective spaces). In particular, we can take $f \in L^{2,s}(\mathbb{R}^n)$ and $g \in L^{2,\sigma}(\mathbb{R}^n)$, so that the continuity property in Ω is obvious.

Consider therefore a sequence $\{z_k\}_{k=1}^{\infty} \subseteq \Omega$ such that $z_k \xrightarrow[k\to\infty]{} z_0 \in [\alpha, \beta]$. The sequence $\{u_k = R(z_k)f\}_{k=1}^{\infty}$ is bounded in $H^{1,-\sigma}(\mathbb{R}^n)$. Therefore there exists a subsequence $\left\{u_{k_j}\right\}_{j=1}^{\infty}$ which converges to a function $u \in L^{2,-\sigma'}$, $\sigma' > \sigma$.

We can further assume that $u_{k_j} \xrightarrow[j\to\infty]{w} u$ in $H^{1,-\sigma}$. It follows that

$$< g, u_{k_j} > \xrightarrow[j\to\infty]{} < g, u > .$$

Passing to the limit in $(H - z_{k_j})u_{k_j} = f$ we see that the limit function satisfies

$$(H - z_0)u = f.$$

We now repeat the argument employed in the proof of Proposition 5.1. If $z_0 \neq 0$ we note that the functions $\{\chi u_k\}_{k=1}^{\infty}$ are radiative functions with uniformly bounded "radiative norms" (4.14) in $|x| > \Lambda_0 + 2$. The same is therefore true for the limit function u.

If $z_0 = 0$ the function $u \in H^{1,-\sigma}$ solves $Hu = f$.

In both cases this function is unique and we get the convergence

$$< g, R(z_k)f >=< g, u_k > \xrightarrow[k\to\infty]{} < g, u > .$$

We can now define

$$R^+(z_0)f = u, \tag{5.22}$$

with an analogous definition for $R^-(z_0)$.

At this point we can readily deduce the following extension of the resolvent $R(z)$ as the inverse of $H - z$.

$$(H - z)R(z)f = f, \quad f \in H^{-1,s}, \quad z \in \overline{\mathcal{C}^{\pm}}, \tag{5.23}$$

where $R(z) = R^{\pm}(\lambda)$ when $z = \lambda \in \mathbb{R}$.

Indeed, observe that if $Im\, z \neq 0$ then $(H - z)R(z)f = f$ for $f \in L^{2,s}(\mathbb{R}^n)$ and $(H - z)R(z) \in B(H^{-1,s}, H^{-1,-\sigma})$, so the assertion follows from the density of $L^{2,s}(\mathbb{R}^n)$ in $H^{-1,s}(\mathbb{R}^n)$. For $z = \lambda \in \mathbb{R}$ we use the (just established) weak continuity of the map $z \hookrightarrow (H - z)R(z)$ from $H^{-1,s}$ into $H^{-1,-\sigma}$ in $\overline{\mathcal{C}^{\pm}}$.

The passage "from weak to uniform continuity" (in the operator topology) is a classical argument due to Agmon [1]. In [8] we have applied it in the case $n = 1$. Here we outline the proof in the case $n > 1$.

We establish first the continuity of the operator-valued function $z \to R(z)$, $\overline{\Omega}$, in the *uniform operator topologoy* of $B(H^{-1,s}(\mathbb{R}^n), L^{2,-\sigma}(\mathbb{R}^n))$.

Let $\{z_k\}_{k=1}^{\infty} \subseteq \overline{\Omega}$ and $\{f_k\}_{k=1}^{\infty} \subseteq H^{-1,s}(\mathbb{R}^n)$ be sequences such that

$z_k \xrightarrow[k\to\infty]{} z \in \overline{\Omega}$ and f_k converges weakly to f in $H^{-1,s}(\mathbb{R}^n)$. It suffices to prove that the sequence $u_k = R(z_k)f_k$, which is bounded in $H^{1,-\sigma}(\mathbb{R}^n)$, converges strongly in $L^{2,-\sigma}(\mathbb{R}^n)$. Since this is clear if $Im\, z \neq 0$, we can take $z \in [\alpha, \beta]$.

Note first that we can take $\frac{1}{2} < \sigma' < \sigma$ so that s, σ' satisfy (4.7). Then the sequence $\{u_k\}_{k=1}^{\infty}$ is bounded in $H^{1,-\sigma'}(\mathbb{R}^n)$ and there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ which converges to a function $u \in L^{2,-\sigma}$.

We can further assume that $u_{k_j} \xrightarrow[j\to\infty]{w} u$ in $H^{1,-\sigma}$.

