

VISCOUS CONSERVATION LAWS IN 1D WITH MEASURE INITIAL DATA

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ABSTRACT. The one-dimensional viscous conservation law is considered on the whole line

$$u_t + f(u)_x = \varepsilon u_{xx}, \quad (x, t) \in \mathbb{R} \times \overline{\mathbb{R}_+}, \quad \varepsilon > 0,$$

subject to positive measure initial data.

The flux $f \in C^1(\mathbb{R})$ is assumed to satisfy a p -condition, a weak form of convexity. In particular, any flux of the form $f(u) = \sum_{i=1}^J a_i u^{m_i}$ is admissible if $a_i > 0$, $m_i > 1$, $i = 1, 2, \dots, J$.

The only case treated hitherto in the literature is $f(u) = u^m$ [10] and the initial data is a “single source”, namely, a multiple of the delta function. The corresponding solutions have been labeled as “source-type” and the treatment made substantial use of the special form of both the flux and the initial data.

In this paper existence and uniqueness of solutions is established. The method of proof relies on sharp decay estimates for the viscous Hamilton-Jacobi equation. Some estimates are independent of the viscosity coefficient, thus leading to new estimates for the (inviscid) hyperbolic conservation law.

1. INTRODUCTION

We consider here the (viscous) nonlinear scalar conservation law in one space dimension, for an unknown real function $u(x, t)$,

$$(1.1) \quad u_t + f(u)_x = \varepsilon u_{xx}, \quad (x, t) \in \mathbb{R} \times \overline{\mathbb{R}_+}, \quad \varepsilon > 0,$$

subject to the initial condition

$$(1.2) \quad u(x, 0) = u_0(x) \in \mathcal{M}_+,$$

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where $\mathcal{M}_+ = \mathcal{M}_+(\mathbb{R})$ is the space of (finite) nonnegative Borel measures on the line.

$$(1.3) \quad \begin{aligned} &\text{We assume that } f \in C^1(\mathbb{R}), f(0) = f'(0) = 0, \\ &f'(u) \text{ is locally Hölder for } u \in \mathbb{R}. \end{aligned}$$

Throughout the paper we fix $\varepsilon > 0$, and omit the obvious dependence of the solution on this parameter (namely, we write u and not u^ε).

In particular, we are interested in the case

$$(1.4) \quad u_0(x) = M\delta_{x_0},$$

where $M > 0$, and δ_{x_0} is the Dirac mass at the point $x_0 \in \mathbb{R}$.

Following the terminology in the linear theory, solutions to (1.1)–(1.4) are called *fundamental solutions*. Another term used for such solutions is *source-type solutions*. The latter is probably better suited, due to the lack of a superposition principle. At any rate, these are solutions evolving from an initial (positive) measure that is located at a single point.

Equations of the type of (1.1) are referred to as “convection-diffusion” equations. The literature concerning such equations, as well as the related “convection-reaction-diffusion” equations, is quite extensive. Thus the references discussed below are restricted to those that are rather closely related to the present paper.

In the special case

$$(1.5) \quad u_t = u_{xx} - |u|^{q-1}u_x, \quad q > 1, \quad x \in \mathbb{R},$$

the existence and uniqueness of the source-type solution (with initial data (1.4)) is proved in [10, Theorem 3]. **To the best of our knowledge, this is the only case that has been treated in the literature up to now.**

This paper is concerned with the construction of nonnegative solutions with measure initial data, generalizing the source-type solutions.

We are primarily interested in estimates depending only on $\|u_0\|_{\mathcal{M}}$, the initial measure norm. For future reference, we make a clear distinction between estimates that depend on $\varepsilon > 0$, and those that do not.

For a general flux $f(u) \in C^1(\mathbb{R})$, we obtain in Sections 2, 3 and the first part of Section 4 estimates that depend on $\varepsilon > 0$. On the other hand in Subsection 4.1 we introduce the p –condition, a sort of “weak

convexity" assumption, that has been used in the study of Hamilton-Jacobi equations [2]. This condition allows us to give estimates that are independent of $\varepsilon > 0$, see Corollary 4.6.

The p -condition is used in Theorem 5.2, where we state an existence and uniqueness theorem for solutions of (1.1), with measure initial data.

In Section 6 we treat the special case (1.5) with any $q > 1$. This flux satisfies the p -condition, so that the general results can be applied, as well as some additional results depending on this special flux. In particular, we obtain (Theorem 6.1) long time estimates for this equation for all $q > 1$ and *all measure initial data*, uniformly in $\varepsilon > 0$. Consequently they can be used (Theorem 6.4) in the limiting inviscid case.

We recall (see [10]) that, in the case of Equation (1.5), for $1 < q < 2$, the solution $u(\cdot, t)$ (and, in fact, the solution for every initial function $u_0 \in L^1(\mathbb{R})$) approaches, as $t \rightarrow \infty$, the (self-similar) source-type solution of the hyperbolic equation $u_t = -|u|^{q-1}u_x$. On the other hand, if $q \geq 2$, the (nonlinear) convection term becomes negligible and the solution approaches, as $t \rightarrow \infty$, the fundamental solution of the heat equation.

In the case of (1.5) with any $1 < q \leq 2$ and sign-changing solutions an Oleinik-type estimate was derived in [17]. It is applicable to any integrable initial function $u_0(x)$, but with a constant that depends on u_0 .

As already mentioned, the treatment of source-type solutions (or more generally, initial measure data) has been quite limited. On the other hand, if the initial function $u_0(x)$ is at least integrable, there is an extensive literature concerning the long time asymptotic behavior of the solutions, both in the one-dimensional and multi-dimensional settings. We list here a few representative studies in this direction. The long time decay of solutions to viscous conservation laws was studied in [13, 16, 25, 26]. The asymptotic behavior of solutions to more general multi-dimensional convection diffusion was studied [3, 9]. For the Burgers equation with stationary source $u_t + uu_x - u_{xx} = \delta$ (but sufficiently regular initial data u_0) an explicit solution was constructed in [7], using the Cole-Hopf transformation. Its uniqueness and convergence to a steady state were proved.

Recall that the long time asymptotic behavior is strongly related to the problem of stability of travelling wave solutions ([27, 28, 29] and references therein).

The asymptotic behavior of source-type solutions in the inviscid case was studied in [18, 21].

Notation.

For a function $h(x, t)$ we denote $h_t = \frac{\partial}{\partial t}h(x, t)$ and $h_x = \frac{\partial}{\partial x}h(x, t)$.

Alternatively we use also $h_t = \partial_t h$ and $h_x = \partial_x h$.

Second-order derivatives are denoted by h_{xx} or $\partial_x^2 h$.

We use $\mathbb{R}_+ = (0, \infty)$.

We denote by $\|\cdot\|_q$ the norm in $L^q(\mathbb{R})$, $1 \leq q \leq \infty$.

The norm in the measure space $\mathcal{M}_+(\mathbb{R})$ is designated as $\|\cdot\|_{\mathcal{M}}$.

$W^{k,q}(\mathbb{R})$, $1 \leq q \leq \infty$, for a nonnegative integer k , is the space of functions having (distributional) derivatives up to order k in $L^q(\mathbb{R})$.

$C_0(\mathbb{R})$ is the space of continuous, compactly supported functions on \mathbb{R} .

$C^k(\mathbb{R})$ is the space of continuously differentiable functions on \mathbb{R} , up to order k .

$C_b^k(\mathbb{R})$ is the subspace of $C^k(\mathbb{R})$ consisting of all functions whose derivatives up to order k are bounded in \mathbb{R} .

We write C_b for C_b^0 .

2. GENERAL FACTS FOR CONSERVATION LAWS ON THE REAL LINE

In this section we do not assume $u_0 \geq 0$, unless this is explicitly imposed.

It is well known that, under the assumption $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and just $f \in C^1(\mathbb{R})$, Equation (1.1) has a unique global classical solution $u(x, t)$, $(x, t) \in \mathbb{R} \times \mathbb{R}_+$, that converges (in the L^1 topology) to $u_0(x)$ as $t \rightarrow 0$. This solution satisfies the maximum-minimum principle, namely, $-\|u_0\|_\infty \leq u(x, t) \leq \|u_0\|_\infty$ [15, Section 2.2]

Another well-known fact is that $\|u(\cdot, t)\|_p$ is nonincreasing, as a function of $t \in [0, \infty)$, for any $p \in [1, \infty]$.

