THE LOCAL THEORY FOR VISCOUS HAMILTON-JACOBI EQUATIONS IN LEBESGUE SPACES

Matania BEN-ARTZI [1,*], Philippe SOUPLET [2,3], Fred B. WEISSLER [4]

[1] *Institute of Mathematics, Hebrew University, Jerusalem 91904, Israel*
E-mail: mbartzi@math.huji.ac.il

[2] *Département de Mathématiques, INSSET, Université de Picardie, 02109 St-Quentin, France*

[3] *Laboratoire de Mathématiques Appliquées, UMR CNRS 7641, Université de Versailles 45 avenue des Etats-Unis, 78035 Versailles, France. E-mail: souplet@math.uvsq.fr*

[4] *LAGA, UMR CNRS 7539, Institut Galilée, Université Paris-Nord 93430 Villetaneuse, France. E-mail: weissler@math.univ-paris13.fr*

Abstract: We consider viscous Hamilton-Jacobi equations of the form

$$(\mathrm{VHJ}) \qquad \begin{cases} u_t - \Delta u = a|\nabla u|^p, & x \in \mathbb{R}^N, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases}$$

where $a \in \mathbb{R}$, $a \neq 0$ and $p \geq 1$. We provide an extensive investigation of the local Cauchy problem for (VHJ) for irregular initial data u_0, namely for u_0 in Lebesgue spaces $L^q = L^q(\mathbb{R}^N)$, $1 \leq q < \infty$. The case of initial data measures or in Sobolev spaces is also considered.

When $p < 2$, we prove well-posedness in L^q for $q \geq q_c = \frac{N(p-1)}{2-p}$. This holds without sign restriction neither on a nor on u_0.

In the case $a > 0$ and $u_0 \geq 0$ (repulsive gradient term) we show that existence fails in all L^q spaces when $p \geq 2$. When $p < 2$, we prove that both existence and uniqueness fail if $1 \leq q < q_c$.

Rather surprisingly, in the case $a < 0$ and $u_0 \geq 0$ (absorbing gradient term), we show that existence holds in L^1 while it may fail in measures. More precisely, we obtain existence in L^q for `any` $q \geq 1$ when $p \leq 2$ (and also for $p > 2$ under some additional assumption on u_0), whereas nonexistence occurs for a large class of measure initial data if $p > \frac{N+2}{N+1}$.

In particular, a critical exponent for existence and uniqueness in the scale of L^q spaces appears if the gradient term is repulsive, while none occurs if it is absorbing.

AMS Classification: 35K15, 35K55, 35B33, 35B40

Keywords: Nonlinear parabolic equations, viscous Hamilton-Jacobi equations, well-posedness, nonexistence, nonuniqueness, Lebesgue spaces, critical exponents.

* This research was partially supported by the Israel Science Foundation, Grant 450/99–2.

1. Introduction

1.1. Statement of the problem

This paper is concerned with viscous Hamilton-Jacobi equations of the form

$$\text{(VHJ)} \qquad \begin{cases} u_t - \Delta u = a|\nabla u|^p, & x \in \mathbb{R}^N, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases}$$

where $a \in \mathbb{R}$, $a \neq 0$ and $p \geq 1$.

The equation (VHJ) possesses both mathematical and physical interest. It can serve as a typical model-case in the theory of parabolic partial differential equations. Indeed, it is the simplest example of a parabolic PDE with a nonlinearity depending on the first order spatial derivatives of u, and it can be considered as an analogue of the extensively studied equation with zero order nonlinearity $u_t - \Delta u = a|u|^{p-1}u$. On the other hand, the equation $u_t - \varepsilon\Delta u = a|\nabla u|^p$, which can be easily reduced to (VHJ) by rescaling, may be viewed as the viscosity approximation (as $\varepsilon \to 0^+$) of Hamilton-Jacobi type equations from stochastic control theory (see [Lio]). Also, equation (VHJ) appears in the physical theory of growth and roughening of surfaces, where it is known as the Kardar-Parisi-Zhang equation (see [KPZ, KS]).

When u_0 is a sufficiently regular function, say $u_0 \in C_b^2$, and $p \geq 1$, the existence of a unique local – and actually global – classical solution of (VHJ) was established in [B1, AB]. This result was recently extended to $u_0 \in C_b$ and $p > 0$ in [GGK].

Our purpose is to provide a rather extensive investigation of the local Cauchy problem for (VHJ) for irregular initial data u_0, namely for u_0 in Lebesgue spaces $L^q = L^q(\mathbb{R}^N)$, $1 \leq q < \infty$. The case of initial data measures or in Sobolev spaces will be also considered. We will present various results on existence, nonexistence, uniqueness and nonuniqueness of solutions. Some of our results will concern positive solutions, while others will apply to solutions of mixed sign. The issue to these questions involve different ranges of values of p, q, together with the sign of a. Many of our results are optimal and we obtain an almost complete classification regarding local (non-)existence and (non-)uniqueness for all $1 \leq p, q < \infty$.

Beside local existence/uniqueness, another interesting question regarding equation (VHJ) is the long time behavior of solutions (recall that all local solutions of (VHJ) exist globally). This question was studied by a number of authors in the past few years, see [AB, BRV1, BRV2, BK, BL1, BGL, GGK, BLS, BGK, BLSS]. A particular attention was given to the question whether solutions decay as $t \to \infty$ when $u_0 \geq 0$ and $a < 0$. In Theorem 2.5 below we obtain some decay properties without sign restrictions on a or u_0.

Results on other aspects of problem (VHJ) and on its generalizations can be found in [BL2, BL3, P, AR1, AR2, AQR]. Also, let us mention that the related equation $u_t - \Delta u = a|\nabla u|^p + bu^p$, first studied in [ChW], has received a lot of attention from the point of view of blow-up and global existence (see [S2] for a recent survey).

Let us briefly summarize our main results. Put

$$p_0 = p_0(N) = \frac{N+2}{N+1} \quad \text{and} \quad q_c = q_c(N,p) = \frac{N(p-1)}{2-p} \quad \text{if } p < 2.$$

The *critical exponent* q_c plays a crucial role in this theory. We will say that q is supercritical, critical or subcritical, according to whether $q > q_c$, $q = q_c$ or $q < q_c$.

(i) When $p < 2$, we prove well-posedness in L^q for *supercritical* and *critical* q. This holds without sign restriction neither on a nor on u_0. Well-posedness holds also for measure data if $p < p_0(N)$ and for $W^{1,q}$ data if $1 \le p < \infty$ and $q \ge N(p-1)$.

We next specialize to the case $a > 0$ and $u_0 \ge 0$ (repulsive gradient term) and we obtain:

(ii) When $p \ge 2$, existence fails in general in L^q for *any* $q \ge 1$.

(iii) Thus returning to $p < 2$, we show that both existence and uniqueness fail in general in L^q for *subcritical* q and in $W^{1,q}$ if $q < N(p-1)$. The nonuniqueness result is extended to some more general nonlinearities depending on u and $|\nabla u|$.

We then examine the situation when $a < 0$ and $u_0 \ge 0$ (absorbing gradient term).

(iv) We obtain existence in L^q for *any* $q \ge 1$ when $p \le 2$. This even extends to $p > 2$ for $u_0 \ge 0$ in a large subset of L^q (including $u_0 \in L^q$ symmetric radially decreasing, possibly singular at 0). However, the uniqueness of this solution is an open question in general, except for $p = 2$ where uniqueness holds.

(v) We introduce a notion of *p-atomic* measure, which contains in particular atomic measures, and we show that the previous existence result cannot be extended to such measure initial data.

One of the consequences of our study is that a critical exponent for existence in the scale of L^q spaces appears if the gradient term is repulsive, while none occurs if it is absorbing. Also, in the absorbing case, it is a rather surprising fact that existence holds in L^1 while it may fail in measures. A heuristic interpretation is that when approaching u_0 by more regular initial data, one "loses" the initial trace in the limiting process if u_0 is a singular measure. On the contrary, if u_0 is an L^1 function, then it is possible to recover the initial trace, by using suitable monotonicity arguments (see Remarks 4.1 and 4.2).

Let us compare our results with previous work on equation (VHJ) with irregular data. It was proved in [BL1] that if $a < 0$, $p < 2$, $u_0 \ge 0$ and $u_0 \in L^1 \cap L^q$ with $q > q_c$, then (VHJ) admits a unique (mild) solution. Note that, as compared with the result (i) above, the signs of a and u_0 seem to be essential in the approach of [BL1]. When u_0 is a bounded and nonnegative measure, it was proved in [BL1] that the existence and uniqueness hold if $a < 0$, $1 < p < p_0(N)$, whereas nonexistence was shown if u_0 is a Dirac mass and $a < 0$, $p \ge p_0(N)$. The result (v) extends this to more general singular measures.

In [An], the more general degenerate equation $u_t - \Delta u^m = |\nabla u^r|^p$ $(m, r, p \ge 1)$ was considered for initial data measures. Conditions for existence and nonexistence of positive weak solutions were obtained in terms of a certain local regularity property of the measure u_0. When applied to the special case $m = r = 1$ (i.e. (VHJ) with $a > 0$) and $u_0 \in L^q$, the results of [An] yield local existence of (at least) a solution of (VHJ) when $q > q_c$ and nonexistence if $q < q_c$. Although the context of [An] is more general than ours, it has to be pointed out that, as a consequence of the completely different approach, the resulting (weak) solution lies only in some local spaces and that both existence in the critical case and uniqueness are left open in this approach. Also the assumption $a > 0$ seems important in the arguments used for existence. On the other hand our nonexistence result in (iii) is close to the nonexistence result of [An] for $m = r = 1$. However the functional frameworks are different: we work with mild solutions which require $u \in C([0,T); L^q(\mathbb{R}^N))$ and $|\nabla u|^p \in L^1(0,T; L^q(\mathbb{R}^N))$, while [An] works with weak solutions which require $u \in C([0,T); L^2_{\mathrm{loc}}(\mathbb{R}^N))$ and $|\nabla u|^2 \in L^1_{\mathrm{loc}}((0,T) \times \mathbb{R}^N)$, and the two sets of hypotheses are not comparable in general for $q \geq 1$ and $1 < p < 2$. Also our method is simpler.