It follows that the limit function satisfies (see Eq. (5.23))

$$(H - z)u = f.$$

Once again we consider separately the cases $z \neq 0$ and $z = 0$.

In the first case, in view of (5.23) and Remark 4.5 the functions χu_k are "radiative functions" . Since they are uniformly bounded in $H^{1,-\sigma}$ their "radiative norms" (4.14) are uniformly bounded, and we conclude that also $\mathcal{R}u < \infty$.

In the second case, we simply note that $u \in H^{1,-\sigma}$ solves $Hu = f$.

As in the proof of Proposition 5.1 we conclude that in both cases the limit is unique, so that the whole sequence $\{u_k\}_{k=1}^{\infty}$ converges to u in $L^{2,-\sigma}(\mathbb{R}^n)$.

Thus, the continuity in the uniform operator topologoy of $B(H^{-1,s}(\mathbb{R}^n), L^{2,-\sigma}(\mathbb{R}^n))$ is proved.

Finally, we claim that the operator-valued function $z \to R(z)$ is continuous in the *uniform operator toplogoy* of $B(H^{-1,s}(\mathbb{R}^n), H^{1,-\sigma}(\mathbb{R}^n))$. Indeed, if we invoke Eq. (5.23) we get that also $z \to HR(z)$ is continuous in the uniform operator topology of $B(H^{-1,s}(\mathbb{R}^n), H^{-1,-\sigma}(\mathbb{R}^n))$.

Since the domain of H in $H^{-1,-\sigma}(\mathbb{R}^n)$ is $H^{1,-\sigma}(\mathbb{R}^n)$, the claim follows. The conclusion of the theorem follows by taking $\sigma = s$. □

Remark 5.2. *In view of* (5.4) *and Remark 4.5 it follows that for* $\lambda > 0$ *the functions* $R^{\pm}(\lambda)f$, $f \in H^{-1,s}$, *are "radiative", i.e., satisfy a Sommerfeld radiation condition.*

6. The Eigenfunction Expansion Theorem

In this section we prove Theorem B stated in Section 3. We first collect some basic properties of the generalized eigenfunctions in the following proposition.

Proposition 6.1. *The generalized eigenfunctions*

$$\varphi_{\pm}(x, \xi) = \exp(i\xi x) + \psi_{\pm}(x, \xi)$$

(see (3.4)*) are in* $H^1_{loc}(\mathbb{R}^n)$ *for each fixed* $\xi \in \mathbb{R}^n$ *and satisfy the equation*

$$(H - |\xi|^2)\varphi_{\pm}(x, \xi) = 0. \tag{6.1}$$

In addition, these functions have the following properties:

(i) The map

$$\mathbb{R}^n \ni \xi \hookrightarrow \psi_{\pm}(\cdot, \xi) \in H^{1,-s}(\mathbb{R}^n), \quad s > 1,$$

is continuous.

(ii) For any compact $K \subseteq \mathbb{R}^n$ the family of functions $\big\{\varphi_\pm(x,\xi), \quad \xi \in K\big\}$ is uniformly bounded and uniformly Hölder continuous in $x \in \mathbb{R}^n$.

Proof. Since $(H-|\xi|^2)\exp(i\xi x) \in H^{-1,s}$, $s>1$, Equation (6.1) follows from the definition (3.3) in view of Equation (5.23).

Furthermore, the map

$$\mathbb{R}^n \ni \xi \hookrightarrow (H-|\xi|^2)\exp(i\xi x) \in H^{-1,s}(\mathbb{R}^n), \quad s>1,$$

is continuous, so the continuity assertion (i) follows from Theorem A.

For $s>1$ the set of functions $\{\psi_\pm(\cdot,\xi), \quad \xi \in K\}$ is uniformly bounded in $H^{1,-s}$. Thus, in view of (6.1), it follows from the De Giorgi-Nash-Moser Theorem [35, Chapter 8] that the set $\big\{\varphi_\pm(x,\xi), \quad \xi \in K\big\}$ is uniformly bounded and uniformly Hölder continuous in $\{|x|<R\}$ for every $R>0$. In particular, we can take $R>\Lambda_0$ (see Equation (1.2)). In the exterior domain $\{|x|>R\}$ the set $\{\psi_\pm(x,\xi),\, \xi \in K\}$ is bounded in $H^{1,-s}$, $s>1$, and we have $(H_0-|\xi|^2)\psi_\pm(x,\xi)=0$.