The “initial mass” of the solution $M = \int_{\mathbb{R}} u_0(x) dx$ is conserved by the evolution,

$$(2.1) \quad \int_{\mathbb{R}} u(x, t) dx = M, \quad t > 0.$$

In order to study initial data beyond $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we shall need estimates for the time decay of the norms $\|u(\cdot, t)\|_p$, using only the initial L^1 norm $\|u_0\|_1$.

A well-known property is the *comparison principle*, as follows.

If $u_0, v_0 \in C_0(\mathbb{R})$ are nonnegative initial data, with corresponding solutions $u(x, t), v(x, t)$, and if $u_0(x) \leq v_0(x)$ for $x \in \mathbb{R}$, then for all $t > 0$,

$$(2.2) \quad u(x, t) \leq v(x, t), \quad x \in \mathbb{R}.$$

Lemma 2.1. *Let $u(x, t)$, $v(x, t)$ be solutions corresponding to initial functions $u_0, v_0 \in C_0(\mathbb{R})$, respectively. Then,*

$$(2.3) \quad \int_{\mathbb{R}} |u(x, t) - v(x, t)| dx \leq \int_{\mathbb{R}} |u_0(x) - v_0(x)| dx, \quad t > 0,$$

and in particular (taking $v_0 = 0$)

$$\int_{\mathbb{R}} |u(x, t)| dx \leq \int_{\mathbb{R}} |u_0(x)| dx.$$

Proof. The properties (2.1) and (2.2) allow us to invoke the Crandall-Tartar lemma ([8], [15, Section 2.5]), which yields the contraction property (2.3). \square

We note that the L^1 contraction property (2.3) satisfied by the solutions to the viscous conservation law can be obtained without resorting to the Crandall-Tartar lemma (and to the order-preserving property (2.2)). Instead, we can use the maximum-minimum principle (for linear equations).

Lemma 2.2. *Let $u(x, t)$ be a solution to (1.1), with initial data $u_0 \in C_0(\mathbb{R})$. Then*

- *The maximum-minimum principle is satisfied by the solution,*

$$(2.4) \quad \inf_{x \in \mathbb{R}} u_0(x) \leq \inf_{x \in \mathbb{R}} u(x, t) \leq \sup_{x \in \mathbb{R}} u(x, t) \leq \sup_{x \in \mathbb{R}} u_0(x), \quad \forall t > 0.$$

- *Let $u(x, t)$, $v(x, t)$ be solutions to (1.1), with respective initial data $u_0, v_0 \in C_0(\mathbb{R})$.*

Then ,

$$(2.5) \quad \|u(\cdot, t) - v(\cdot, t)\|_1 \leq \|u_0 - v_0\|_1, \quad t > 0.$$

Proof. The maximum-minimum principle is obtained by invoking its validity for the linear convection-diffusion equation. Indeed, consider the linear equation

$$z_t + f'(u(x, t))z_x = \varepsilon z_{xx}, \quad z(x, 0) = u_0(x),$$

and apply the linear maximum-minimum principle to it.

To establish the contraction property, let $w(x, t) = u(x, t) - v(x, t)$. It satisfies the equation

$$(2.6) \quad w_t + (b(u, v)w)_x = \varepsilon w_{xx}, \quad (x, t) \in \mathbb{R} \times \overline{\mathbb{R}_+},$$

where $b(u, v) = \frac{f(u) - f(v)}{u - v}$.

Fix $T > 0$. The dual equation to (2.6) in the strip $\mathbb{R} \times [0, T]$ is the linear parabolic equation

$$(2.7) \quad \phi_t + b(u, v)\phi_x = -\varepsilon\phi_{xx}, \quad (x, t) \in \mathbb{R} \times [0, T],$$

subject to the “terminal” condition

$$\phi(x, T) = \phi_T(x) \in C_0^\infty(\mathbb{R}),$$

as well as the boundary condition that

$$\lim_{R \rightarrow \infty} \sup_{|x| > R, t \in [0, T]} |\phi(x, t)| = 0.$$

Clearly ϕ satisfies the maximum-minimum principle

$$\|\phi(\cdot, 0)\|_\infty \leq \|\phi_T\|_\infty,$$

which implies by a standard duality argument that

$$\|w(\cdot, T)\|_1 \leq \|w(\cdot, 0)\|_1.$$

Since $T > 0$ is arbitrary, (2.5) is established. \square

Remark 2.3. *Lemma 2.2 implies that the solution operator*

$$S(t)u_0 = u(\cdot, t), \quad t > 0,$$

is a contraction in L^1 , hence can be extended to any $u_0 \in L^1(\mathbb{R})$. However, we have very little information about this extension. In particular, it is not even clear if it is indeed a solution, even in a weak sense, of Equation (1.1).

2.1. FURTHER ESTIMATES. In deriving the following estimates, we assume that the initial function u_0 is smooth and compactly supported. This ensures that the the solution $u(x, t)$ decays at infinity, for any fixed $t > 0$.

The estimates for general initial data will follow by a standard density argument.

In addition to the contraction property (Lemma 2.1) we have also

Lemma 2.4. *Assume that $u_0(x) \geq 0$. Then for every $p \in [2, \infty)$,*

$$(2.8) \quad \int_{\mathbb{R}} u(x, t)^p dx \leq \int_{\mathbb{R}} u_0(x)^p dx.$$

The following spacetime estimate holds,

$$(2.9) \quad 2\varepsilon \int_0^\infty \int_{\mathbb{R}} |u_x(x, t)|^2 dx dt \leq \int_{\mathbb{R}} u_0(x)^2 dx.$$

Proof. By the maximum principle the solution $u(x, t)$ is nonnegative.

To obtain the estimate (2.8) we multiply Equation (1.1) by u^{p-1} and integrate over \mathbb{R} . Noting that

$$\int_{\mathbb{R}} u^{p-1} f(u)_x dx = \int_{\mathbb{R}} g(u)_x dx = 0,$$

where $g'(u) = u^{p-1} f'(u)$, we get

$$(2.10) \quad d/dt \int_{\mathbb{R}} u(x, t)^p dx = -\varepsilon p(p-1) \int_{\mathbb{R}} u(x, t)^{p-2} |u_x(x, t)|^2 dx \leq 0.$$

To obtain the spacetime estimate (2.9) we take $p = 2$ in (2.10) and integrate with respect to time. \square

3. $L^p(\mathbb{R})$ ESTIMATES BY THE NASH INEQUALITY

Our treatment of the L^p estimates is based on the Nash inequality [1, 5, 22] restricted to the one-dimensional case over the whole line. It can be stated as follows.

Let ϕ be an integrable Lipschitz function on \mathbb{R} . Then

$$(3.1) \quad \left(\int_{\mathbb{R}} |\phi|^2 dx \right)^3 \leq C \int_{\mathbb{R}} |\phi_x|^2 dx \cdot \left(\int_{\mathbb{R}} |\phi| dx \right)^4.$$

In the context of convection-diffusion equations, the Nash inequality is stated as Lemma 1 in [9].

Even though it is not strictly needed, we shall assume in what follows that $u_0 \geq 0$. This will simplify the estimates, as powers of the solution u can be taken without absolute values.

Using the Nash inequality we obtain the following lemma.

Lemma 3.1. *Let $u(x, t)$ be a solution to (1.1), with $u_0 \in L^1(\mathbb{R})$.*

There exists a constant $C > 0$, independent of u_0 , p , ε , such that, for any $2 \leq p < \infty$

$$(3.2) \quad \|u(\cdot, t)\|_p \leq \left(\frac{Cp}{\varepsilon} \right)^{\frac{p-1}{2p}} \|u_0\|_1 t^{-\frac{p-1}{2p}}, \quad t > 0.$$

Proof. The proof uses the Nash inequality in a way that is essentially identical to the proof of Proposition 1 in [9]. Since it plays an important role in what follows, we bring it here for the convenience of the reader.

By the well-posedness of (1.1) in $L^1(\mathbb{R})$ we can assume $u_0 \in C_0^1(\mathbb{R})$.

Multiplying (1.1) by $pu(x, t)^{p-1}$, where $p \geq 2$, and integrating over \mathbb{R} we get

$$\begin{aligned}
 (3.3) \quad & \frac{d}{dt} \int_{\mathbb{R}} u(x, t)^p dx + p \int_{\mathbb{R}} u(x, t)^{p-1} f(u(x, t))_x dx \\
 &= -\varepsilon p(p-1) \int_{\mathbb{R}} u(x, t)^{p-2} |u_x(x, t)|^2 dx \\
 &= -4\varepsilon \frac{p-1}{p} \int_{\mathbb{R}} |(u(x, t)^{\frac{p}{2}})_x|^2 dx.
 \end{aligned}$$

The second integral in the left-hand side of the equality vanishes, being the integral of $F(u(x, t))_x$, with $F'(\xi) = pf'(\xi)\xi^{p-1}$.