Remark 1.1. Let us point out that the situation for the Cauchy problem is rather different from that for the Cauchy-Dirichlet problem associated with (VHJ) on a bounded domain. This is due to the fact that solutions of the latter problem may exhibit finite time gradient blowup whehever $p > 2$ (see, e.g., [FL, S3]), a phenomenon which does not occur for the Cauchy problem. This is the reason – besides simplicity – why we have restricted our attention to the Cauchy problem, although many of the results discussed here would certainly apply to the bounded domain case (with homogeneous Dirichlet conditions) when $p < 2$. For some related existence/nonexistence results in the bounded domain case, let us mention the papers [BMP, Al]. For results in the case of periodic boundary conditions, see [BGL, GGK, BLSS].

The outline of the paper is as follows.

Section 1.2 of the Introduction contains the necessary notation and definitions of solutions.

Section 2 is devoted to well-posedness for supercritical and critical q ($a > 0$ or < 0). We also consider initial data in measures and in Sobolev spaces.

In Section 3 we specialize to the case $a > 0$, $u_0 \geq 0$. After showing nonexistence in L^q for $p \geq 2$, we prove both nonexistence and nonuniqueness results for $p < 2$ and q subcritical, and we give extensions of the nonuniqueness results to different equations.

Then in Section 4, we turn to the case $a < 0$, $u_0 \geq 0$. We prove existence in all L^q for all $1 < p < 2$ (and for all $p > 1$ for a large subset of L^q). We then show existence and uniqueness in all L^q for $p = 2$. Finally we study the nonexistence for singular measures when $p > p_0(N)$.

Some of the results of this paper have been announced in [BSW] and [B3].

1.2. Notation and definitions of solutions

In what follows, $L^q = L^q(\mathbb{R}^N)$, $1 \leq q \leq \infty$, denotes the usual Lebesgue spaces of real valued functions, with norm denoted by $\|.\|_q$. $W^{1,q} = W^{1,q}(\mathbb{R}^N)$ is the usual Sobolev space. $\mathcal{M} = \mathcal{M}(\mathbb{R}^N)$ denotes the Banach space of bounded Borel measures on $\mathbb{R}^N$, the dual space of $C_0(\mathbb{R}^N)$. Also, throughout the paper, we will denote by C, c, C_1, $C_2, \ldots$ various positive constants which may vary from line to line. The dependence of these constants will be made precise when necessary.

For all $t > 0$, $e^{t\Delta}$ denotes the convolution operator with the standard heat kernel, that is

$$\big(e^{t\Delta} f\big)(x) = \int_{\mathbb{R}^N} G(x-y,t) f(y)\, dy,$$

where

$$G(x,t) = (4\pi t)^{-N/2} e^{-\frac{|x|^2}{4t}}, \quad t > 0, \quad x \in \mathbb{R}^N$$

and f is either a nonnegative measurable function, or $f \in L^q$ for some $q \in [1, \infty]$. If f is a finite Borel measure or, more generally, if $f \in \mathcal{S}'$, then $\big(e^{t\Delta} f\big)(x)$ is understood as $\langle f, G(t, x - .)\rangle$.

Let $a \in \mathbb{R}$, $a \neq 0$, $1 \leq p < \infty$ and $1 \leq q < \infty$ be real numbers. We are primarily interested in the existence and uniqueness of *mild* solutions of the equation (VHJ) i.e., solutions of the integral equation

$$u(t) = e^{t\Delta} u_0 + a \int_0^t e^{(t-s)\Delta} |\nabla u(s)|^p \, ds, \quad 0 \leq t < T \tag{1.1}$$

for some $T \in (0, \infty]$, where $u_0 \in \mathcal{S}'$ and the unknown function $u = u(x,t)$ is a real valued measurable function on $Q_T := \mathbb{R}^N \times (0, T)$. We will use interchangeably $u(t)$ for $u(\cdot, t)$ when there is no risk of confusion. Also, for $1 \leq p < 2$ we put

$$q_c = \frac{N(p-1)}{2-p}.$$

The function u being a solution of (1.1) can be defined in several ways. In view of the uniqueness and nonexistence results that we will develop, it is natural to work with reasonable notions of solutions that are as general as possible.

Our basic definition of solution is the following.

Definition 1.1. *Let $u_0 \in \mathcal{S}'$. A pointwise mild solution of (VHJ) is a function $u \in L^1_{\mathrm{loc}}(Q_T)$ such that $\nabla u \in L^p_{\mathrm{loc}}(Q_T)$ and such that*

$$u(x,t) = \big(e^{t\Delta} u_0\big)(x) + a \int_0^t \int_{\mathbb{R}^N} G(x-y, t-s) \, |\nabla u(y,s)|^p \, dy \, ds \qquad \textit{for a.e. } (x,t) \in Q_T. \tag{1.2}$$

Note that the time-space integral term in (1.2) makes sense since $|\nabla u(y,s)|^p$ is a nonnegative measurable function in Q_T and that since $u \in L^1_{\mathrm{loc}}(Q_T)$, (1.2) implies that the time-space integral term is finite for almost every $(x,t) \in Q_T$.

We will make use also of the following notion of mild L^q solution.

Definition 1.2. *Let $q \in [1, \infty)$ and $u_0 \in L^q$. A mild L^q solution of (VHJ) is a function $u \in C([0,T); L^q)$ such that*

$$|\nabla u|^p \in L^1(0, T; L^q) \tag{1.3}$$

and

$$u(t) = e^{t\Delta} u_0 + a \int_0^t e^{(t-s)\Delta} |\nabla u(s)|^p \, ds \qquad \textit{in } L^q \textit{ for all } 0 \leq t < T. \tag{1.4}$$

(If $T = \infty$, the condition (1.3) is replaced by $|\nabla u|^p \in L^1(0, T_0; L^q)$ for all $T_0 \in (0, \infty)$.)

It is clear that any mild L^q solution is a pointwise mild solution. Conversely, for $q = 1$, we have:

Proposition 1.1. *Let $T \in (0,\infty)$, $q = 1$ and $u_0 \in L^1$ and let u be a pointwise mild solution of (VHJ). Assume that either*

$$a < 0, \quad u \geq 0 \text{ a.e. in } Q_T, \tag{1.5}$$

or

$$\operatorname{ess\,lim\,inf}_{t\to T} \|u(t)\|_1 < \infty \tag{1.6}$$

Then

$$|\nabla u|^p \in L^1(0,T;L^1) \tag{1.7}$$

and u is a mild L^1 solution.

Proof. Using Fubini's theorem and the preservation of the integral by $e^{t\Delta}$, we have

$$\int_0^t \int_{\mathbb{R}^N} |\nabla u(y,s)|^p \, dy \, ds = \int_{\mathbb{R}^N} \int_0^t e^{(t-s)\Delta} |\nabla u(s)|^p \, dy \, ds. \tag{1.8}$$

First assume (1.5). Integrating (1.2) in space and using (1.8), $u \geq 0$ and $u_0 \in L^1$, we get

$$|a| \int_0^t \int_{\mathbb{R}^N} |\nabla u(y,s)|^p \, dy \, ds \leq \int_{\mathbb{R}^N} e^{t\Delta} u_0(y) \, dy \leq \|u_0\|_1 < \infty.$$

Since $e^{t\Delta}u_0 \in C([0,T);L^1)$, this easily implies that $u \in C([0,T);L^1)$ and that (1.3) and (1.4) are satisfied.

Now assuming (1.6), we obtain similarly that

$$|a| \int_0^T \int_{\mathbb{R}^N} |\nabla u(y,s)|^p \, dy \, ds \leq \|u_0\|_1 + \operatorname{ess\,lim\,inf}_{t\to T} \|u(t)\|_1 < \infty$$

and we conclude as before. □

Denote by $C_b^2 = C_b^2(\mathbb{R}^N)$ the space of functions with bounded continuous partial derivatives up to second order and by $C^{2,1}(Q_T)$ the space of functions which are continuously differentiable in Q_T up to order two in x and one in t.

Definition 1.3. *Let $u_0 \in L^q$. A classical solution of (VHJ) in Q_T is a function $u \in C([0,T);L^q) \cap C^{2,1}(Q_T)$ such that $u(0) = u_0$,*

$$u \in C((0,T);C_{\mathrm{b}}^2)$$

and $u_t - \Delta u = a|\nabla u|^p$ for all $(x,t) \in Q_T$.

When considering the issue of local existence-uniqueness in $\mathcal{M}$, we will use the following definition.

Definition 1.4. *Let $u_0 \in \mathcal{M}$. A mild $\mathcal{M}$ solution of (VHJ) is a function $u \in C_{\mathrm{b}}((0,T);L^1)$ such that $|\nabla u|^p \in L^1(0,T;L^1)$, (1.1) holds in L^1 for all $t \in (0,T)$ and $u(t) - e^{t\Delta}u_0$ converges to 0 in L^1 as $t \to 0$. In particular, $u(t) \rightharpoonup u_0$ weak star in $\mathcal{M}$ as $t \to 0$.*

Remark 1.2. If $u_0 \in \mathcal{M}$ and u is a pointwise mild solution of (VHJ), then u is a mild $\mathcal{M}$ solution whenever (1.5) or (1.6) holds. This follows from the proof of Proposition 1.1.

2. Well-posedness in supercritical and critical L^q spaces.

2.1. Main results

Our main result on well-posedness is the following theorem.