In addition the boundary values $\{\psi_\pm(x,\xi),\, |x|=R,\, \xi \in K\}$ are uniformly bounded. From well-known properties of solutions of the Helmholtz equation we conclude that this set is uniformly bounded and therefore, invoking once again the De Giorgi-Nash-Moser Theorem , uniformly Hölder continuous. □

Proof of Theorem B. We use the LAP proved in Theorem A, adapting the methodology of Agmon's proof [1] for the eigenfunction expansion in the case of Schrödinger operators with short-range potentials. To simplify notation, we prove for $\mathbb{F}_+$.

Let $u \in H^1$ be compactly supported. For any z such that $Imz \neq 0$ we can write its Fourier transform as

$$\hat{u}(\xi) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} u(x)\exp(-i\xi x)dx = \frac{(2\pi)^{-\frac{n}{2}}}{|\xi|^2-z} \int\limits_{\mathbb{R}^n} u(x)(H_0-z)\exp(-i\xi x)dx.$$

Let $\theta \in C_0^\infty(\mathbb{R}^n)$ be a (real) cutoff function such that $\theta(x)=1$ for x in a neighborhood of the support of u.

We can rewrite the above equality as

$$\hat{u}(\xi) = \frac{(2\pi)^{-\frac{n}{2}}}{|\xi|^2-z} < (H_0-z)u(x), \theta(x)\exp(i\xi x) >,$$

where $< \cdot,\cdot >$ is the $(H^{-1,s}, H^{1,-s})$ bilinear pairing (conjugate linear with respect to the second term).

We have therefore, with $f = (H - z)u$,

$$\hat{u}(\xi) = \tag{6.2}$$

$$\frac{(2\pi)^{-\frac{n}{2}}}{|\xi|^2 - z}(< (H - z)u(x), \theta(x)\exp(i\xi x) > + \overline{< (H_0 - H)\exp(i\xi x), u(x) >})$$

$$= \frac{(2\pi)^{-\frac{n}{2}}}{|\xi|^2 - z}(< f(x), \theta(x)\exp(i\xi x) > + < f(x), R(\bar{z})(H_0 - H)\exp(i\xi x) >).$$

Introducing the function

$$\widetilde{f}(\xi, z) = \hat{f}(\xi) + (2\pi)^{-\frac{n}{2}} < f(x), R(\bar{z})(H_0 - H)\exp(i\xi x) >,$$

we have

$$\hat{u}(\xi) = \widehat{R(z)f}(\xi) = \frac{\widetilde{f}(\xi, z)}{|\xi|^2 - z}, \quad Im\, z \neq 0, \tag{6.3}$$

We now claim that this equation is valid for all compactly supported $f \in H^{-1}$.

Indeed, let $u = R(z)f \in H^{1,-s}$, $s > 1$. Let $\psi(x) = 1 - \chi(x)$, where $\chi(x)$ is defined in (5.3).

We set

$$u_k(x) = \psi(k^{-1}x)u(x), \quad f_k(x) = (H - z)(\psi(k^{-1}x)u(x)),\ k = 1, 2, 3...$$

The equality (6.3) is satisfied with u, f replaced, respectively, by u_k, f_k.

Since

$$\psi(k^{-1}x)u(x) \xrightarrow[k\to\infty]{} u(x)$$

in $H^{1,-s}$, we have

$$(H - z)(\psi(k^{-1}x)u(x)) \xrightarrow[k\to\infty]{} (H - z)u = f(x)$$

in $H^{-1,-s}$, where in the last step we have used Equation (5.23).

In addition, since $(H_0 - H)\exp(i\xi x)$ is compactly supported

$$< f_k(x), R(\bar{z})(H_0 - H)\exp(i\xi x) >= \overline{< (H_0 - H)\exp(i\xi x), R(z)f_k(x) >}$$

$$\xrightarrow[k\to\infty]{} \overline{< (H_0 - H)\exp(i\xi x), R(z)f >} =< f, R(\bar{z})(H_0 - H)\exp(i\xi x) > .$$

Combining these considerations with the continuity of the Fourier transform (on tempered distributions) we establish that (6.3) is valid for all compactly supported $f \in H^{-1}$.