Invoking the Nash inequality (3.1) with $\phi = u(\cdot, t)^{\frac{p}{2}}$ to get

$$(3.4) \quad \int_{\mathbb{R}} |(u(x, t)^{\frac{p}{2}})_x|^2 dx \geq C^{-1} \left(\int_{\mathbb{R}} u(x, t)^p dx \right)^3 \cdot \left(\int_{\mathbb{R}} |u(x, t)|^{\frac{p}{2}} dx \right)^{-4}.$$

We now use the interpolation inequality

$$\|g\|_{\frac{p}{2}} \leq \|g\|_p^{\frac{p-2}{p-1}} \|g\|_1^{\frac{1}{p-1}},$$

with $g = u(\cdot, t)$. Inserting this in (3.4) results in

$$\begin{aligned}
 (3.5) \quad & \int_{\mathbb{R}} (u(x, t)^{\frac{p}{2}})_x^2 dx \\
 & \geq C^{-1} \left(\int_{\mathbb{R}} u(x, t)^p dx \right)^{1+\frac{2}{p-1}} \cdot \left(\int_{\mathbb{R}} |u(x, t)| dx \right)^{-\frac{2p}{p-1}}.
 \end{aligned}$$

We let $z(t) = \|u(\cdot, t)\|_p^p$. Recall that $\|u(\cdot, t)\|_1 \leq \|u_0\|_1$. Thus, we obtain from (3.3) and (3.5),

$$(3.6) \quad z'(t) \leq -4C^{-1}\varepsilon \frac{p-1}{p} \|u_0\|_1^{-\frac{2p}{p-1}} z(t)^{1+\frac{2}{p-1}}.$$

Comparison with the solution of

$$w'(t) = -4C^{-1}\varepsilon \frac{p-1}{p} \|u_0\|_1^{-\frac{2p}{p-1}} w(t)^{1+\frac{2}{p-1}}, \quad w(0) = \infty,$$

yields

$$z(t) \leq \left(\frac{L_p}{\varepsilon} \right)^{\frac{p-1}{2}} \|u_0\|_1^p t^{-\frac{p-1}{2}},$$

where

$$L_p = \frac{pC}{8}.$$

This concludes the proof of (3.2). \square

4. $L^\infty(\mathbb{R})$ ESTIMATES

In this section we turn to $L^\infty(\mathbb{R})$ estimates for solutions of (1.1). Here and throughout the rest of the paper we assume that

$$(4.1) \quad u_0(x) \geq 0.$$

We first recall the sharp estimate of Carlen and Loss [6, Theorem 1], in the case of a scalar conservation law in \mathbb{R}^n :

$$(4.2) \quad u_t + \nabla \cdot f(u) = \varepsilon \Delta u, \quad (x, t) \in \mathbb{R}^n \times \overline{\mathbb{R}_+}, \quad \varepsilon > 0.$$

subject to

$$(4.3) \quad u_0(x) \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n).$$

In their estimate there is no need to impose a convexity assumption on the flux function $f(u)$, but they impose a regularity assumption that can be roughly described as:

$$(4.4) \quad \frac{f(\xi)}{\xi} \in C^1(\mathbb{R}^n).$$

Lemma 4.1 (Carlen-Loss). *Assume $u_0(x) \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and that the flux function satisfies (4.4).*

Then

$$(4.5) \quad \|u(\cdot, t)\|_p \leq K(p) \cdot (4\pi\varepsilon t)^{-\frac{n}{2}(1-\frac{1}{p})} \|u_0\|_1, \quad 1 \leq p \leq \infty, \quad t > 0,$$

where $K(p) = \left(\frac{4\pi}{p}\right)^{\frac{n}{2p}}$, $1 < p < \infty$, and $K(1) = K(\infty) = 1$.

Taking $n = 1$ in the Carlen-Loss estimate we get

$$(4.6) \quad \|u(\cdot, t)\|_\infty \leq (4\pi\varepsilon t)^{-\frac{1}{2}} \|u_0\|_1, \quad t > 0.$$

We note that for the prototypical example $f(u) = u^q$ the assumption (4.4) requires $q \geq 2$.

On the other hand a less optimal (in terms of the coefficient) estimate is obtained in [12, Lemma 3.1] under the sole assumption $f(u) \in C^1(\mathbb{R})$ (where in fact a much wider class of degenerate convection-diffusion equations is considered):

Lemma 4.2. [12, Lemma 3.1] *Suppose that $f \in C^1(\mathbb{R})$, then for some $C > 0$, depending on ε ,*

$$(4.7) \quad \|u(\cdot, t)\|_\infty \leq Ct^{-\frac{1}{2}} \|u_0\|_1, \quad t > 0.$$

Thus, actually we can take any $q > 1$ in the case $f(u) = u^q$.

We observe that this estimate depends on ε . Furthermore, it is not clear what are the optimal decay estimates, depending possibly on the special features of $f(u)$. This is manifested in Theorem 6.1 and

Theorem 6.4 below. Observe that in these theorems the estimates are independent of ε , in contrast to the estimates (4.6) and (4.7). The decay estimates in the above mentioned theorems are due to the effect of the nonlinear convective term, an effect that is completely absent in the application of the Nash or logarithmic Sobolev inequalities.

4.1. CONVEXITY and the p -CONDITION. To establish an L^∞ estimate (that is independent of $\varepsilon > 0$), we use the equivalence of the one-dimensional conservation law and the Hamilton-Jacobi equation. The main shortcoming in this approach is that it is based on a **convexity hypothesis** imposed on $f(u)$.

Slightly more generally, we begin by introducing a certain class of continuously differentiable functions on $[0, \infty)$ as follows (taken from [2]). Let $\Psi(r) \in C^1[0, \infty)$ be a nonnegative function, having the following property.

$$(4.8) \quad \left\{ \begin{array}{l} \bullet \text{ There exists a family of nonnegative smooth functions} \\ \quad \{\Phi_\eta\}_{\eta>0} \text{ defined in } [0, \infty) \text{ such that} \\ (i) \quad \Phi_\eta(0) = 0 \text{ for all } \eta > 0. \\ (ii) \quad \Phi_\eta(r^2) \xrightarrow[\eta \rightarrow 0^+]{} \Psi(r), \text{ uniformly in compact intervals of } [0, \infty). \end{array} \right.$$

Definition 4.3. Consider the family of functions $\{\Theta_\eta\}_{\eta>0}$ defined by

$$\Theta_\eta(r) = 2r\Phi'_\eta(r) - \Phi_\eta(r), \quad (r, \eta) \in [0, \infty) \times (0, \infty).$$

Let $p \in (1, \infty)$. We say that Ψ satisfies the **p -condition** if there exist $\gamma > 0$, $a > 0$, $b > 0$, such that, for $r > 0$ and sufficiently small $\eta > 0$,

$$(4.9) \quad \Theta_\eta(r) \geq ar^{\frac{p}{2}} - b\eta^\gamma, \quad \text{if } p \in (1, \infty).$$

Now in addition to our basic assumption (1.3) on f , we impose the following assumption.

$$(4.10) \quad \text{For some } p \in (1, \infty), \quad f \text{ satisfies the } p\text{-condition.}$$

As was shown in [2], the prototypical example

$$(4.11) \quad f(r) = r^p, \quad p \in (1, \infty),$$

satisfies the above assumptions with $\Phi_\eta(r) = (r + \eta^2)^{\frac{p}{2}} - \eta^p$. In fact, the same argument shows that one can take

$$(4.12) \quad f(r) = \sum_{k=1}^m \mu_k r^{p_k}, \quad \mu_k > 0,$$

where $\{p_1, \dots, p_m\} \in (1, \infty)^m$, and $p = \max\{p_1, \dots, p_m\}$.