Theorem 2.1. *Assume $1 \le p < 2$. Let $1 \le q < \infty$ satisfy $q > q_c$ or $q = q_c > 1$, and let $u_0 \in L^q$.*

(i) There exists a global solution

$$u \in C([0,\infty); L^q) \cap C((0,\infty); W^{1,r}), \quad q \le r \le \infty \tag{2.1}$$

of (1.1). The function u is a mild L^q solution if $q > q_c$ and a pointwise mild solution if $q = q_c$. Moreover, u is a classical solution of (VHJ) in $\mathbb{R}^N \times (0,\infty)$.

(ii) Assume $q > q_c$. For all $T > 0$, u is the unique local in time (pointwise mild) solution of (1.1) in the class

$$C([0,T]; L^q) \cap C((0,T]; W^{1,pq}). \tag{2.2}$$

(iii) Assume $q = q_c$. For all $T > 0$, u is the unique local in time (pointwise mild) solution of (1.1) in the class

$$C([0,T]; L^{q_c}) \cap C((0,T]; W^{1,r}), \tag{2.3}$$

for any $r \ge p$ such that $q_c < r < pq_c$.

In the case of initial data measures, we have the following result.

Theorem 2.2. *Let $1 \le p < \frac{N+2}{N+1}$ hence, $q_c < 1$. For every $u_0 \in \mathcal{M}$ there exists a function*

$$u \in C_b((0,\infty); L^1) \cap C((0,\infty); W^{1,r}), \quad 1 \le r \le \infty, \tag{2.4}$$

which is a global mild $\mathcal{M}$ solution of (1.1). Moreover, u is a classical solution of (VHJ) in $\mathbb{R}^N \times (0,\infty)$. Furthermore, for all $T > 0$, u is the unique pointwise mild solution of (1.1) in the class $C_b((0,T); L^1) \cap C((0,T]; W^{1,p})$.

As a corollary to the proof of Theorems 2.1 and 2.2, we obtain that the solutions given there satisfy the following smoothing properties for small t.

Proposition 2.3. *There exist T, $C > 0$ such that the solution given in Theorems 2.1 and 2.2 satisfy*

$$\sup_{(0,T]} t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})} \|u(t)\|_r \le C, \quad q \le r \le \infty \tag{2.5}$$

(with $q=1$ in the case of Theorem 2.2.).

$$\sup_{(0,T]} t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})+\frac{1}{2}} \|\nabla u(t)\|_r \le C, \quad q \le r \le \infty. \tag{2.6}$$

Moreover, in the case of Theorem 2.1 with $q > q_c$ (resp., of Theorem 2.2), T and C actually depend only on $\|u_0\|_q$ (resp., $\|u_0\|_{\mathcal{M}}$). In the case of Theorem 2.1 with $q = q_c$, if $\|u_0\|_q$ is sufficiently small, then (2.5) and (2.6) hold with $T = \infty$ and C independent of u_0.

In the next proposition, we consider the solvability of (1.4) in the Sobolev spaces $W^{1,q}$ instead of the Lebesgue spaces L^q. We will show existence and uniqueness of local solutions to (1.1) for all $u_0 \in W^{1,q}$ where $q > N(p-1)$ or $q = N(p-1) > 1$. Note that we no longer need assume $p < 2$. When $1 \le q < N(p-1)$, we will show in Section 3 that local uniqueness is no longer true in general, and some nonexistence results will be given in Section 4.

Proposition 2.4. *Assume $p \ge 1$ and let $1 \le q < \infty$ satisfy $q > N(p-1)$ or $q = N(p-1) > 1$. Let $u_0 \in W^{1,q}$.*

(i) There exists a global pointwise mild solution

$$u \in C([0,\infty); W^{1,q}) \cap C((0,\infty); W^{1,r}), \quad q \le r \le \infty,$$

of (1.1). Moreover, u is a classical solution of (VHJ) in $\mathbb{R}^N \times (0,\infty)$.

(ii) Assume $q > N(p-1)$. For all $T > 0$, u is the unique local in time (pointwise mild) solution of (1.1) in the class

$$C([0,T); W^{1,q}) \cap C((0,T); W^{1,pq}).$$

(iii) Assume $q = N(p-1) > 1$. For all $T > 0$, u is the unique local in time (pointwise mild) solution of (1.1) in the class

$$C([0,T); W^{1,q}) \cap C((0,T); W^{1,r}),$$

for all $r \ge p$ such that $N(p-1) < r < Np(p-1)$.

(iv) There exist T, $C > 0$ such that the solution given in (i) satisfies

$$\sup_{(0,T]} t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})} \big(\|u(t)\|_r + \|\nabla u(t)\|_r\big) \le C, \quad q \le r \le \infty.$$

In particular if $N(p-1) < q < N$, then

$$|\nabla u|^p \in L^1(0,T; L^{q^*}) \qquad (q^* = Nq/(N-q)). \tag{2.8}$$

Moreover, if $q > N(p-1)$, then T and C depend only on $\|u_0\|_{W^{1,q}}$.

Remarks 2.1. (a) The local L^q theory for (VHJ) with $a > 0$, that we describe in Theorem 2.1 (and in Theorems 3.1 and 3.2 below), has many common features with the known L^q theory of the equation

$$u_t - \Delta u = |u|^{p-1}u. \tag{2.7}$$

For the latter equation, the critical exponent is $N(p-1)/2$. Well-posedness for $q \geq N(p-1)/2$ (with $q > 1$ if $q = N(p-1)/2$) was proved in [W2], Theorem 1. The uniqueness class was improved in [BC]. For $q < N(p-1)/2$, nonexistence results were obtained in [W2, W4, BP]) and examples of nonuniqueness in [HW, Ba] (see also [NS]).

(b) If $q > q_c$ and $u_0 \in L^q$, the solution given by Theorem 2.1 is actually unique in the larger class $L^\infty(0,T;L^q) \cap L^\infty_{\mathrm{loc}}(0,T;W^{1,pq})$. This follows from slight modifications of the proofs below (see Remark 2.5).

(c) If $q > q_c$, the arguments of the proof of Theorem 2.1 show that for all finite t_0, the solution u on $[0,t_0]$ depends continuously in L^q on the initial data (see also Remarks 2.3 and 2.6).

(d) The conclusions of Theorem 2.1 (i) and (ii) and Proposition 2.3 remain valid for $q = \infty$ and any $1 \leq p < 2$. In this case one has to replace (2.1) and (2.2) by $u(t) - e^{t\Delta}u_0 \in C([0,\infty);L^\infty)$, $u \in C((0,\infty);W^{1,\infty})$.

(e) The conclusions of Theorems 2.1 and 2.2 remain true (except perhaps for u being a classical solution) if the coefficient a is replaced by any function $a(x) \in L^\infty(\mathbb{R}^N)$.

We conclude this section by a result concerning the large time behavior of solutions of (VHJ). In the critical case $q = q_c > 1$, one has the following decay property for small initial data, which shows that $u \equiv 0$ is a stable and asymptotically stable equilibrium of (VHJ) in L^{q_c}.

Theorem 2.5. *Assume $q = q_c > 1$ (hence $p_0 < p < 2$) and $u_0 \in L^{q_c}$. There exists $\varepsilon_0 = \varepsilon_0(p,N) > 0$ such that the solution of (1.1) given by Theorem 2.1 satisfies*

$$\sup_{[0,\infty)} \|u(t)\|_{q_c} \leq 2\|u_0\|_{q_c} \qquad \text{and} \qquad \lim_{t\to\infty} \|u(t)\|_{q_c} = 0,$$

whenever $\|u_0\|_{q_c} \leq \varepsilon_0$. Moreover one also has $\lim_{t\to\infty} \|u(t)\|_k = 0$ for all $k \in (q_c,\infty]$.

Remarks 2.2. (a) A similar result was proved in [S1] for the nonlinear heat equation (2.7). Namely, if $q = N(p-1)/2 > 1$ and the initial data is small in L^q norm, then u is global and decays in L^q. See also [Ka1] for a related result concerning the Navier-Stokes equations.

(b) The smallness condition on $\|u_0\|_{q_c}$ in Theorem 2.5 cannot be removed. Indeed, the (self-similar) solution constructed in Theorem 3.3 satisfies $\|u(t)\|_{q_c} = \|u_0\|_{q_c} > 0$ for all $t \in [0,\infty)$.

(c) No extension of Theorem 2.5 to $q = 1$ is possible when $a > 0$: if $u_0 \geq 0$ (with, say, $u_0 \in L^1 \cap C_b$), then $\|u(t)\|_1 \geq \|e^{t\Delta}u_0\|_1 = \|u_0\|_1$.

2.2 Proofs

Our proof of local existence and uniqueness of solutions to (1.1) in L^q and in $W^{1,q}$ uses ideas which go back to [KF, W1, W2]. These arguments have been carried out in a number of contexts, in particular for the Navier-Stokes equations. In [ChW], in the case where the nonlinear part of (1.1) also includes a power term, well-posedness of the Cauchy problem for (1.1) was proved in $W_0^{1,q}(\Omega)$, where Ω is a smooth bounded domain in $\mathbb{R}^N$, under the hypotheses $q > N(p-1)$ and $q \geq p$ plus other conditions related to the power term. The proof is based on the abstract theory in [W1]. Also, [ChW] includes a brief remark on how the methods of [W2] can be applied to give well-posedness in certain $L^q(\Omega)$ spaces. Later, in [AW] it is observed that the same results carry over if Ω is replaced by $\mathbb{R}^N$. More recently, in [SnTW1] the integral equation (1.1) with an invariant power term added is studied in the "critical" case (corresponding to $q = q_c$ in Theorem 2.1 above). Here the ideas are ultimately based on the treatment of the critical case in [W2], but follow more closely the treatment in [CaW] of the pure power nonlinear heat equation. The spaces X, Y, ... that we use below are in some sense analogous to those used in [GMO, Ka2] for the treatment of the 2 dimensional Navier-Stokes equations in vorticity formulation.