Let $\{E(\lambda),\ \lambda \in \mathbb{R}\}$ be the spectral family associated with H. Let $A(\lambda) = \frac{d}{d\lambda}E(\lambda)$ be its weak derivative. More precisely, we use the

well-known formula,

$$A(\lambda) = \frac{1}{2\pi i} \lim_{\epsilon \to 0+} (R(\lambda + i\epsilon) - R(\lambda - i\epsilon)),$$

to get (using Theorem A), for any $f \in H^{-1,s}$, $s > 1$,

$$< f, A(\lambda) f >= \frac{1}{2\pi i} < f, (R^+(\lambda) - R^-(\lambda)) f > .$$

We now take $f \in L^2$ and compactly supported. From the resolvent equation we infer

$$R(\lambda + i\epsilon) - R(\lambda - i\epsilon) = 2i\epsilon R(\lambda + i\epsilon) R(\lambda - i\epsilon), \quad \epsilon > 0,$$

so that

$$< f, A(\lambda) f >= \lim_{\epsilon \to 0+} \frac{\epsilon}{\pi} \|R(\lambda + i\epsilon) f\|_0^2, \quad \epsilon > 0.$$

Using Equation (6.3) and Parseval's theorem we therefore have,

$$\begin{aligned} &< f, A(\lambda) f > \\ &= \lim_{\epsilon \to 0+} \frac{\epsilon}{\pi} \|(|\xi|^2 - (\lambda + i\epsilon))^{-1} \widetilde{f}(\xi, \lambda + i\epsilon)\|_0^2, \quad \epsilon > 0. \end{aligned} \tag{6.4}$$

Note that $\widetilde{f}(\xi, z)$ can be extended continuously as $z \to \lambda + i \cdot 0$ by

$$\widetilde{f}(\xi, \lambda) = \hat{f}(\xi) + (2\pi)^{-\frac{n}{2}} < f(x), R^-(\lambda)(H_0 - H) \exp(i\xi x) > . \tag{6.5}$$

In order to study properties of $\widetilde{f}(\xi, z)$ as a function of ξ we compute

$$\begin{aligned} \widetilde{f}(\xi, z) &= \hat{f}(\xi) + (2\pi)^{-\frac{n}{2}} \overline{< (\sum_{l,j=1}^{n} \partial_l (a_{l,j}(x) - \delta_{l,j}) \partial_j) \exp(i\xi x), R(z) f(x) >} \\ &= \hat{f}(\xi) + (2\pi)^{-\frac{n}{2}} i \sum_{l,j=1}^{n} \xi_j \int_{\mathbb{R}^n} (a_{l,j}(x) - \delta_{l,j}) \partial_l (R(z) f(x)) \exp(-i\xi x) dx, \end{aligned} \tag{6.6}$$

where in the last step we have used that both $\partial_l (R(z) f(x))$ and $(a_{l,j}(x) - \delta_{l,j}) \exp(-i\xi x)$ are in L^2.

Consider now the integral

$$g(\xi, z) = \int_{\mathbb{R}^n} (a_{l,j}(x) - \delta_{l,j}) \partial_l (R(z) f(x)) \exp(-i\xi x) dx, \quad z \in \Omega,$$

where Ω is as in (5.1).

In view of Theorem A the family $\{\partial_l R(z) f(x)\}_{z \in \Omega}$ is uniformly bounded in $L^{2,-s}$, $s > 1$, so by Parseval's theorem we get

$$\|g(\cdot, z)\|_0 < C, \quad z \in \Omega,$$

where C only depends on f.

This estimate and (6.6) imply that, if $f \in L^2$ is compactly supported:

(i) The function

$$\mathbb{R}^n \times \overline{\Omega} \ni (\xi, z) \to \widetilde{f}(\xi, z)$$

is continuous. For real z it is given by (6.5).

(ii)

$$\lim_{k\to\infty} \int\limits_{|\xi|>k} (|\xi|^2 - z)^{-1} |\widetilde{f}(\xi, z)|^2 d\xi = 0,$$

uniformly in $z \in \Omega$.

As $z \to |\xi|^2 + i \cdot 0$, we have by Theorem A and Equation (3.4),

$$\lim_{z\to|\xi|^2+i\cdot 0} \widetilde{f}(\xi, z) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} f(x)\overline{\varphi_+(x,\xi)}dx = \mathbb{F}_+ f(\xi), \tag{6.7}$$

so that , taking (i) and (ii) into account we obtain from (6.4), for any compactly supported $f \in L^2$,

$$< f, A(\lambda) f >= \frac{1}{2\sqrt{\lambda}} \int\limits_{|\xi|^2=\lambda} |\mathbb{F}_+ f(\xi)|^2 d\sigma, \quad \lambda > 0, \tag{6.8}$$

where $d\sigma$ is the surface Lebesgue measure.