Remark 4.4. *In the paper [2] the case $0 < p < 1$ is also considered. However we note that in this case $f \notin C^1(\mathbb{R})$.*

Theorem 4.5. [2] *Consider the equation*

$$(4.13) \quad v_t + f(v_x) = \varepsilon v_{xx}, \quad (x, t) \in \mathbb{R} \times \overline{\mathbb{R}_+}, \quad \varepsilon > 0,$$

subject to the initial condition

$$(4.14) \quad v(x, 0) = \varphi(x).$$

Let $p \in (1, \infty)$, and let $\varphi \in C_b(\mathbb{R})$. Assume that f satisfies the p -condition (4.10). Then the solution $v(x, t)$ satisfies

$$(4.15) \quad |\partial_x v(x, t)| \leq \|\varphi\|_\infty^{\frac{1}{p}} (at)^{-\frac{1}{p}}.$$

The application of this theorem to Equation (1.1) is straightforward.

Corollary 4.6. *Let $u(x, t)$ be the solution to Equation (1.1), subject to the initial condition*

$$(4.16) \quad 0 \leq u(x, 0) = u_0(x) \in L^1(\mathbb{R}).$$

Assume that f satisfies the p -condition (4.10).

Then for all $t > 0$,

$$(4.17) \quad \|u(\cdot, t)\|_\infty \leq \|u_0\|_1^{\frac{1}{p}} (at)^{-\frac{1}{p}}.$$

Proof. Taking $v(x, t) = \int_{-\infty}^x u(y, t) dy$, we observe that v satisfies (4.13), subject to the initial condition $\varphi(x) = \int_{-\infty}^x u_0(y) dy \in C_b(\mathbb{R})$. Thus, the estimate (4.17) follows directly from (4.15). \square

5. SOLUTIONS WITH MEASURE INITIAL DATA

We consider the scalar viscous conservation law (1.1), assuming that,

$$(5.1) \quad u_0(x) \in \mathcal{M}_+.$$

The comprehensive texts [19, 20] do not cover this problem, due to the generality of the initial data (see also Remark 5.4 below). We refer also to [30] and references therein for a related treatment of nonlinear parabolic equations.

Concerning (1.1), for the special case $f(u) = u^q$, $q > 1$, and $u_0 = M\delta_{x_0}$, $M > 0$, the existence and uniqueness of a source-type solution were established in [10, Theorem 3].

Definition 5.1. *A continuous function $u(x, t)$, $(x, t) \in \mathbb{R} \times \mathbb{R}_+$, is a **classical solution** to the general conservation law (1.1) if the partial derivatives u_t , u_x and u_{xx} are continuous and the equation is satisfied pointwise.*

Theorem 5.2. *Consider the general conservation law (1.1) with initial data (5.1), where $u_0 \geq 0$ is a compactly supported Borel measure on \mathbb{R} .*

Assume that

- **ASSUMPTION 1:** *f satisfies the p -condition (Definition 4.3), for some $p > 1$.*
- **ASSUMPTION 2:** *There exists a constant $C > 0$, such that (for the same p),*

$$(5.2) \quad |f'(r)| \leq C(1 + r^{p-1}), \quad r > 0.$$

Observe that it implies the growth condition

$$(5.3) \quad f(r) \leq C(r + r^p), \quad r > 0.$$

Then there exists a nonnegative classical solution $u(x, t)$, $(x, t) \in \mathbb{R} \times \mathbb{R}_+$, so that

$$(5.4) \quad u(\cdot, t) \xrightarrow[t \rightarrow 0+]{} u_0,$$

the convergence being in the sense of measures.

In addition, this solution has the following properties.

•

$$(5.5) \quad \int_{\mathbb{R}} u(x, t) dx \equiv \|u_0\|_{\mathcal{M}}, \quad t > 0.$$

- *There exists a constant $C > 0$, such that*

$$(5.6) \quad \|u(\cdot, t)\|_{\infty} \leq C \|u_0\|_{\mathcal{M}}^{\frac{1}{p}} t^{-\frac{1}{p}}, \quad t > 0.$$

Uniqueness: *Let $v(x, t)$ be a classical solution satisfying, in the sense of measures,*

$$(5.7) \quad v(\cdot, t) \xrightarrow[t \rightarrow 0+]{} u_0.$$

Assume further that, for some constant $C_1 > 0$ it satisfies the estimate

$$(5.8) \quad \|v(\cdot, t)\|_{\infty} \leq C_1 \|u_0\|_{\mathcal{M}}^{\frac{1}{p}} t^{-\frac{1}{p}}, \quad t > 0.$$

Define the functions

$$(5.9) \quad U(x, t) = \int_{-\infty}^x u(y, t) dy, \quad V(x, t) = \int_{-\infty}^x v(y, t) dy.$$

Assume that the difference $Z(x, t) = U(x, t) - V(x, t)$ is continuous in the strip $\mathbb{R} \times [0, \eta]$ for some $\eta > 0$.

Then

$$(5.10) \quad u(x, t) \equiv v(x, t).$$

Proof. The solution will be constructed as a limit of regular solutions, obtained by regularizing the singular initial data.

In the first part of the proof, including Claim 5.5,

we do not use the p -condition, just the fact that $f \in C^1(\mathbb{R})$. This means that we shall need to deal carefully with estimating spatial and temporal derivatives of the approximating sequence.

Let $\{u_0^{(k)}\}_{k=1}^\infty \subseteq C_0^\infty(\mathbb{R})$ be a sequence of nonnegative test functions such that

$$(5.11) \quad u_0(x) = \lim_{k \rightarrow \infty} u_0^{(k)}(x),$$

where the limit is taken in the sense of measures. If the approximating sequence is obtained by convolving u_0 with a compactly supported (nonnegative) mollifier, then we can further impose the condition

$$(5.12) \quad \sup_{1 \leq k < \infty} \|u_0^{(k)}\|_1 \leq M := \|u_0\|_{\mathcal{M}}.$$

In addition, due to the compact support of u_0 , we can assume

$$\text{supp } u_0^{(k)} \subseteq K,$$

where $K \Subset \mathbb{R}$ is a compact interval.

Let $u^{(k)}(x, t)$ be the solution to (1.1) subject to the initial condition $u_0^{(k)}$, namely

$$(5.13) \quad u_t^{(k)} + f(u^{(k)})_x = \varepsilon u_{xx}^{(k)}, \quad (x, t) \in \mathbb{R} \times \overline{\mathbb{R}_+}, \quad \varepsilon > 0,$$

$$u^{(k)}(x, 0) = u_0^{(k)}.$$

In view of Lemma 2.2 we first have

$$(5.14) \quad \|u^{(k)}(\cdot, t)\|_1 \leq M, \quad k = 1, 2, \dots$$

Let us fix $\tau > 0$. It follows from the L^∞ estimate in Lemma 4.2 that

$$(5.15) \quad \Upsilon(\tau) = \sup_{k=1,2,\dots} \|u^{(k)}(\cdot, \tau)\|_\infty < \infty.$$

The maximum principle yields

$$(5.16) \quad \sup_{k=1,2,\dots} \|u^{(k)}(\cdot, t)\|_\infty \leq \Upsilon(\tau), \quad t > \tau.$$

Remark 5.3. *Note that some of the estimates in Section 4 depend on $\varepsilon > 0$, while some others are ε -independent. For simplicity in what follows, we shall not explicitly mention this dependence.*

It follows that

$$(5.17) \quad \sup_{k=1,2,\dots} \|u^{(k)}(\cdot, \tau)\|_2 < \infty.$$

In view of (2.9)

$$(5.18) \quad \varepsilon \sup_{k=1,2,\dots} \int_{\tau}^{\infty} \int_{\mathbb{R}} |u_x^{(k)}(x, t)|^2 dx dt < \infty,$$

hence, by the uniform boundedness (5.16),

$$(5.19) \quad \varepsilon \sup_{k=1,2,\dots} \int_{\tau}^{\infty} \int_{\mathbb{R}} |f(u^{(k)})_x(x, t)|^2 dx dt < \infty.$$

The standard L^2 theory [20, Section VII.3] now implies that, in every domain P such that \overline{P} is compact in $\mathbb{R} \times [\tau, \infty)$,

$$(5.20) \quad \sup_{k=1,2,\dots} \int_P \left[|u_{xx}^{(k)}(x, t)|^2 + |u_t^{(k)}(x, t)|^2 \right] dx dt < \infty.$$

The uniform estimates (5.16) and (5.20) by themselves do not imply the (local) convergence of the sequence of solutions $\{u^{(k)}\}_{k=1}^{\infty}$ and their derivatives. In general, the L^{∞} estimates should yield, by “standard parabolic estimates”, the fact that this set of solutions, along with their first and second-order x -derivatives, as well as the first-order t -derivatives, are uniformly Hölder continuous in every domain P such that \overline{P} is compact in $\mathbb{R} \times \mathbb{R}_+$ (see e.g. [20, Lemma 4.17]). However, for such estimates to hold one must rely on the fact that the nonlinear term $f(u^{(k)})_x$ in (5.13) is itself Hölder continuous in (x, t) , and that such estimates are uniform with respect to k . Furthermore, we are not assuming the existence of a second derivative $f''(u)$. Thus, a direct argument seems to be desirable.