Since both the details of the proofs and the statements of the results for well-posedness of (1.1) in L^q and in $W^{1,q}$ are different from in the case where a power term is present, and since not all of our results can be directly deduced from the abstract framework in [W2], we present the proofs in enough detail as to minimize explicit references to previous results. Moreover, we will improve the result in [ChW] on well-posedness in $W^{1,q}$ by eliminating the requirement that $q \geq p$.

If $u : (0,T] \to W^{1,r}$, for some $r \geq p$, is a continuous function, we formally define

$$\mathcal{G}u(t) = \int_0^t e^{(t-s)\Delta} |\nabla u(s)|^p \, ds. \tag{2.9}$$

Our basic approach is to prove existence of solutions to (1.1) by showing that the application $\mathcal{F}$ given by

$$\mathcal{F}u(t) = e^{t\Delta} u_0 + a\mathcal{G}u(t)$$

is a strict contraction on an appropriate complete metric space of curves.

In all Section 2.2, C denotes a generic positive constant depending only on N, p, q, r and a.

Proof of Theorem 2.1 for $q > q_c$.

For $0 < T < \infty$, let $X = X(T)$ be the Banach space of continuous curves $u : (0,T] \to W^{1,pq}$ such that

$$\|u\|_X = \max[\sup_{(0,T]} t^\alpha \|u(t)\|_{pq}, \sup_{(0,T]} t^{\alpha+\frac{1}{2}} \|\nabla u(t)\|_{pq}] < \infty,$$

where

$$\alpha = \frac{N}{2}\Big(\frac{1}{q} - \frac{1}{pq}\Big). \tag{2.10}$$

We denote by $X_K(T)$ the closed ball of X with radius K.

The first step (Lemma 2.1) is to use a contraction mapping argument to obtain existence and uniqueness of a local (and actually global) solution in a more restricted class than (2.2), namely, $u \in C([0,T]; L^q) \cap X_K(T)$ for suitable K, $T > 0$. In a second step (Lemma 2.2), we will then show that uniqueness actually holds in the larger class (2.3).

Lemma 2.1. *Assume $q > q_c$, $q \geq 1$ and $u_0 \in L^q$.*

(i) Let K, $T > 0$ satisfy

$$K \geq C_1(\|u_0\|_q + K^p T^\gamma) \tag{2.11}$$

where $C_1 = C_1(N,p,q,a) > 0$ and $\gamma = 1 - p(\alpha + \frac{1}{2}) > 0$. Then there exists a unique function $u \in X_K(T)$ which is a (pointwise mild) solution of (1.1) on $(0,T)$. Moreover, $u \in C([0,T]; L^q)$ and u is actually a mild L^q solution.

(ii) For all $T' > 0$, there is at most one (pointwise mild) solution of (1.1) in the class $X(T')$.

Note that Lemma 2.1 guarantees the existence of a unique maximal solution of (1.1) in $X(T_{\max})$ for some $T_{\max} \in (0,\infty]$, with $u \in X(\infty)$ meaning $u \in X(T)$ for all $T > 0$. This solution will be referred to as *the* solution given by Lemma 2.1. We will see later that this solution is actually global, i.e. $T_{\max} = \infty$.

Proof of Lemma 2.1. (i) If $u \in X_K(T)$, using

$$\|e^{t\Delta}\phi\|_{pq} \leq Ct^{-\alpha}\|\phi\|_q \quad \text{and} \quad \|\nabla e^{t\Delta}\phi\|_{pq} \leq Ct^{-\alpha-1/2}\|\phi\|_q,$$

we have, for all $t \in [0,T]$,

$$\begin{aligned}
\|\mathcal{G}u(t)\|_{pq} &\leq C\int_0^t (t-s)^{-\alpha}\||\nabla u(s)|^p\|_q\,ds = C\int_0^t (t-s)^{-\alpha}\|\nabla u(s)\|_{pq}^p\,ds \\
&\leq CK^p\int_0^t (t-s)^{-\alpha}s^{-p(\alpha+\frac{1}{2})}ds = CK^p t^{1-\alpha-p(\alpha+\frac{1}{2})}\int_0^1 (1-s)^{-\alpha}s^{-p(\alpha+\frac{1}{2})}\,ds \\
&\leq CK^p t^{-\alpha}T^{1-p(\alpha+\frac{1}{2})},
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla\mathcal{G}u(t)\|_{pq} &\leq C\int_0^t (t-s)^{-\alpha-\frac{1}{2}}\||\nabla u(s)|^p\|_q\,ds = C\int_0^t (t-s)^{-\alpha-\frac{1}{2}}\|\nabla u(s)\|_{pq}^p\,ds \\
&\leq CK^p\int_0^t (t-s)^{-\alpha-\frac{1}{2}}s^{-p(\alpha+\frac{1}{2})}\,ds \\
&= CK^p t^{-\alpha-\frac{1}{2}+1-p(\alpha+\frac{1}{2})}\int_0^1 (1-s)^{-\alpha-\frac{1}{2}}s^{-p(\alpha+\frac{1}{2})}\,ds \\
&\leq CK^p t^{-\alpha-\frac{1}{2}}T^{1-p(\alpha+\frac{1}{2})}.
\end{aligned}$$

In particular, it follows that

$$\|\mathcal{G}u\|_X \le CK^pT^{1-p(\alpha+\frac{1}{2})} \tag{2.12}$$

(where C is independent of T, and in fact depends only on p and q). The fact that $q > q_c$ guarantees that all the integrals above are convergent and that $1 - p(\alpha + \frac{1}{2}) > 0$.

Moreover, we note that $u_0 \in L^q$ implies

$$\max\big[\sup_{(0,T]} t^\alpha \|e^{t\Delta}u_0\|_{pq}, \sup_{(0,T]} t^{\alpha+\frac{1}{2}}\|\nabla e^{t\Delta}u_0\|_{pq}\big] \le C\|u_0\|_q. \tag{2.13}$$

Choose K, $T > 0$ such that (2.11) holds. It follows from (2.12) and (2.13) that $\mathcal{F}$ maps $X_K(T)$ into itself.

Now using

$$\||\nabla u|^p - |\nabla v|^p\|_{r/p} \le p(\|\nabla u\|_r^{p-1} + \|\nabla v\|_r^{p-1})\|\nabla u - \nabla v\|_r \quad \text{(valid for } r \ge p) \tag{2.14}$$

with $r = pq$, we obtain for all $t \in [0, T)$,

$$\begin{aligned}
\|\mathcal{F}u(t) - \mathcal{F}v(t)\|_{pq} &\le C\int_0^t (t-s)^{-\alpha}\||\nabla u(s)|^p - |\nabla v(s)|^p\|_q\,ds \\
&\le CK^{p-1}\int_0^t (t-s)^{-\alpha}s^{-(\alpha+\frac{1}{2})(p-1)}\|\nabla u(s) - \nabla v(s)\|_{pq}\,ds \\
&\le CK^{p-1}\|u-v\|_X\int_0^t (t-s)^{-\alpha}s^{-(\alpha+\frac{1}{2})p}\,ds \\
&\le CK^{p-1}t^{-\alpha}T^{1-p(\alpha+\frac{1}{2})}\|u-v\|_X,
\end{aligned}$$

and similarly

$$\|\nabla\mathcal{F}u(t) - \nabla\mathcal{F}v(t)\|_{pq} \le CK^{p-1}t^{-\alpha-1/2}T^{1-p(\alpha+\frac{1}{2})}\|u-v\|_X.$$

Therefore,

$$\|\mathcal{F}u - \mathcal{F}v\|_X \le CK^{p-1}T^{1-p(\alpha+\frac{1}{2})}\|u-v\|_X$$

and assuming (2.11) (with C_1 perhaps replaced by a slightly larger value), it follows that $\mathcal{F}$ is indeed a strict contraction on X_K, and thus has a unique fixed point u. This fixed point is a (pointwise mild) solution of (1.1).

Finally, if $m \ge q$, we can modify the calculation leading to (2.12) as follows. (This was not needed for the contraction argument, but will be useful to obtain additional properties of the solution, in particular Proposition 2.3.)

$$\begin{aligned}
\|\mathcal{G}u(t)\|_m &\le C\int_0^t (t-s)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{m})}\||\nabla u(s)|^p\|_q\,ds = C\int_0^t (t-s)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{m})}\|\nabla u(s)\|_{pq}^p\,ds \\
&\le CK^p\int_0^t (t-s)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{m})}s^{-p(\alpha+\frac{1}{2})}\,ds \\
&= CK^pt^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{m})+1-p(\alpha+\frac{1}{2})}\int_0^1 (1-s)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{m})}s^{-p(\alpha+\frac{1}{2})}\,ds.
\end{aligned}$$

Thus, if

$$\frac{1}{q} - \frac{2}{N} < \frac{1}{m} \le \frac{1}{q},$$

then $\mathcal{G}u : (0,T] \to L^m$ is continuous and

$$t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{m})}\|\mathcal{G}u(t)\|_m \le CK^p t^{1-p(\alpha+\frac{1}{2})} \le CK^p T^{1-p(\alpha+\frac{1}{2})}. \tag{2.15}$$