It follows that for any $[\alpha, \beta] \subseteq [0, \infty)$,

$$\begin{aligned} ((E(\beta) - E(\alpha))f, f) &= \int_\alpha^\beta < f, A(\lambda) f > d\lambda \\ &= \int\limits_{\alpha\le|\xi|^2\le\beta} |\mathbb{F}_+ f(\xi)|^2 d\xi. \end{aligned} \tag{6.9}$$

Letting $\alpha \to 0$, $\beta \to \infty$, we get

$$\|f\|_0 = \|\mathbb{F}_+ f\|_0. \tag{6.10}$$

Thus $f \to \mathbb{F}_+ f \in L^2(\mathbb{R}^n)$ is an isometry for compactly supported functions, which can be extended by density to all $f \in L^2(\mathbb{R}^n)$.

Furthermore, since the spectrum of H is entirely absolutely continuous, it follows that for every $f \in L^2$, Equation (6.8) holds for almost all $\lambda > 0$ (with respect to the Lebesgue measure).

Let $f \in D(H)$. By the spectral theorem

$$< Hf, A(\lambda) Hf >= \lambda^2 < f, A(\lambda) f >= \frac{1}{2\sqrt{\lambda}} \int\limits_{|\xi|^2=\lambda} ||\xi|^2 \mathbb{F}_+ f(\xi)|^2 d\sigma, \quad \lambda > 0.$$

In particular,

$$\|Hf\|_0^2 = \int\limits_{\mathbb{R}^n} ||\xi|^2 \mathbb{F}_+ f(\xi)|^2 d\xi. \tag{6.11}$$

Conversely, if the right-hand side of (6.11) is finite, then $\int_0^\infty \lambda^2 < f, A(\lambda) f > d\lambda < \infty$, so $f \in D(H)$.

The adjoint operator $\mathbb{F}_+^*$ is a partial isometry (on the range of $\mathbb{F}_+$). If $f(x) \in L^2(\mathbb{R}^n)$ is compactly supported and $g(\xi) \in L^2(\mathbb{R}^n)$ is likewise compactly supported then

$$\begin{aligned}(\mathbb{F}_+ f, g) &= (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} (\int\limits_{\mathbb{R}^n} f(x) \overline{\varphi_+(x,\xi)} dx) \overline{g(\xi)} d\xi \\ &= (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} f(x) (\int\limits_{\mathbb{R}^n} \overline{g(\xi)\varphi_+(x,\xi)} d\xi) dx,\end{aligned}$$

where in the change of order of integration Proposition 6.1 was taken into account.

It follows that for a compactly supported $g(\xi) \in L^2(\mathbb{R}^n)$,

$$(\mathbb{F}_+^* g)(x) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} g(\xi) \varphi_+(x,\xi) d\xi, \tag{6.12}$$

and the extension to all $g \in L^2(\mathbb{R}^n)$ is obtained by the fact that $\mathbb{F}_+^*$ is a partial isometry.

Now if $f \in D(H)$, $g \in L^2(\mathbb{R}^n)$, we have

$$(Hf, g) = \int_{\mathbb{R}^n} |\xi|^2 \mathbb{F}_+ f(\xi) \overline{\mathbb{F}_+ g(\xi)} d\xi = \int_{\mathbb{R}^n} \mathbb{F}_+^* (|\xi|^2 \mathbb{F}_+ f(\xi)) \overline{g(\xi)} d\xi,$$

which is the statement (3.6) of the theorem.

It follows from the spectral theorem that for every interval $J = [\alpha, \beta] \subseteq [0, \infty)$ and for every $f \in L^2(\mathbb{R}^n)$ we have, with $E_J = E(\beta) - E(\alpha)$ and χ_J the characteristic function of J,

$$E_J f(x) = \mathbb{F}_+^* (\chi_J(|\xi|^2) \mathbb{F}_+ f(\xi)),$$

or

$$\mathbb{F}_+ E_J f(\xi) = \chi_J(|\xi|^2) \mathbb{F}_+ f(\xi).$$

It remains to prove that the isometry $\mathbb{F}_+$ is onto (and hence unitary). So, suppose to the contrary that for some nonzero $g(\xi) \in L^2(\mathbb{R}^n)$

$$(\mathbb{F}_+^* g)(x) = 0.$$

In particular, for any $f \in L^2(\mathbb{R}^n)$ and any interval J as above,

$$0 = (E_J f, \mathbb{F}_+^* g) = (\mathbb{F}_+ E_J f, g) = (\chi_J(|\xi|^2) \mathbb{F}_+ f(\xi), g(\xi)) = (\mathbb{F}_+ f(\xi), \chi_J(|\xi|^2) g(\xi)),$$

so that $\mathbb{F}^*_+(\chi_J(|\xi|^2)g(\xi)) = 0$.