Remark 5.4. *Note that an application of the theory of general (second-order) parabolic equation to the special case of the viscous conservation law (1.1) is not straightforward. Specifically, we need first to establish uniform (with respect to k) Hölder continuity in $(x, t) \in \mathbb{R} \times [\beta, \infty)$, for every $\beta > 0$. For example, if we take the general nonlinear equation (restricted to one space dimension) in [19, Chapter V], it reads*

$$u_t - \partial_x a(x, t, u, u_x) + b(x, t, u, u_x) = 0,$$

then it covers (1.1), with the possibility of $a(x, t, u, u_x) = \varepsilon u_x - f(u)$ and $b \equiv 0$ or $a(x, t, u, u_x) = \varepsilon u_x$ and $b(x, t, u, u_x) = f'(u)u_x$. However, taking either choice, the constraints imposed in [19, Chapter V, Section 1] are not (a priori) satisfied, since they need to hold uniformly for the

sequence of derivatives $\left\{u_x^{(k)}\right\}_{k=1}^{\infty}$. The following claim establishes such uniform estimates.

We formulate the pointwise estimates in the following claim.

Claim 5.5. *The sequences*

$$\left\{\partial_t u^{(k)}(x, t)\right\}_{k=1}^{\infty}, \quad \left\{\partial_x u^{(k)}(x, t)\right\}_{k=1}^{\infty}, \quad \left\{\partial_{xx} u^{(k)}(x, t)\right\}_{k=1}^{\infty},$$

are uniformly bounded in $(x, t) \in \mathbb{R} \times [4\tau, \infty)$.

In addition, the sequence $\left\{\partial_x u^{(k)}(x, t)\right\}_{k=1}^{\infty}$ is uniformly Hölder continuous in $(x, t) \in \mathbb{R} \times [4\tau, \infty)$, with respect to the two variables x, t .

Proof of Claim 5.5. Let

$$(5.21) \quad G_{\varepsilon}(x, t) = (4\pi\varepsilon t)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{4\varepsilon t}\right)$$

be the heat kernel in \mathbb{R} , so that, for $t > \tau$,

$$(5.22) \quad \begin{aligned} u^{(k)}(x, t) &= \int_{\mathbb{R}} G_{\varepsilon}(x - y, t - \tau) u^{(k)}(y, \tau) dy \\ &\quad - \int_{\tau}^t \int_{\mathbb{R}} G_{\varepsilon}(x - y, t - s) f(u^{(k)}(y, s))_y dy ds \\ &= \int_{\mathbb{R}} G_{\varepsilon}(x - y, t - \tau) u^{(k)}(y, \tau) dy \\ &\quad - \int_{\tau}^t \int_{\mathbb{R}} \partial_x G_{\varepsilon}(x - y, t - s) f(u^{(k)}(y, s)) dy ds. \end{aligned}$$

Differentiating with respect to x ,

$$(5.23) \quad \begin{aligned} \partial_x u^{(k)}(x, t) &= \int_{\mathbb{R}} \partial_x G_{\varepsilon}(x - y, t - \tau) u^{(k)}(y, \tau) dy \\ &\quad - \int_{\tau}^t \int_{\mathbb{R}} \partial_x G_{\varepsilon}(x - y, t - s) f(u^{(k)}(y, s))_y dy ds. \end{aligned}$$

Let, for $t > \tau$,

$$A_k(t) = \|\partial_x u^{(k)}(\cdot, t)\|_{\infty}, \quad k = 1, 2, \dots$$

Using the equalities (where $C > 0$ is a universal generic constant),

$$(5.24) \quad \|\partial_x G_{\varepsilon}(\cdot, t)\|_{\infty} = C(\varepsilon t)^{-1}, \quad \|\partial_x G_{\varepsilon}(\cdot, t)\|_1 = C(\varepsilon t)^{-\frac{1}{2}}, \quad t > 0,$$

we get from (5.23), for $k = 1, 2, \dots$

$$(5.25) \quad A_k(t) \leq A_0(t) + C_1 C \varepsilon^{-\frac{1}{2}} \int_{\tau}^t (t-s)^{-\frac{1}{2}} A_k(s) ds, \quad t > \tau,$$

where, in view of (5.16),

$$(5.26) \quad A_0(t) = C \Upsilon(\tau) [\varepsilon(t-\tau)]^{-\frac{1}{2}}, \quad t > \tau,$$

and

$$C_1 = \sup_{|v| \leq \Upsilon(\tau)} |f'(v)|.$$

Shifting the variable $t = \tilde{t} + \tau$ and defining $\tilde{A}_k(\tilde{t}) = A_k(t)$ the last estimate can be written as

$$(5.27) \quad \tilde{A}_k(\tilde{t}) \leq \tilde{A}_0(\tilde{t}) + C_1 C \varepsilon^{-\frac{1}{2}} \int_0^{\tilde{t}} (\tilde{t}-\tilde{s})^{-\frac{1}{2}} \tilde{A}_k(\tilde{s}) d\tilde{s}, \quad \tilde{t} > 0.$$

Defining $\Theta_k(\tilde{t}) = \sup_{0 < \tilde{s} \leq \tilde{t}} [(\varepsilon \tilde{s})^{\frac{1}{2}} \tilde{A}_k(\tilde{s})]$, $k = 0, 1, 2, \dots$,

we have

$$\Theta_k(\tilde{t}) \leq \Theta_0(\tilde{t}) + C_1 C \tilde{t}^{\frac{1}{2}} \int_0^{\tilde{t}} (\tilde{t}-\tilde{s})^{-\frac{1}{2}} \tilde{s}^{-\frac{1}{2}} \Theta_k(\tilde{s}) d\tilde{s}.$$

We now take $\tilde{T} > 0$ so that

$$2C_1 C \tilde{T}^{\frac{1}{2}} \int_0^1 (1-u)^{-\frac{1}{2}} u^{-\frac{1}{2}} du < \frac{1}{2}.$$

The last estimate now yields, for $0 < \tilde{t} \leq \tilde{T}$,

$$\Theta_k(\tilde{t}) \leq 2\Theta_0(\tilde{t}) \leq C \Upsilon(\tau), \quad k = 1, 2, \dots,$$

We conclude that, with $T = \tau + \tilde{T}$,

$$(5.28) \quad \sup_{k=1,2,\dots} \|\partial_x u^{(k)}(\cdot, t)\|_{\infty} \leq C \Upsilon(\tau) [\varepsilon(t-\tau)]^{-\frac{1}{2}}, \quad \tau < t < T.$$

In particular, the sequence $\{\partial_x u^{(k)}(x, t)\}_{k=1}^{\infty}$ is uniformly bounded in $\mathbb{R} \times [\frac{T+\tau}{2}, T]$. Note that \tilde{T} depends only on $\Upsilon(\tau)$, which is a nonincreasing function of τ . Thus, we can proceed by uniform t -intervals of length $\frac{1}{2}(T-\tau)$ and obtain a uniform limit

$$(5.29) \quad \Upsilon_1(\tau) = \sup_{\substack{k=1,2,\dots \\ 2\tau \leq t < \infty}} \|\partial_x u^{(k)}(\cdot, t)\|_{\infty} < \infty.$$

Now in addition to (5.24) we have

$$(5.30) \quad \|\partial_x^2 G_\varepsilon(\cdot, t)\|_1 = C(\varepsilon t)^{-1}, \quad t > 0,$$

that can be used to estimate

$$\begin{aligned} & \int_{\mathbb{R}} |\partial_x G_\varepsilon(x+h, t) - \partial_x G_\varepsilon(x, t)| dx \\ & \leq h \int_{\mathbb{R}} \int_0^1 |\partial_x^2 G_\varepsilon(x + \mu h, t)| dx d\mu \leq Ch(\varepsilon t)^{-1}, \quad t > 0. \end{aligned}$$

Hence by interpolation, for any $0 \leq \alpha \leq 1$,

$$(5.31) \quad \int_{\mathbb{R}} |\partial_x G_\varepsilon(x+h, t) - \partial_x G_\varepsilon(x, t)| dx \leq C(\varepsilon t)^{-\frac{1}{2}(1+\alpha)} |h|^\alpha, \quad t > 0.$$