In particular, $\mathcal{G}u : (0,T] \to L^q$ is continuous and $\lim_{t\to 0}\|\mathcal{G}u(t)\|_q = 0$. Since $e^{t\Delta}u_0 \in C([0,T];L^q)$, it follows that $\mathcal{F}u \in C([0,T);L^q)$ hence,

$$u \in C([0,T);L^q). \tag{2.16}$$

Also we note that since $p(\alpha+1/2) < 1$, $u \in X(T)$ implies that $|\nabla u|^p \in L^1(0,T;L^q)$, so that u is indeed a mild L^q solution. Moreover,

$$\begin{aligned}
\|\nabla\mathcal{G}u(t)\|_m &\le C\int_0^t (t-s)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{m})-\frac{1}{2}}\||\nabla u(s)|^p\|_q\,ds \\
&= C\int_0^t (t-s)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{m})-\frac{1}{2}}\|\nabla u(s)\|_{pq}^p\,ds \\
&\le CK^p\int_0^t (t-s)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{m})-\frac{1}{2}}s^{-p(\alpha+\frac{1}{2})}\,ds \\
&= CK^p t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{m})-\frac{1}{2}+1-p(\alpha+\frac{1}{2})}\int_0^1 (1-s)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{m})-\frac{1}{2}}s^{-p(\alpha+\frac{1}{2})}\,ds.
\end{aligned}$$

Thus, if

$$\frac{1}{q} - \frac{1}{N} < \frac{1}{m} \le \frac{1}{q}, \tag{2.17}$$

then $\nabla\mathcal{G}u : (0,T] \to L^m$ is continuous and

$$t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{m})+\frac{1}{2}}\|\nabla\mathcal{G}u(t)\|_m \le CK^p t^{1-p(\alpha+\frac{1}{2})} \le CK^p T^{1-p(\alpha+\frac{1}{2})}. \tag{2.18}$$

(ii) Let u and v be two solutions of (1.1) in $X(T')$ for some $T' > 0$. It follows from (2.16) that u, $v \in C([0,T'];L^q)$. Since u and v both belong to $X_{K'}(T')$ for some $K' > C_1\|u_0\|_q$, by taking $T_0 \in (0,T')$ so small that

$$K' \ge C_1(\|u_0\|_q + K'^p T_0^\gamma),$$

we can invoke the above fixed point argument to conclude that u and v coincide on $[0,T_0]$. Letting $T_1 = \sup\{t \in (0,T'];\ u = v \text{ on } [0,t]\} \in (0,T']$, then necessarily $T_1 = T'$. Indeed, otherwise, since $u(T_1) = v(T_1) \in L^q$ and $u(T_1 + .)$, $v(T_1+.) \in X(T'-T_1)$, one could reproduce the same argument on $[T_1, T_1+\varepsilon]$ for $\varepsilon > 0$ small. We conclude that $u = v$ on $[0,T']$. The proof of Lemma 2.1 is complete. □

Remark 2.3. The time t maps of the semiflow generated by these solutions satisfy various continuity properties, which can be proved by modifications to the contraction mapping argument, as is done for example in [W2, CaW, SnTW1, SnTW2]. For example, assuming (2.11), if u_0, $v_0 \in L^q$, and if u, v denote the corresponding solutions of (1.1) in $X_K(T)$, then

$$\max\big[\sup_{(0,T]} t^\alpha \|u(t)-v(t)\|_{pq}\,,\, \sup_{(0,T]} t^{\alpha+\frac12}\|\nabla(u(t)-v(t))\|_{pq}\big] \le C\|u_0-v_0\|_q$$

and

$$\sup_{(0,T]} \|(u(t)-v(t))\|_q \le C\|u_0-v_0\|_q,$$

where $C = C(p,q,N,a) > 0$.

Remark 2.4. The existence and uniqueness result of Lemma 2.1 (i) remains valid for more general initial data, namely for all $u_0 \in \mathcal{S}'$ such that (2.13) holds (except, of course, for the continuity of u in L^q at $t=0$).

The next step is to improve the uniqueness class for local solutions using ideas from [B2, Br, BC].

Lemma 2.2. *Assume $q > q_c$ and let $u_0 \in L^q$. Let*

$$u \in C([0,T];L^q) \cap C((0,T);W^{1,pq}) \tag{2.19}$$

be a (pointwise mild) solution of (1.1) on $(0,T)$. Then u coincides with the solution given by Lemma 2.1.

Proof of Lemma 2.2. Let $M = \sup_{t\in(0,T)}\|u(t)\|_q$ and fix $K_1 > 0$, $T_1 \in (0,T/2)$ such that (with the notation of Lemma 2.1) $K_1 \ge C_1(M + K_1^p T_1^\gamma)$. It follows from Lemma 2.1 that for every $\tau \in (0,T/2)$, there is a unique solution $v_\tau \in X_{K_1}(T_1)$ of

$$v_\tau(t) = e^{t\Delta}u(\tau) + a\int_0^t e^{(t-s)\Delta}|\nabla v_\tau(s)|^p\,ds, \quad 0 \le t \le T_1.$$

Letting $u_\tau(t) = u(\tau+t)$ for $t \in [0,T_1)$, the fact that $u \in C((0,T);W^{1,pq})$ implies $u_\tau \in X(T_1)$. By uniqueness in $X(T_1)$ (Lemma 2.1 (ii)), we deduce that

$$u(\tau+t) = v_\tau(t), \quad 0 \le t < T_1, \quad 0 < \tau < T/2. \tag{2.20}$$

Using the fact that $v_\tau \in X_{K_1}(T_1)$, we see that, for all $\tau \in (0,T/2)$,

$$\max\big[\sup_{(0,T_1)} t^\alpha\|u(\tau+t)\|_{pq},\, \sup_{(0,T_1)} t^{\alpha+\frac12}\|\nabla u(\tau+t)\|_{pq}\big] \le K_1.$$

Letting $\tau \to 0$, it follows that $u \in X_{K_1}(T_1)$, hence $u \in X(T)$. By uniqueness in $X(T)$ (using Lemma 2.1 (ii) again), we conclude that u coincides with the solution given by Lemma 2.1. □

Remark 2.5. The conclusion and the proof of Lemma 2.2 are still valid if one only assumes $L^\infty(0,T;L^q)\cap L^\infty_{\mathrm{loc}}(0,T;W^{1,pq})$. Alternatively, under the assumption (2.19) of Lemma 2.2, one can conclude the proof of Lemma 2.2 after (2.20) as follows. Denote by v_0 the solution of (1.1) given by Lemma 2.1. For each fixed $t\in(0,T_1)$, upon letting $\tau\to 0$, we get $u(\tau+t)\to u(t)$ in L^q (by continuity of u) and $v_\tau(t)\to v_0(t)$ in L^q (by continuous dependence in $X(T)$ – see Remark 2.3). Therefore $u(t)=v_0(t)$ on $(0,T_1)$.

Before completing the proof of Theorem 2.1, it will be useful to obtain the higher regularity and smoothing properties of the solution (Proposition 2.3).

Proof of Proposition 2.3 for $u_0\in L^q$, $q>q_c$. The proof is based on similar arguments in [SnTW2].

Let us first note that (1.1) implies that

$$u(t)=e^{(t-\tau)\Delta}u(\tau)+a\int_\tau^t e^{(t-s)\Delta}|\nabla u(s)|^p\,ds,\quad 0<\tau<t<T. \tag{2.21}$$

Fix m and r with $p\le m<r\le\infty$. Suppose we know that, for some $L>0$,

$$\max\big[\sup_{(0,T]} t^{\frac N2(\frac1q-\frac1m)}\|u(t)\|_m,\ \sup_{(0,T]} t^{\frac N2(\frac1q-\frac1m)+\frac12}\|\nabla u(t)\|_m\big]\le L. \tag{2.22}$$

Using (2.21) with $\tau=t/2$, we see that

$$\begin{aligned}
\|u(t)\|_r &\le \|e^{\frac t2\Delta}u(t/2)\|_r + C\int_{\frac t2}^t (t-s)^{-\frac N2(\frac pm-\frac1r)}\||\nabla u(s)|^p\|_{m/p}\,ds\\
&\le Ct^{-\frac N2(\frac1m-\frac1r)}\|u(t/2)\|_m + C\int_{\frac t2}^t (t-s)^{-\frac N2(\frac pm-\frac1r)}\|\nabla u(s)\|_m^p\,ds\\
&\le CLt^{-\frac N2(\frac1q-\frac1r)} + CL^p\int_{\frac t2}^t (t-s)^{-\frac N2(\frac pm-\frac1r)}s^{-p(\frac N2(\frac1q-\frac1m)+\frac12)}\,ds\\
&= CLt^{-\frac N2(\frac1q-\frac1r)} + CL^p t^{-\frac N2(\frac1q-\frac1r)+1-p(\alpha+\frac12)}\\
&\qquad\times\int_{\frac12}^1 (1-s)^{-\frac N2(\frac pm-\frac1r)}s^{-p(\frac N2(\frac1q-\frac1m)+\frac12)}\,ds
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla u(t)\|_r &\le \|e^{\frac t2\Delta}\nabla u(t/2)\|_r + C\int_{\frac t2}^t (t-s)^{-\frac N2(\frac pm-\frac1r)-\frac12}\||\nabla u(s)|^p\|_{m/p}\,ds\\
&\le Ct^{-\frac N2(\frac1m-\frac1r)}\|\nabla u(t/2)\|_m + C\int_{\frac t2}^t (t-s)^{-\frac N2(\frac pm-\frac1r)-\frac12}\|\nabla u(s)\|_m^p\,ds\\
&\le CLt^{-\frac N2(\frac1q-\frac1r)-\frac12} + CL^p\int_{\frac t2}^t (t-s)^{-\frac N2(\frac pm-\frac1r)-\frac12}s^{-p(\frac N2(\frac1q-\frac1m)+\frac12)}\,ds\\
&= CLt^{-\frac N2(\frac1q-\frac1r)-\frac12} + CL^p t^{-\frac N2(\frac1q-\frac1r)-\frac12+1-p(\alpha+\frac12)}\\
&\qquad\times\int_{\frac12}^1 (1-s)^{-\frac N2(\frac pm-\frac1r)-\frac12}s^{-p(\frac N2(\frac1q-\frac1m)+\frac12)}\,ds.
\end{aligned}$$

The finiteness of the integrals is guaranteed if $\frac{p}{m}-\frac{1}{N}<\frac{1}{r}$. (The power of s in the integrand is of no importance for convergence since the interval of integration stays away from 0. Also, $q>q_c$ implies that $1-p(\alpha+\frac{1}{2})>0$.) If this condition is met, then we may conclude that

$$\tag{2.23} \begin{aligned} \max\big[\sup_{(0,T]} t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}\|u(t)\|_r,\ \sup_{(0,T]} t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})+\frac{1}{2}}\|\nabla u(t)\|_r\big] \\ \le L'(L,p,q)T^{1-p(\alpha+\frac{1}{2})}=L''(L,p,q,T). \end{aligned}$$

Note that one can choose $r>m$ such that $\frac{p}{m}-\frac{1}{N}<\frac{1}{r}$ if and only if $m>N(p-1)$. One may then prove (2.5) and (2.6) in Proposition 2.3 for all $pq\le r\le\infty$ by an iterative procedure. Indeed, start with $r=m=pq$, for which (2.22) is a consequence of the contraction mapping argument used in Lemma 2.1 (i) to prove existence. Then use the calculations above to prove (2.22) for larger and larger values of r. One can easily check that $r=\infty$ is reached in a finite number of iterations.