By Equation (6.12) we have, for any $0 \leq \alpha < \beta$,

$$\int\limits_{\alpha<|\xi|^2<\beta} g(\xi)\varphi_+(x,\xi)d\xi = 0,$$

so that, in view of the continuity properties of $\varphi_+(x,\xi)$ (see Proposition 6.1) , for a.e. $\lambda \in (0,\infty)$,

$$\int\limits_{|\xi|^2=\lambda} g(\xi)\varphi_+(x,\xi)d\sigma = 0. \tag{6.13}$$

From the definition (3.4) we get,

$$\int\limits_{|\xi|^2=\lambda} g(\xi)\exp(i\xi x)d\sigma - \int\limits_{|\xi|^2=\lambda} g(\xi)R^-(\lambda)((H-\lambda)\exp(i\xi x))d\sigma = 0. \tag{6.14}$$

Since $(H-\lambda)\exp(i\xi x)$ is compactly supported (when $|\xi|^2 = \lambda$), the continuity property of $R^-(\lambda)$ enables us to write

$$\int\limits_{|\xi|^2=\lambda} g(\xi)R^-(\lambda)((H-\lambda)\exp(i\xi x))d\sigma = R^-(\lambda)\int\limits_{|\xi|^2=\lambda} g(\xi)(H-\lambda)\exp(i\xi x)d\sigma,$$

which, by Remark 5.2, satisfies a Sommerfeld radiation condition. We conclude that the function

$$G(x) = \int\limits_{|\xi|^2=\lambda} g(\xi)\exp(i\xi x)d\sigma \in H^{1,-s}, \quad s > \frac{1}{2},$$

is a radiative solution (see Remark 4.5) of $(-\Delta-\lambda)G = 0$, and hence must vanish. Since this holds for a.e. $\lambda > 0$, we get $\hat{g}(\xi) = 0$, hence $g = 0$. □

7. **Global Spacetime Estimates**

Proof of Theorem C. .

(a) Define , with $G = H^{\frac{1}{2}}$,

$$u_\pm = \frac{1}{2}(Gu \pm i\partial_t u). \tag{7.1}$$

Then

$$\partial_t u_\pm = \mp iGu_\pm \pm \frac{i}{2}f. \tag{7.2}$$

Defining

$$U(t) = \begin{pmatrix} u_+(t) \\ u_-(t) \end{pmatrix} \tag{7.3}$$

we have

$$i^{-1}U'(t) = -KU + F, \tag{7.4}$$

$$K = \begin{pmatrix} G & 0 \\ 0 & -G \end{pmatrix}, \quad F(t) = \begin{pmatrix} \frac{1}{2}f(\cdot,t) \\ -\frac{1}{2}f(\cdot,t) \end{pmatrix}.$$

Note that, as is common when treating evolution equations, we write $U(t), F(t)...$ for $U(x,t), F(x,t)...$ when there is no risk of confusion.

The operator K is a self adjoint operator on $\mathcal{D} = L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$. Its spectral family $E_K(\lambda)$ is given by $E_K(\lambda) = E_G(\lambda) \oplus (I - E_G(-\lambda))$, $\lambda \in \mathbb{R}$, where E_G is the spectral family of G.

Let $E(\lambda)$ be the spectral family of H, and let $A(\lambda) = \frac{d}{d\lambda}E(\lambda)$ be its weak derivative (3.2). By the definition of G we have

$$E_G(\lambda) = E(\lambda^2),$$

hence its weak derivative is given by

$$A_G(\lambda) = \frac{d}{d\lambda}E_G(\lambda) = 2\lambda A(\lambda^2), \quad \lambda > 0. \tag{7.5}$$

In view of the LAP (Theorem A) we therefore have that the operator-valued function

$$A_G(\lambda) \in B(L^{2,s}(\mathbb{R}^n), L^{2,-s}(\mathbb{R}^n)),$$

is continuous for $\lambda \geq 0$.

Denoting $\mathcal{D}^s = L^{2,s}(\mathbb{R}^n) \oplus L^{2,s}(\mathbb{R}^n)$, it follows that

$$A_K(\lambda) = \frac{d}{d\lambda}E_K(\lambda) = A_G(\lambda) \oplus A_G(-\lambda), \ \lambda \in \mathbb{R},$$

is continuous with values in $B(\mathcal{D}^s, \mathcal{D}^{-s})$ for $s > 1$.