From (5.23) we obtain, for $t > 2\tau$,

$$\begin{aligned} (5.32) \quad & \partial_x u^{(k)}(x+h, t) - \partial_x u^{(k)}(x, t) \\ & = \int_{\mathbb{R}} [\partial_x G_\varepsilon(x+h-y, t-2\tau) - \partial_x G_\varepsilon(x-y, t-2\tau)] u^{(k)}(y, 2\tau) dy \\ & - \int_{2\tau}^t \int_{\mathbb{R}} [\partial_x G_\varepsilon(x+h-y, t-s) - \partial_x G_\varepsilon(x-y, t-s)] f(u^{(k)}(y, s))_y dy ds. \end{aligned}$$

The estimate (5.31) now yields

$$\begin{aligned} (5.33) \quad & \sup_{k=1,2,\dots} |\partial_x u^{(k)}(x+h, t) - \partial_x u^{(k)}(x, t)| \leq \\ & C\Upsilon(\tau) \varepsilon^{-\frac{1}{2}(1+\alpha)} (t-2\tau)^{-\frac{1}{2}(1+\alpha)} |h|^\alpha \\ & + C_1 \Upsilon_1(\tau) \varepsilon^{-\frac{1}{2}(1+\alpha)} |h|^\alpha \int_{2\tau}^t (t-s)^{-\frac{1}{2}(1+\alpha)} ds, \quad t > 2\tau. \end{aligned}$$

In particular, we obtain the Hölder continuity property of the first-order derivative with respect to x ,

$$\begin{aligned} (5.34) \quad & \sup_{k=1,2,\dots} |\partial_x u^{(k)}(x+h, t) - \partial_x u^{(k)}(x, t)| \\ & \leq [C\Upsilon(\tau) \tau^{-\frac{1}{2}(1+\alpha)} + C_1 \Upsilon_1(\tau) \tau^{\frac{1}{2}(1-\alpha)}] \varepsilon^{-\frac{1}{2}(1+\alpha)} |h|^\alpha, \\ & x \in \mathbb{R}, \quad t > 3\tau, \quad 0 < \alpha < 1. \end{aligned}$$

Equation (5.13) can now be written in the half-plane $t \geq 3\tau$ as

$$(5.35) \quad u_t^{(k)} - \varepsilon u_{xx}^{(k)} = -f(u^{(k)})_x = a_k(x, t), \quad (x, t) \in \mathbb{R} \times [3\tau, \infty) \quad \varepsilon > 0,$$

where the sequence of continuous functions $\{a_k(x, t)\}$, $(x, t) \in \mathbb{R} \times [3\tau, \infty)$ is uniformly bounded and uniformly Hölder continuous with respect to x . In fact, recalling that $f'(u)$ is assumed to be locally Hölder continuous (say with exponent $\gamma > 0$), we have, for $(x, t) \in \mathbb{R} \times [3\tau, \infty)$,

$$(5.36) \quad \begin{aligned} |a_k(x+h, t) - a_k(x, t)| &\leq |f'(u^{(k)}(x+h, t)) - f'(u^{(k)}(x, t))| \Upsilon_1(\tau) \\ &\quad + C_1 |\partial_x u^{(k)}(x+h, t) - \partial_x u^{(k)}(x, t)| \\ &\leq C \Upsilon_1(\tau)^{1+\gamma} |h|^\gamma \\ &\quad + C_1 [C \Upsilon(\tau) \tau^{-\frac{1}{2}(1+\alpha)} + C_1 \Upsilon_1(\tau) \tau^{\frac{1}{2}(1-\alpha)}] \varepsilon^{-\frac{1}{2}(1+\alpha)} |h|^\alpha. \end{aligned}$$

that can be written, for some $\delta > 0$,

$$(5.37) \quad \sup_{k=1,2,\dots} |a_k(x+h, t) - a_k(x, t)| \leq L |h|^\delta, \quad (x, t) \in \mathbb{R} \times [3\tau, \infty),$$

where $L > 0$ depends on τ, ε .

The boundedness and uniform Hölder continuity of the right-hand side terms in (5.35) (with respect to x) enables us to establish the uniform boundedness of the sequence $\{\partial_t u^{(k)}(x, t)\}_{k=1}^\infty$ (see [19, Ch. IV, Sec.1] for the local version), using the Duhamel representation.

Indeed, writing

$$(5.38) \quad \begin{aligned} u^{(k)}(x, t) &= \int_{\mathbb{R}} G_\varepsilon(x-y, t-3\tau) u^{(k)}(y, 3\tau) dy \\ &\quad + \int_{3\tau}^t \int_{\mathbb{R}} G_\varepsilon(x-y, t-s) a_k(y, s) dy ds, \end{aligned}$$

a formal differentiation with respect to t yields,

$$\begin{aligned}
(5.39) \quad \partial_t u^{(k)}(x, t) &= \int_{\mathbb{R}} \partial_t G_\varepsilon(x - y, t - 3\tau) u^{(k)}(y, 3\tau) dy \\
&+ a_k(x, t) + \int_{3\tau}^t \int_{\mathbb{R}} \partial_t G_\varepsilon(x - y, t - s) [a_k(y, s) - a_k(x, s)] dy ds,
\end{aligned}$$

where we have used

$$\lim_{\eta \rightarrow 0} \int_{\mathbb{R}} G_\varepsilon(x - y, \eta) a_k(y, t - \eta) dy = a_k(x, t).$$

Thus we need only establish the boundedness of the spacetime integral of

$$I(y, s) = \partial_t G_\varepsilon(x - y, t - s) [a_k(y, s) - a_k(x, s)],$$

in $\mathbb{R} \times [3\tau, t]$. We have

$$\partial_t G_\varepsilon(x, t) = (4\pi\varepsilon t)^{-\frac{1}{2}} \left\{ \frac{x^2}{4\varepsilon t^2} - \frac{1}{2t} \right\} e^{-\frac{x^2}{4\varepsilon t}},$$

so in view of the estimate (5.37) we obtain, with $z = \frac{x-y}{2\sqrt{\varepsilon(t-s)}}$,

$$\int_{3\tau}^t \int_{\mathbb{R}} |I(y, s)| dy ds \leq CL \int_{3\tau}^t (t-s)^{-1+\frac{\delta}{2}} ds \int_{\mathbb{R}} [z^2 + 1] e^{-z^2} |z|^\delta dz.$$

From (5.39) we now infer that, for $t \in [4\tau, 5\tau]$,

$$\sup_{k=1,2,\dots} |\partial_t u^{(k)}(x, t)| < \infty, \quad (x, t) \in \mathbb{R} \times [4\tau, 5\tau].$$

Since the bound depends only on τ (and the initial mass M), we can proceed by τ -steps to get the uniform boundedness of the sequence of time-derivatives $\{\partial_t u^{(k)}(x, t)\}_{k=1}^\infty$, $(x, t) \in \mathbb{R} \times [4\tau, \infty)$:

$$(5.40) \quad \sup_{k=1,2,\dots} |\partial_t u^{(k)}(x, t)| < \infty, \quad (x, t) \in \mathbb{R} \times [4\tau, \infty).$$

The uniform boundedness of the sequence of second-order spatial derivatives $\{u_{xx}^{(k)}\}_{k=1}^\infty$ now follows from Equation (5.35).

The uniform Hölder continuity of the set of spatial derivatives $\{u_x^{(k)}\}_{k=1}^\infty$ with respect to t follows from the uniform Hölder continuity with respect to x by a well-known argument [19, Chapter II, Section 3]. We give here the details, since we need to verify that this continuity is uniform in the full half-plane $\mathbb{R} \times [4\tau, \infty)$.

Pick $t_2 > t_1 > 4\tau$, $x_1 < x_2$ and define

$$(5.41) \quad J^{(k)}(x_1, x_2; t) = \int_{x_1}^{x_2} [u_x^{(k)}(s, t) - u_x^{(k)}(x_1, t_1)] ds.$$

Clearly

$$(5.42) \quad \begin{aligned} & J^{(k)}(x_1, x_2; t_2) - J^{(k)}(x_1, x_2; t_1) \\ &= u^{(k)}(x_2, t_2) - u^{(k)}(x_1, t_2) - u^{(k)}(x_2, t_1) + u^{(k)}(x_1, t_1). \end{aligned}$$

The uniform boundedness of $\left\{u_t^{(k)}\right\}_{k=1}^{\infty}$ (5.40) entails, in view of (5.42)

$$(5.43) \quad |J^{(k)}(x_1, x_2; t_2) - J^{(k)}(x_1, x_2; t_1)| \leq A|t_2 - t_1|,$$

where $A > 0$ depends on τ, ε , but not on k, x_1, x_2, t_1, t_2 .