Next, the properties (2.5) and (2.6) for $q\le r\le pq$ follow from (2.15), (2.18) and the fact that $u_0\in L^q$ (note that $q>q_c$ and $q\le r\le pq$ imply that (2.17) is satisfied with $m=r$).

The proof of Proposition 2.3 for $u_0\in L^q$, $q>q_c$ is thus complete. □

Completion of proof of Theorem 2.1 for $q>q_c$. The local existence and uniqueness part follows from Lemmas 2.1 and 2.2. Moreover, from Proposition 2.3, one easily deduces that $u\in C((0,T_{\max});W^{1,r})$ for $q\le r\le\infty$. It only remains to show that u is classical and global. By standard arguments using interior parabolic regularity theory (see, e.g., [Lie, Theorem 7.13]), along with $u\in C((0,T_{\max});W^{1,\infty})$, one easily obtains that $u\in C^{2,1}(Q_T)$ and $u\in C((0,T);C^2_{\rm b})$, so that that u is a classical solution of (VHJ) on $\mathbb{R}^N\times(0,T_{\max})$. It then follows from [AB, Theorem A and estimate (2.14)] that u satisfies

$$\tag{2.24} \sup_{(t_0,T_{\max})}\|u(t)\|_\infty+\|\nabla u(t)\|_\infty\le\|u(t_0)\|_\infty+\|\nabla u(t_0)\|_\infty<\infty,\quad 0<t_0<T_{\max}.$$

But (1.1) then implies that

$$\tag{2.25} u(t_0+t)=\mathrm{e}^{t\Delta}\,u(t_0)+\int_0^t \mathrm{e}^{(t-s)\Delta}\,b(s,y)|\nabla u(t_0+s)|\,ds,$$

where b is bounded on $\mathbb{R}^N\times(t_0,T_{\max})$. From (2.25), one easily deduces that $\|\nabla u(t)\|_q$, and then $\|u(t)\|_q$, remain bounded on (t_0,T') for all finite $T'\le T_{\max}$. It follows from the contraction mapping argument of Lemma 2.1 (i) that u can be extended to a global solution of (1.1), with $u\in C([0,\infty);L^q)\cap C((0,\infty);W^{1,pq})$, and so u is a classical solution of (VHJ) in $\mathbb{R}^N\times(0,\infty)$. The proof of Theorem 2.1 in the case $q>q_c$ is thus complete. □

We turn to well-posedness in L^q, where $q=q_c>1$. Our proofs are very closely modeled on [SnTW1, SnTW2] for existence, uniqueness and regularity, and on [BC] for extending the uniqueness class.

Proof of Theorem 2.1 for $q = q_c$. Let us first remark that we can no longer work in the space $X(T)$ that we used in the case $q > q_c$. Indeed taking $q = q_c$ would lead to $p(\alpha + 1/2) = 1$ and the integrals involved in the proof of Lemma 2.1 would be infinite. Instead of this, we fix r such that

$$1 \le \frac{r}{p} < q_c < r. \tag{2.26}$$

Such an r is certainly not unique, and what follows is valid for any choice of r, which we for the moment consider as fixed. For $0 < T < \infty$, let $Y = Y(T)$ be the Banach space of continuous curves $u : (0, T] \to W^{1,r}$ such that

$$\|u\|_Y = \max\big[\sup_{(0,T]} t^{\beta}\|u(t)\|_r, \sup_{(0,T]} t^{\beta+\frac{1}{2}}\|\nabla u(t)\|_r\big] < \infty,$$

where

$$\beta = \frac{N}{2}\Big(\frac{1}{q_c} - \frac{1}{r}\Big).$$

We denote by $Y_K(T)$ the closed ball of $Y(T)$ with radius K.

As in the supercritical case $q > q_c$, the first step (Lemma 2.3) is to use a contraction mapping argument to obtain existence and uniqueness of a local solution in a more restricted class than (2.3), namely, $u \in C([0, T]; L^q) \cap Y_K(T)$ for suitable K, $T > 0$. It will sometimes be possible to carry out the contraction mapping argument all at once for all $t > 0$. Thus, if $T = \infty$, we interpret the interval $(0, T]$ as $(0, \infty)$. In a second step (Lemma 2.4), we will then show that uniqueness actually holds in the larger class (2.2).

Lemma 2.3. *Assume $q = q_c > 1$ and let $u_0 \in L^q$. For all $T > 0$, define*

$$M_0(u_0, T) = \max\big[\sup_{(0,T)} t^{\beta}\|e^{t\Delta}u_0\|_r, \sup_{(0,T)} t^{\beta+\frac{1}{2}}\|\nabla e^{t\Delta}u_0\|_r\big] \le C\|u_0\|_{q_c}. \tag{2.27}$$

(i) We have

$$\lim_{T\to 0} M_0(u_0, T) = 0. \tag{2.28}$$

(ii) There exists $C_0 = C_0(p, q, r, a) > 0$, such that for all K, $T > 0$ satisfying

$$K > M_0(u_0, T) + C_0K^p, \tag{2.29}$$

there exists a unique function $u \in Y_K(T)$ which is a (pointwise mild) solution of (1.1) on $(0, T)$. Moreover $u \in C([0, T]; L^q)$. (Note that K, T satisfying (2.29) exist in view of (i).)

(iii) Let $K, T > 0$ satisfy (2.29). Then, for all $T' > 0$, there is at most one solution of (1.1) in the class $C((0, T']; L^q \cap W^{1,r}) \cap Y_K(T)$.

It follows from Lemma 2.3 that there exists a maximal existence time $T_{\max} \in (0, \infty]$ and a unique maximal solution u of (1.1) in the class

$$C([0, T_{\max}); L^q) \cap C((0, T_{\max}); W^{1,r}) \cap Y_K(T),$$

where K, T satisfy (2.29) (u does not depend on the choice of K, T). This solution will be referred to as *the* solution given by Lemma 2.3.

Proof of Lemma 2.3. (i) The family of operators $t^\beta e^{t\Delta}$ and $t^{\beta+1/2}\nabla e^{t\Delta}$, $t>0$, are uniformly bounded from L^q into L^r. Moreover, (2.28) is true for all u_0 in the dense subset $W^{1,r}\cap L^q$ of L^q. It follows that (2.28) holds for all $u_0\in L^q$.

(ii) If $u\in Y_K(T)$, we have, for all $t\in[0,T]$,

$$
\begin{aligned}
\|\mathcal{G}u(t)\|_r &\le C\int_0^t (t-s)^{-\frac{N(p-1)}{2r}}\||\nabla u(s)|^p\|_{r/p}\,ds = C\int_0^t (t-s)^{-\frac{N(p-1)}{2r}}\|\nabla u(s)\|_r^p\,ds\\
&\le CK^p\int_0^t (t-s)^{-\frac{N(p-1)}{2r}} s^{-p(\beta+\frac12)}ds\\
&= CK^p t^{-\beta}\int_0^1 (1-s)^{-\frac{N(p-1)}{2r}} s^{-p(\beta+\frac12)}ds = CK^p t^{-\beta},
\end{aligned}
$$

and that

$$
\begin{aligned}
\|\nabla\mathcal{G}u(t)\|_r &\le C\int_0^t (t-s)^{-\frac{N(p-1)}{2r}-\frac12}\||\nabla u(s)|^p\|_{r/p}\,ds\\
&= C\int_0^t (t-s)^{-\frac{N(p-1)}{2r}-\frac12}\|\nabla u(s)\|_r^p\,ds\\
&\le CK^p\int_0^t (t-s)^{-\frac{N(p-1)}{2r}-\frac12} s^{-p(\beta+\frac12)}ds\\
&= CK^p t^{-\beta-\frac12}\int_0^1 (1-s)^{-\frac{N(p-1)}{2r}-\frac12} s^{-p(\beta+\frac12)}ds = CK^p t^{-\beta-\frac12}.
\end{aligned}
$$

In particular, it follows that

$$
\|\mathcal{G}u\|_Y \le C_0K^p, \tag{2.30}
$$

where $C_0=C_0(p,r,a)>0$ (note that C_0 is independent of T). The relation (2.26) guarantees that all the integrals above are convergent.

Moreover, choosing $K\in(0,C_0^{-1/(p-1)})$, (2.29) is then satisfied for $T>0$ small enough in view of (2.29). It follows that $\mathcal{F}$ maps $Y_K(T)$ into itself.