Making use of Hypotheses (H1)-(H2), we invoke [65, Theorem 5.1] to conclude that $\limsup_{\mu\to\infty} \mu^{\frac{1}{2}} \|A(\mu)\|_{B(L^{2,s},L^{2,-s})} < \infty$, so that by (7.5) there exists a constant $C > 0$, such that

$$\|A_G(\lambda)\|_{B(L^{2,s},L^{2,-s})} < C, \quad \lambda \geq 0. \tag{7.6}$$

It follows that also

$$\|A_K(\lambda)\|_{B(\mathcal{D}^s,\mathcal{D}^{-s})} < C, \quad \lambda \in \mathbb{R}, \ s > 1, \ \lambda \in \mathbb{R}. \tag{7.7}$$

Let $<,>$ be the bilinear pairing between $\mathcal{D}^{-s}$ and $\mathcal{D}^s$ (conjugate linear with respect to the second term).

For any $\psi, \chi \in \mathcal{D}^s$ we have, in view of the fact that $A_K(\lambda)$ is a weak derivative of a spectral measure,

$$\begin{aligned} &(i) \qquad \left| < A_K(\lambda)\psi, \chi > \right|^2 \leq < A_K(\lambda)\psi, \psi > \cdot < A_K(\lambda)\chi, \chi >, \\ &(ii) \qquad \int_{-\infty}^{\infty} < A_K(\lambda)\psi, \psi > d\lambda = \|\psi\|^2_{L^2(\mathbb{R}^n)\oplus L^2(\mathbb{R}^n)}. \end{aligned} \tag{7.8}$$

We first treat the pure Cauchy problem, i.e., $f \equiv 0$.

To estimate $U(x,t) = e^{-itK}U(x,0)$ we use a duality argument. Some of the following computations will be rather formal, but they can easily be justified by a density argument, as in [7, 17]. We shall use $((,))$ for the scalar product in $L^2(\mathbb{R}^{n+1}) \oplus L^2(\mathbb{R}^{n+1})$.

Take $w(x,t) \in C_0^\infty(\mathbb{R}^{n+1}) \oplus C_0^\infty(\mathbb{R}^{n+1})$. Then,

$$\begin{aligned} ((U,w)) &= \int_{-\infty}^{\infty} e^{-itK}U(x,0) \cdot \overline{w(x,t)} dx dt \\ &= \int_{-\infty}^{\infty} < A_K(\lambda)U(x,0), \int_{-\infty}^{\infty} e^{it\lambda} w(\cdot,t) dt > d\lambda \\ &= (2\pi)^{1/2} \int_{-\infty}^{\infty} < A_K(\lambda)U(x,0), \tilde{w}(\cdot,\lambda) > d\lambda, \end{aligned}$$

where

$$\tilde{w}(x,\lambda) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} w(x,t) e^{it\lambda} dt.$$

Noting (7.8) , (7.7) and using the Cauchy-Schwartz inequality

$$\begin{aligned} |((U,w))| &\leq (2\pi)^{1/2} \|U(x,0)\|_0 \cdot \left(\int_{-\infty}^{\infty} < A_K(\lambda)\tilde{w}(\cdot,\lambda), \tilde{w}(\cdot,\lambda) > d\lambda \right)^{1/2} \\ &\leq C\|U(x,0)\|_0 \cdot \left(\int_{-\infty}^{\infty} \|\tilde{w}(\cdot,\lambda)\|^2_{\mathcal{D}^s} \, d\lambda \right)^{\frac{1}{2}}. \end{aligned}$$

It follows from the Plancherel theorem that

$$|((U,w))| \leq C\|U(x,0)\|_0 \left(\int_{\mathbb{R}} \|w(\cdot,t)\|^2_{\mathcal{D}^s} \, dt \right)^{1/2}.$$

Let $\phi(x,t) \in C_0^\infty(\mathbb{R}^{n+1}) \oplus C_0^\infty(\mathbb{R}^{n+1})$, and take $w(x,t) = (1+|x|^2)^{-\frac{s}{2}}\phi(x,t)$, so that

$$\left|(((1+|x|^2)^{-\frac{s}{2}}U,\phi))\right| \leq C \cdot \|U(x,0)\|_0 \cdot \|\phi\|_{L^2(\mathbb{R}^{n+1})}.$$

This concludes the proof of the part involving the Cauchy data in (3.14), in view of (7.3).

To prove the part concerning the inhomogeneous equation, it suffices to take $u_0 = v_0 = 0$.

In this case the Duhamel principle yields, for $t > 0$,

$$U(t) = \int_0^t e^{-i(t-\tau)K} F(\tau) d\tau,$$

where we have used the form (7.4) of the equation.