From (5.41) we derive two facts, where we use $A_1 > 0$ as a generic constant depending on τ, ε , but not on k, x_1, x_2, t_1, t_2 .

- The uniform Hölder continuity of the derivatives $\left\{u_x^{(k)}\right\}_{k=1}^{\infty}$ (5.34) yields

$$|J^{(k)}(x_1, x_2; t_1)| \leq A_1|x_2 - x_1|^{1+\alpha}.$$

- By the mean value theorem

$$J^{(k)}(x_1, x_2; t_2) = [u_x^{(k)}(\sigma, t_2) - u_x^{(k)}(x_1, t_1)](x_2 - x_1), \quad \sigma \in [x_1, x_2],$$

so again by the uniform Hölder continuity of the derivatives $\left\{u_x^{(k)}\right\}_{k=1}^{\infty}$ with respect to x ,

$$\begin{aligned} |J^{(k)}(x_1, x_2; t_2)| &\geq \left[|u_x^{(k)}(x_1, t_2) - u_x^{(k)}(x_1, t_1)| - |u_x^{(k)}(\sigma, t_2) - u_x^{(k)}(x_1, t_2)| \right] |x_2 - x_1| \\ &\geq \left[|u_x^{(k)}(x_1, t_2) - u_x^{(k)}(x_1, t_1)| - A_1|x_2 - x_1|^\alpha \right] |x_2 - x_1|. \end{aligned}$$

Incorporating these estimates in (5.43) yields

$$\begin{aligned} |u_x^{(k)}(x_1, t_2) - u_x^{(k)}(x_1, t_1)| &\leq \frac{|J^{(k)}(x_1, x_2; t_2)|}{|x_2 - x_1|} + A_1|x_2 - x_1|^\alpha \\ &\leq \frac{|J^{(k)}(x_1, x_2; t_2) - J^{(k)}(x_1, x_2; t_1)| + |J^{(k)}(x_1, x_2; t_1)|}{|x_2 - x_1|} + A_1|x_2 - x_1|^\alpha \\ &\leq A \frac{|t_2 - t_1|}{|x_2 - x_1|} + 2A_1|x_2 - x_1|^\alpha. \end{aligned}$$

Selecting x_2 such that $A \frac{|t_2 - t_1|}{|x_2 - x_1|} = 2A_1|x_2 - x_1|^\alpha$, we obtain the uniform Hölder continuity of the derivatives $u_x^{(k)}$ with respect to t ,

$$(5.44) \quad |u_x^{(k)}(x, t_2) - u_x^{(k)}(x, t_1)| \leq 2A|t_2 - t_1|^{\frac{\alpha}{1+\alpha}}, \quad x \in \mathbb{R}, \quad t_2 > t_1 > 4\tau.$$

End of Proof of Claim 5.5

□

We now turn back to the proof of Theorem 5.2.

From Claim 5.5 we infer that the sequence $\{a_k(x, t)\}_{k=1}^\infty$ in Equation (5.35) is uniformly Hölder continuous, with respect to (x, t) , in the half-plane $(x, t) \in \mathbb{R} \times [4\tau, \infty)$.

We are now able to use the classical Schauder estimates for the heat equation [19, Chapter 4, Section 2] or [20, Chapter 4], in order to obtain the uniform Hölder continuity of the sequences

$$\left\{ u_t^{(k)}(x, t) \right\}_{k=1}^\infty, \quad \left\{ u_{xx}^{(k)}(x, t) \right\}_{k=1}^\infty.$$

By a diagonal process we can therefore extract a subsequence converging to a function $u(x, t)$, $(x, t) \in \mathbb{R} \times \mathbb{R}_+$. The convergence is uniform, together with all relevant derivatives, in every compact domain $P \Subset \mathbb{R} \times \mathbb{R}_+$. It follows that $u(x, t)$ is a classical solution.

Applying Fatou's lemma to the sequence of nonnegative pointwise converging functions (for every fixed $t > 0$) $\{u^{(k)}(x, t)\}_{k=1}^\infty$ and noting (5.14) we obtain

$$(5.45) \quad \int_{\mathbb{R}} u(x, t) dx \leq M = \|u_0\|_{\mathcal{M}}.$$

It follows that $u(\cdot, t) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for every $t > 0$, so that all the properties mentioned in Section 2 can be applied. In particular, combining (2.1) with (5.45)

$$(5.46) \quad \int_{\mathbb{R}} u(x, t) dx = \text{const} \leq M = \|u_0\|_{\mathcal{M}}, \quad t > 0.$$

We now establish the convergence to the initial data in the sense of measures, as in (5.4).

In the case $f(u) = |u|^q$, $q > 1$, and $u_0(x) = M\delta_0$, such a proof is given in [10, Section 4].

Let $\zeta(x) \in C_0^\infty(\mathbb{R})$. For every $k = 1, 2, \dots$ we have, by integrating Equation (5.13),

$$(5.47) \quad \begin{aligned} & \int_{\mathbb{R}} u^{(k)}(x, t) \zeta(x) dx - \int_{\mathbb{R}} u_0^{(k)}(x) \zeta(x) dx \\ &= \varepsilon \int_{\mathbb{R}} \int_0^t u^{(k)}(x, s) \zeta_{xx}(x) dx ds + \int_{\mathbb{R}} \int_0^t f(u^{(k)}(x, s)) \zeta_x(x) dx ds. \end{aligned}$$

By the contraction estimate (5.14) we have

$$\left| \int_{\mathbb{R}} \int_0^t u^{(k)}(x, s) \zeta_{xx}(x) dx ds \right| \leq M \|\zeta_{xx}\|_\infty t.$$

Thus

$$(5.48) \quad \left| \int_{\mathbb{R}} u^{(k)}(x, t) \zeta(x) dx - \int_{\mathbb{R}} u_0^{(k)}(x) \zeta(x) dx \right| \leq M \|\zeta_{xx}\|_\infty t + \int_{\mathbb{R}} \int_0^t \left| f(u^{(k)}(x, s)) \zeta_x(x) \right| dx ds.$$

Let us show that, uniformly in k ,

$$(5.49) \quad \lim_{t \rightarrow 0} \left| \int_{\mathbb{R}} \int_0^t f(u^{(k)}(x, s)) \zeta_x(x) dx ds \right| = 0.$$

In order to prove it, the growth assumption (5.3) is invoked. **Note that it is used here for the first time in the proof.**

In view of (5.14) we have

$$\|u^{(k)}(\cdot, t)\|_1 \leq M, \quad k = 1, 2, \dots$$

Furthermore, the estimate (4.17) yields

$$(5.50) \quad \|u^{(k)}(\cdot, t)\|_\infty \leq M^{\frac{1}{p}} (at)^{-\frac{1}{p}},$$

so that

$$\begin{aligned} \int_{\mathbb{R}} \left| f(u^{(k)}(x, s)) \zeta_x(x) \right| dx &\leq C \|\zeta_x\|_\infty \int_{\mathbb{R}} \left[u^{(k)}(x, s) (1 + u^{(k)}(x, s)^{p-1}) \right] dx \\ &\leq CM \|\zeta_x\|_\infty \left[1 + M^{\frac{p-1}{p}} (as)^{-\frac{p-1}{p}} \right], \end{aligned}$$

from which (5.49) follows.

Thus, passing to the limit as $k \rightarrow \infty$ in (5.48) yields,

$$(5.51) \quad \lim_{t \rightarrow 0} \left| \int_{\mathbb{R}} u(x, t) \zeta(x) dx - \int_{\mathbb{R}} \zeta(x) du_0(x) \right| = 0,$$

so that the convergence in measure to the initial data is established.

A well-known fact about weak convergence of functionals entails

$$(5.52) \quad \|u_0\|_{\mathcal{M}} \leq \liminf_{t \rightarrow 0} \int_{\mathbb{R}} u(x, t) dx,$$

and in conjunction with (5.46) we get (5.5).

The estimate (5.6) now follows from (5.50).

Finally, we address the uniqueness of the solution. Let $v(x, t)$ be another classical solution for the same initial data.

Let $U(x, t)$, $V(x, t)$ be as in (5.9).