Now using (2.14), we obtain for all $t\in[0,T)$,

$$
\begin{aligned}
\|\mathcal{F}u(t)-\mathcal{F}v(t)\|_r &\le C\int_0^t (t-s)^{-\frac{N(p-1)}{2r}}\||\nabla u(s)|^p-|\nabla v(s)|^p\|_{r/p}\,ds\\
&\le CK^{p-1}\int_0^t (t-s)^{-\frac{N(p-1)}{2r}} s^{-(\beta+\frac12)(p-1)}\|\nabla u(s)-\nabla v(s)\|_r\,ds\\
&\le CK^{p-1}\|u-v\|_Y\int_0^t (t-s)^{-\frac{N(p-1)}{2r}} s^{-(\beta+\frac12)p}\,ds\\
&\le CK^{p-1}t^{-\beta}\|u-v\|_Y,
\end{aligned}
$$

and similarly

$$\|\nabla \mathcal{F}u(t) - \nabla \mathcal{F}v(t)\|_r \le CK^{p-1} t^{-\beta-1/2} \|u-v\|_Y. \tag{2.31}$$

Therefore,

$$\|\mathcal{F}u - \mathcal{F}v\|_Y \le C_0 K^{p-1} \|u-v\|_Y$$

(with $C_0 = C_0(p,r,a)$ perhaps replaced by a slightly larger value than in (2.30)). Assuming (2.29), it follows that $\mathcal{F}$ is indeed a strict contraction on X_K, and thus has a unique fixed point u. This fixed point is a (pointwise mild) solution of (1.1).

Finally, if $m \ge r/p$, we can modify the calculation leading to (2.30) as follows. (Again, this was not needed for the contraction argument, but will be useful to obtain additional properties of the solution, in particular Proposition 2.3.)

$$\begin{aligned}
\|\mathcal{G}u(t)\|_m &\le C \int_0^t (t-s)^{-\frac{N}{2}(\frac{p}{r}-\frac{1}{m})} \| |\nabla u(s)|^p \|_{r/p}\, ds = C \int_0^t (t-s)^{-\frac{N}{2}(\frac{p}{r}-\frac{1}{m})} \|\nabla u(s)\|_r^p \, ds \\
&\le CK^p \int_0^t (t-s)^{-\frac{N}{2}(\frac{p}{r}-\frac{1}{m})} s^{-p(\beta+\frac{1}{2})}\, ds \\
&= CK^p t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{m})} \int_0^1 (1-s)^{-\frac{N}{2}(\frac{p}{r}-\frac{1}{m})} s^{-p(\beta+\frac{1}{2})}\, ds.
\end{aligned}$$

Thus, if

$$\frac{p}{r} - \frac{2}{N} < \frac{1}{m} \le \frac{p}{r},$$

then $\mathcal{G}u : (0,T] \to L^m$ is continuous and

$$t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{m})} \|\mathcal{G}u(t)\|_m \le CK^p. \tag{2.32}$$

In particular, $\lim_{t\to 0} \|\mathcal{G}u(t)\|_m = 0$ if $r/p \le m < q_c$.

Moreover,

$$\begin{aligned}
\|\nabla \mathcal{G}u(t)\|_m &\le C \int_0^t (t-s)^{-\frac{N}{2}(\frac{p}{r}-\frac{1}{m})-\frac{1}{2}} \| |\nabla u(s)|^p \|_{r/p}\, ds \\
&= C \int_0^t (t-s)^{-\frac{N}{2}(\frac{p}{r}-\frac{1}{m})-\frac{1}{2}} \|\nabla u(s)\|_r^p\, ds \\
&\le CK^p \int_0^t (t-s)^{-\frac{N}{2}(\frac{p}{r}-\frac{1}{m})-\frac{1}{2}} s^{-p(\beta+\frac{1}{2})}\, ds \\
&= CK^p t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{m})-\frac{1}{2}} \int_0^1 (1-s)^{-\frac{N}{2}(\frac{p}{r}-\frac{1}{m})-\frac{1}{2}} s^{-p(\beta+\frac{1}{2})}\, ds.
\end{aligned}$$

Thus, if

$$\frac{p}{r} - \frac{1}{N} < \frac{1}{m} \le \frac{p}{r},$$

then $\nabla \mathcal{G}u : (0,T] \to L^m$ is continuous and

$$t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{m})+\frac{1}{2}} \|\nabla \mathcal{G}u(t)\|_m \le CK^p. \tag{2.33}$$

In particular, $\lim_{t\to 0}\|\nabla\mathcal{G}u(t)\|_m = 0$ if $r/p \le m < N(p-1)$.

Among the various additional properties of the fixed point u, we note right away, as a consequence of (2.32), that

$$u(t) - e^{t\Delta}u_0 \in C([0,T]; L^m),$$

if $r/p \le m < q_c$. To prove continuity in L^{q_c}, note that as $T \to 0$, in view of (2.28), one may choose K in (2.29) as small as we wish. Thus, again by (2.32), $\sup_{(0,T]}\|\mathcal{G}u(t)\|_{q_c} \to 0$, as $T \to 0$. This proves $u \in C([0,T]; L^{q_c})$.

(iii) Let u and v be two solutions of (1.1) in $Y_K(T) \cap C((0,T']; L^q \cap W^{1,r})$ for some $T' > 0$. By part (ii), they coincide on $[0,T]$. Letting

$$T_1 = \sup\bigl\{t \in (0,T'];\ u = v \text{ on } [0,t]\bigr\} \in (0,T'],$$

then necessarily $T_1 = T'$. Indeed, otherwise, since $u(T_1) = v(T_1) \in L^q$ and since $u(T_1 + .)$ and $u(T_1 + .) \in C([0, T' - T_1]; W^{1,r}) \subset Y_{\hat K}(\hat T)$ for some $\hat K, \hat T > 0$ satisfying

$$\hat K > M_0(u(T_1), \hat T) + C_0 \hat K^p,$$

we would deduce from part (ii) that $u = v$ on $[0, T_1 + \hat T]$. □

Remark 2.6. (a) A straightforward modification of the above contraction mapping argument can be used to show that if u_0, $v_0 \in L^{q_c}$ both satisfy (2.29) for the same K, $T > 0$, and if the corresponding solutions of (1.1) are given by u, $v \in Y_K(T)$, then

$$\max\Bigl[\sup_{(0,T]} t^\beta\|u(t)-v(t)\|_r\,,\ \sup_{(0,T]} t^{\beta+\frac12}\|\nabla(u(t)-v(t))\|_r\Bigr] \le C_2\|u_0 - v_0)\|_{q_c},$$

where $C_2 = C_2(p,r,N,a) > 0$. Further calculations show that

$$\sup_{(0,T]}\|u(t)-v(t)\|_{q_c} \le C_2\|u_0 - v_0\|_{q_c}$$

(see Step 1 in the proof of Theorem 2.5 for details).

(b) If $\|u_0\|_{q_c}$ is sufficiently small, then, in view of (2.27), one can choose $T = \infty$ in Lemma 2.3 and in Remark (a) above.

(c) Denote by $\overline{Y}$ the space corresponding to another value $\overline{r}$ satisfying (2.26). Then the solution constructed in Lemma 2.3 (for the value of r that we have fixed) also belongs to $\overline{Y}_{K'}(T')$ for some K', $T' > 0$ satisfying the analogue of (2.29), and it is the unique solution of (1.1) in that class. This follows from the observation that the contraction argument can be carried out in the intersection $Y_K(T) \cap \overline{Y}_{K'}(T')$.

(d) The existence and uniqueness result of Lemma 2.3 (ii) remains valid for more general initial data, namely for all $u_0 \in \mathcal{S}'$ such that $M_0(u_0, T)$ is sufficiently small for some $T > 0$ (except, of course, for the continuity of u in L^{q_c} at $t = 0$).

Next, we turn to the stronger uniqueness results, as in the subcritical case, modeled after the work of [B2, Br, BC].

Lemma 2.4. *Assume $q = q_c > 1$ and let $u_0 \in L^q$. Let $u \in C([0,T];L^q) \cap C((0,T);W^{1,r})$ be a pointwise mild solution of (1.1) on $(0,T)$. Then u coincides with the solution given by Lemma 2.3.*

Following [B2, BC], in view of the proof of Lemma 2.4, we prepare the following Lemma.

Lemma 2.5. *Let $1 < q < \infty$. Let $\mathcal{K}$ be a compact subset of L^q and define*

$$\delta(t;\mathcal{K}) = \sup_{\phi \in \mathcal{K}} M_0(\phi, t), \quad t > 0,$$

where M_0 is defined in Lemma 2.3. Then

$$\delta(t;\mathcal{K}) \to 0, \quad \text{as } t \to 0^+.$$

Proof of Lemma 2.5. First we note that the families of operators, $t^\beta e^{t\Delta}$ and $t^{\beta+\frac{1}{2}}\nabla e^{t\Delta}$, for $t > 0$, are uniformly bounded from L^{q_c} into L^r. Moreover, they converge pointwise to 0 as $t \to 0$ in view of (2.28). Thus, they converge uniformly to 0 on any compact subset of L^q. The conclusion follows. □

Proof of Lemma 2.4. Since the image $\mathcal{K}$ of $[0,T/2]$ under the continuous function u is compact in L^{q_c}, Lemma 2.5 implies that

$$\delta(t;\mathcal{K}) = \sup_{\tau \in [0,T/2]} M_0(u(\tau), T_1) \to 0, \quad \text{as } T_1 \to 0.$$

Therefore there exist $K > 0$ and $T_1 \in (0,T/2)$ such that

$$K \geq M_0(u(\tau), T_1) + C_0 K^p, \quad 0 < \tau < T/2. \tag{2.34}$$

It follows from Lemma 2.3 (ii) that for every $\tau \in (0,T/2)$, there is a unique solution $v_\tau \in Y_K(T_1)$ of

$$v_\tau(t) = e^{t\Delta}u(\tau) + a\int_0^t e^{(t-s)\Delta}|\nabla v_\tau(s)|^p \, ds, \quad 0 \leq t < T_1.$$