Integrating the inequality

$$\|U(t)\|_{\mathcal{D}^{-s}} \leq \int_0^t \|e^{-i(t-\tau)K} F(\tau)\|_{\mathcal{D}^{-s}} d\tau,$$

we get

$$\int_0^\infty \|U(t)\|_{\mathcal{D}^{-s}}\, dt \leq \int_0^\infty \int_\tau^\infty \|e^{-i(t-\tau)K} F(\tau)\|_{\mathcal{D}^{-s}}\, dt\, d\tau.$$

Invoking the first part of the proof we obtain

$$\int_0^\infty \|U(t)\|_{\mathcal{D}^{-s}}\, dt \leq C \int_0^\infty \|F(\tau)\|_0\, d\tau,$$

which proves the part related to the inhomogeneous term in (3.14).

(b) Define

$$v_\pm(x,t) = \exp(\pm itG)\phi_\pm(x),$$

where

$$\phi_\pm(x) = \frac{1}{2}[u_0(x) \mp G^{-1}v_0(x)].$$

Then clearly

$$u(x,t) = v_+(x,t) + v_-(x,t). \tag{7.9}$$

We establish the estimate (3.15) for v_+.

Taking $w(x,t) \in C_0^\infty(\mathbb{R}^{n+1})$ we proceed as in the first part of the proof. Let $<,>$ be the $L^{2,-s}(\mathbb{R}^n), L^{2,s}(\mathbb{R}^n)$ pairing. Then

$$\begin{aligned}(v_+, w) &= \int_{-\infty}^{\infty} e^{itG}\phi_+(x) \cdot \overline{w(x,t)}dxdt \\ &= \int_0^\infty < A_G(\lambda)\phi_+, \int_{-\infty}^{\infty} e^{-it\lambda}w(\cdot,t)dt > d\lambda \\ &= (2\pi)^{1/2}\int_0^\infty < A_G(\lambda)\phi_+, \tilde{w}(\cdot,\lambda) > d\lambda,\end{aligned}$$

where

$$\tilde{w}(x,\lambda) = (2\pi)^{-\frac{1}{2}}\int_{\mathbb{R}} w(x,t)e^{-it\lambda}dt.$$

Noting (7.6) as well as the inequalities (7.8) (with A_G replacing A_K) and using the Cauchy-Schwartz inequality

$$\begin{aligned}|(v_+, w)| &\leq (2\pi)^{1/2}\|\phi_+\|_0 \cdot \left(\int_0^\infty < A_G(\lambda)\tilde{w}(\cdot,\lambda), \tilde{w}(\cdot,\lambda) > d\lambda\right)^{1/2} \\ &\leq C\|\phi_+\|_0 \cdot \left(\int_0^\infty \|\tilde{w}(\cdot,\lambda)\|_{0,s}^2\, d\lambda\right)^{\frac{1}{2}}.\end{aligned}$$

The Plancherel theorem yields

$$|(v_+, w)| \leq C\|\phi_+\|_0 \left(\int_{\mathbb{R}} \|w(\cdot,t)\|_{0,s}^2\, dt\right)^{1/2}.$$

Let $\omega \in C_0^\infty(\mathbb{R}^{n+1})$, and take $w(x,t) = (1+|x|^2)^{-\frac{s}{2}}\omega(x,t)$, so that

$$\left|((1+|x|^2)^{-\frac{s}{2}}v_+, \omega)\right| \leq C \cdot \|\phi_+\|_0 \cdot \|\omega\|_{L^2(\mathbb{R}^{n+1})}.$$

This (with the similar estimate for v_-) concludes the proof of the estimate (3.15). □

Remark 7.1 (optimality of the requirement $s > 1$)**.** *A key point in the proof was the use of the uniform bound* (7.6)*. In view of the relation* (7.5)*, this is reduced to the uniform boundedness of* $\lambda A(\lambda^2)$, $\lambda \geq 0$, *in* $B(L^{2,s}, L^{2,-s})$*. By* [65, Theorem 5.1] *the boundedness at infinity,*

$\limsup_{\mu\to\infty} \mu^{\frac{1}{2}} \|A(\mu)\| < \infty$, *holds already with* $s > \frac{1}{2}$. *Thus the further restriction* $s > 1$ *is needed in order to ensure the boundedness at* $\lambda = 0$ *(Theorem A).*

Remark 7.2. *Clearly we can take* $[0, T]$ *as the time interval, instead of* $\mathbb{R}$, *for any* $T > 0$.

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