The functions U , V are classical solutions to the viscous Hamilton-Jacobi equation

$$(5.53) \quad W_t + f(W_x) = \varepsilon W_{xx}, \quad x \in \mathbb{R}, \quad t > 0.$$

From the convergence in measure (5.4) and (5.7) it follows that

$$(5.54) \quad \lim_{t \rightarrow 0} U(x, t) = \lim_{t \rightarrow 0} V(x, t) = U_0(x), \quad a.e. x \in \mathbb{R},$$

where

$$U_0(x) = \int_{-\infty}^x du_0,$$

is a monotone nondecreasing function.

Since u_0 is compactly supported, the convergence in measure also implies that for every small $\delta > 0$ there exist $r > 0$, $\theta > 0$, such that

$$(5.55) \quad \begin{cases} \max(U(x, t), V(x, t)) \leq \delta, & x < -r, \quad 0 < t < \theta, \\ \min(U(x, t), V(x, t)) \geq 1 - \delta, & x > r, \quad 0 < t < \theta. \end{cases}$$

Let $Z(x, t) = U(x, t) - V(x, t)$. Noting (5.54) it follows by Helly's theorem [23, Section VIII.4] that there exists a decreasing subsequence $\{t_k \downarrow 0\}$ such that

$$\lim_{k \rightarrow \infty} Z(x, t_k) = 0, \quad x \in \mathbb{R}.$$

By assumption $Z(x, t)$ is continuous in $\mathbb{R} \times [0, \eta]$ hence $Z(x, 0) \equiv 0$.

The difference $f(U_x) - f(V_x)$ can be written as

$$(5.56) \quad f(U_x) - f(V_x) = A(x, t)Z_x(x, t),$$

where

$$A(x, t) = \int_0^1 f'(\lambda U_x + (1 - \lambda)V_x) d\lambda.$$

Thus $Z(x, t)$ satisfies the linear parabolic equation

$$(5.57) \quad Z_t + A(x, t)Z_x = \varepsilon Z_{xx}, \quad x \in \mathbb{R}, \quad t > 0.$$

On account of assumption (5.8) the coefficient $A(x, t)$ is bounded in every strip of the form $\mathbb{R} \times [\tau, \eta]$ for $0 < \tau < \eta$.

Let $\{t_k \downarrow 0\}$ be a sequence as above and let

$$M_k = \sup \{|Z(x, t)|, \quad x \in \mathbb{R}, \quad t_k \leq t \leq \eta\}.$$

The maximum principle [24, Section 3.2] implies that there exists a point $x_k \in \mathbb{R}$ such that $Z(x_k, t_k) = M_k$. In view of (5.55) we may assume that $x_k \in [-r, r]$, $k = 1, 2, \dots$. Hence there is a subsequence (without changing notation) such that $\lim_{k \rightarrow \infty} x_k = \bar{x} \in [-r, r]$ and by the assumed continuity of $Z(x, t)$

$$\lim_{k \rightarrow \infty} M_k = Z(\bar{x}, 0) = 0.$$

Since $\{M_k\}_{k=1}^\infty$ is non-decreasing, we must have $M_k = 0$, $k = 1, 2, \dots$ □

6. THE SPECIAL CASE $f(u) = u^p$

In this section we consider the special case of a “power-law” flux. As has already been mentioned in the Introduction, this “proto-typical” case is the only one that has been studied in the literature, in the context of *measure initial data*. We point out that even in this case our results are new in their generality (compare Remark 6.2 below). Furthermore, even in the better studied case of $u_0 \in L^1(\mathbb{R})$ there are various estimates that yield decay rates that are not necessarily compatible with the ones obtained in this paper. For the convenience of the reader we have documented some of them in Remark 6.3 below.

The equation to be considered here is

$$(6.1) \quad u_t + (u^p)_x = \varepsilon u_{xx}, \quad x \in \mathbb{R}, \quad p > 1, \quad \varepsilon > 0,$$

subject to the nonnegative measure initial condition

$$(6.2) \quad 0 \leq u(x, 0) = u_0 \in \mathcal{M}_{+..}$$

We assume that the measure u_0 is compactly supported.

The flux certainly satisfies the hypotheses imposed in Theorem 5.2, so all the conclusions of the theorem are valid here. In particular, since it clearly satisfies the p -condition, it satisfies the decay estimate (5.6).

We summarize the decay estimates in the following theorem.

Theorem 6.1. *Let $u(x, t)$ be the solution to (6.1), with $1 < p < \infty$. Then*

(1) *With some constant $C = C(p) > 0$, independent of $\varepsilon > 0$,*

$$(6.3) \quad \|u(\cdot, t)\|_\infty \leq C \|u_0\|_{\mathcal{M}}^{\frac{1}{p}} t^{-\frac{1}{p}}, \quad t > 0.$$

(2) *With some constant $C = C(p) > 0$, independent of $\varepsilon > 0$,*

$$(6.4) \quad \|u(\cdot, t)\|_2 \leq C \|u_0\|_{\mathcal{M}}^{\frac{p+1}{2p}} t^{-\frac{1}{2p}}, \quad t > 0.$$

Proof. As already noted, the estimate (6.3) is just the decay estimate (5.6).

The estimate (6.4) is obtained by interpolating (6.3) with the contraction property $\|u(\cdot, t)\|_1 \leq \|u_0\|_{\mathcal{M}}$.

□

Remark 6.2. *The estimate (6.3) is proved in [10, Lemma 1.2], for $1 < p < 2$, where the initial data is a point-source (1.4). However, as was seen above, the validity of this estimate also for $p \geq 2$ was useful in studying the behavior of the solution near the initial data.*

For $p \geq 2$ one has the estimate (4.6), which gives a faster decay as $t \rightarrow \infty$, but depends on ε .

Remark 6.3. (a) *It is interesting to compare the L^2 estimate (6.4) (for the Equation (6.1)) to the "dispersive estimate" [29, Section 1.1], limited to initial data $u_0 \in L^1(\mathbb{R})$,*

$$\|u(\cdot, t)\|_2 \leq \alpha(\varepsilon) \|u_0\|_1 t^{-\frac{1}{4}}, \quad t > 0.$$

The time decay $t^{-\frac{1}{4}}$ is identical for the case $p = 2$ but is different otherwise. The dependence on $\|u_0\|_1$ is different. Also note that the constant $C > 0$ in (6.4) is independent of $\varepsilon > 0$. We note, on the other hand, that the "dispersive estimate" is independent of the non-linear term (which is integrated out) and can therefore be applied in other situations (see its derivation for the vorticity in two-dimensional Navier-Stokes equations in [4, Section 3]).

(b) *The rate of decay (in time) given by (6.4) was first derived by Schonbek, in the multi-dimensional case [25]. However, the dependence on u_0 is different, as well as the fact that here the coefficient C is independent of ε (see (6.8) below).*

(c) *In the case of the viscous Burgers equation ($p = 2$) sharp constants for both (6.3) and (6.4) were given in [6, Theorem 1]. The dependence there on $\|u_0\|_1$ is linear, as in the case of the heat equation. However, once again, the coefficients depend on ε .*

The fact that the constants in (6.3)-(6.4) are independent of $\varepsilon > 0$ yields immediately the following result.

Theorem 6.4. *Consider the (inviscid) conservation law*

$$(6.5) \quad u_t + (|u|^p)_x = 0, \quad x \in \mathbb{R}, \quad p > 1,$$

subject to the initial condition

$$(6.6) \quad u(x, 0) = u_0(x) \in L^1(\mathbb{R}).$$

Then, with some constant $C > 0$,

$$(6.7) \quad \|u(\cdot, t)\|_\infty \leq C \|u_0\|_1^{\frac{1}{p}} t^{-\frac{1}{p}}, \quad t > 0,$$

$$(6.8) \quad \|u(\cdot, t)\|_2 \leq C \|u_0\|_1^{\frac{p+1}{2p}} t^{-\frac{1}{2p}}, \quad t > 0.$$

Proof. Denoting by u^ε the solution to (6.1), we know from the theory of viscous approximations to hyperbolic conservation laws [15] that $u^\varepsilon \rightarrow u$ pointwise for a.e. $t > 0$. Therefore (6.7) follows from (6.3). In particular, the set $\{u^\varepsilon(\cdot, t)\}_\varepsilon$ is uniformly bounded (for a.e. t), so that by the dominated convergence theorem

$$\int_{-N}^N |u(x, t)|^2 dx \leq C^2 \|u_0\|_1^{\frac{p+1}{p}} t^{-\frac{1}{p}}, \quad t > 0,$$

and (6.8) follows by letting $N \rightarrow \infty$. □

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