Moreover, $v_\tau \in C([0,T_1];L^q)$. Let $u_\tau(t) = u(\tau + t)$ for $t \in [0,T_1]$. Since $u_\tau \in C([0,T_1];W^{1,r})$, it follows that $\|u_\tau\|_{Y(t)} \to 0$ as $t \to 0$. Therefore, there exists $T_\tau \in (0,T_1]$ such that $u_\tau \in Y_K(T_{\ tau})$. Moreover, since $T_\tau \leq T_1$, (2.34) implies that

$$K \geq M_0(u(\tau), T_\tau) + C_0 K^p, \quad 0 < \tau < T/2$$

and $v_\tau \in Y_K(T_\tau)$. Since $u_\tau(0) = v_\tau(0) = u(\tau) \in L^q$ and u_τ, $v_\tau \in C([0,T_1];W^{1,r} \cap L^q)$, we may apply Lemma 2.3 (iii) to deduce that $u_\tau = v_\tau$ on $[0,T_1]$ that is,

$$u(\tau + t) = v_\tau(t), \quad 0 \leq t \leq T_1, \quad 0 < \tau < T/2.$$

Using the fact that $v_\tau \in Y_K(T_1)$, we see that, for all $\tau \in (0, T/2)$,

$$\max\big[\sup_{(0,T_1)} t^\beta \|u(\tau+t)\|_r,\ \sup_{(0,T_1)} t^{\beta+\frac{1}{2}} \|\nabla u(\tau+t)\|_r\big] \le K.$$

Letting $\tau \to 0$, it follows that $u \in Y_K(T_1)$. Applying Lemma 2.3 (iii) again, one concludes that u and v coincide on $[0, T]$. □

As in the supercritical case, before completing the proof of Theorem 2.1, we establish the higher regularity and smoothing properties of the solution (Proposition 2.3).

Proof of Proposition 2.3 for $u_0 \in L^q$, $q = q_c$. Instantaneous smoothing of solutions into $W^{1,m}$ for $m > r$ is proved exactly as in the case $q > q_c$. Start with $m = r$, the value used in the contraction mapping argument which verifies (2.26), and then follow the same iterative procedure used in the case $q > q_c$. The only difference is that since here $q = q_c$, the factor $T^{1-p(\alpha+\frac{1}{2})}$ does not appear in formula (2.23) i.e., L'' does not depend on T. As a result, the iterative step is independent of T. Of course, as in the case $q > q_c$, if $u_0 \in L^{q_c}$, then

$$\max\big[\sup_{(0,T]} t^{\frac{N}{2}(\frac{1}{q_c}-\frac{1}{m})} \|u(t)\|_m,\ \sup_{(0,T]} t^{\frac{N}{2}(\frac{1}{q_c}-\frac{1}{m})+\frac{1}{2}} \|\nabla u(t)\|_m\big] < \infty.$$

for $q_c \le m < r$ by the properties (2.32) and (2.33) of $\mathcal{G}u(t)$. Finally, if $\|u_0\|_{q_c}$ is sufficiently small, then the previous inequality is valid with $T = \infty$ by Remark 2.6 (b). Thus (2.5) and (2.6) are valid with $T = \infty$. □

Completion of proof of Theorem 2.1 for $q = q_c$. The solution given by Lemma 2.3 was constructed for a particular value of r, say r_0, fixed in (2.26). However, the uniqueness result of Lemma 2.4 holds not only in the class $C([0,T]; L^q) \cap C((0,T); W^{1,r_0})$, but actually in $C([0,T]; L^q) \cap C((0,T); W^{1,r})$ for any r such that $q_c/p < r < q_c$. Indeed, in view of Remark 2.6 (c), the proof of Lemma 2.4 works for all such r. The local existence and uniqueness statements of Theorem 2.1 in the case $q = q_c$ are thus proved.

Arguing exactly as in the case $q > q_c$, we obtain that u is classical on $(0, T_{\max})$ and satisfies (2.24) and $\sup_{(t_0,T')} \|u(t)\|_q + \|\nabla u(t)\|_q < \infty$ for all finite $T' \le T_{\max}$. Therefore, we have $\sup_{(t_0,T')} \|u(t)\|_{W^{1,r}} < \infty$ for all finite $T' \le T_{\max}$ and $q \le r \le \infty$. It follows from the contraction mapping argument of Lemma 2.1 (i) that u can be extended to a global solution of (1.1), with $u \in C([0,\infty); L^q) \cap C((0,\infty); W^{1,r})$, $q \le r \le \infty$, and so u is a classical solution of (VHJ) in $\mathbb{R}^N \times (0,\infty)$. The proof of Theorem 2.1 in the case $q = q_c$ is complete. □

Remark 2.7. Note that one can also use the space Y_K to prove local existence and uniqueness of solutions to (1.1) in the case $q > q_c$. Of course, in that case, one has to replace q_c by q in (2.26) and in the definition of β. This gives slightly better uniqueness results.

Proof of Theorem 2.2. It follows along the lines of proof of Theorem 2.2 for $q = 1 > q_c$. In particular one works with the same space X and uses (2.11) and (2.13) with $\|u_0\|_{\mathcal{M}}$ instead of $\|u_0\|_q$ and one gets $u(t) - e^{t\Delta}u_0 \in C([0,T); L^1)$ instead of (2.16). □

Finally, we turn to well-posedness in $W^{1,q}$, with $q > N(p-1)$ or $q = N(p-1) > 1$.

Proof of Proposition 2.4 for $q > N(p-1)$. Let $Z = Z(T)$ be the Banach space of continuous curves $u : (0,T] \to W^{1,pq}$ such that

$$\|u\|_Z = \max\big[\sup_{(0,T]} t^\alpha \|u(t)\|_{pq}, \sup_{(0,T]} t^\alpha \|\nabla u(t)\|_{pq}\big] < \infty,$$

where

$$\alpha = \frac{N}{2}\Big(\frac{1}{q} - \frac{1}{pq}\Big).$$

Note the difference between Z and X: both terms in the norms of Z have the same power of t. We denote by $Z_K = Z_K(T)$ the closed ball of Z with radius K. If $u \in Z_K(T)$, it follows that

$$\begin{aligned}
\|\mathcal{G}u(t)\|_{pq} &\le C\int_0^t (t-s)^{-\alpha} \||\nabla u(t)|^p\|_q \, ds = C\int_0^t (t-s)^{-\alpha}\|\nabla u(t)\|_{pq}^p \, ds \\
&\le CK^p \int_0^t (t-s)^{-\alpha} s^{-p\alpha}\, ds \\
&= CK^p t^{1-(p+1)\alpha}\int_0^1 (1-s)^{-\alpha} s^{-p\alpha}\, ds \le CK^p t^{-\alpha} T^{1-p\alpha},
\end{aligned}$$

and that

$$\begin{aligned}
\|\nabla\mathcal{G}u(t)\|_{pq} &\le C\int_0^t (t-s)^{-\alpha-\frac{1}{2}} \||\nabla u(t)|^p\|_q \, ds \\
&\le CK^p \int_0^t (t-s)^{-\alpha-\frac{1}{2}} s^{-p\alpha}\, ds = CK^p t^{1-\frac{1}{2}-(p+1)\alpha}\int_0^1 (1-s)^{-\alpha-\frac{1}{2}} s^{-p\alpha}\, ds \\
&\le CK^p t^{-\alpha} T^{\frac{1}{2}-p\alpha}.
\end{aligned}$$

In particular, it follows that

$$\|\mathcal{G}u\|_Z \le CK^p \max[T^{\frac{1}{2}-p\alpha}, T^{1-p\alpha}]$$

(where C is independent of T, and in fact depends only on p and q). The fact that $q > N(p-1)$ guarantees that all the integrals above are convergent and that $\frac{1}{2} - p\alpha > 0$.

Turning now to the contraction mapping argument, since $u_0 \in W^{1,q}$, we have

$$\max\big[\sup_{(0,T]} t^\alpha \|e^{t\Delta}u_0\|_{pq}, \sup_{(0,T]} t^\alpha \|\nabla e^{t\Delta} u_0\|_{pq}\big] \le M.$$

Choose $K > M$ and $T > 0$ so that

$$M + CK^p \max[T^{\frac{1}{2}-p\alpha}, T^{1-p\alpha}] \le K.$$

It follows that $\mathcal{F}$ maps Z_K into itself. As in Lemma 2.1, an easy modification of the above calculations shows that, with C perhaps replaced by a slightly larger value, $\mathcal{F}$ is indeed a strict contraction on Z_K, and thus has a unique fixed point u. This fixed point is a solution of (1.1). Moreover, since $\alpha p < 1/2$, $u \in Z_K$ implies $|\nabla u|^p \in L^1(0,T;L^q)$ and $u \in C([0,T);W^{1,q})$.

The rest of the proof, in particular the uniqueness statement (ii) and the regularity (iv), is very similar to the corresponding proof in Theorem 2.1 and Proposition 2.3 and is thus omitted.

Proof of Proposition 2.4 for $q = N(p-1) > 1$. Fix r such that

$$1 \le \frac{r}{p} < N(p-1) < r. \tag{2.26}$$

For $0 < T < \infty$, let $W = W(T)$ be the Banach space of continuous curves $u : (0,T] \to W^{1,r}$ such that

$$\|u\|_W = \max\big[\sup_{(0,T]} t^\beta \|u(t)\|_r, \sup_{(0,T]} t^\beta \|blau(t)\|_r\big] < \infty,$$

where

$$\beta = \frac{N}{2}\Big(\frac{1}{q_c} - \frac{1}{r}\Big).$$

We denote by $W_K(T)$ the closed ball of $W(T)$ with radius K.

If $u \in W_K(T)$, we have, for all $t \in [0,T]$,

$$\begin{aligned}\|\mathcal{G}u(t)\|_r &\le C\int_0^t (t-s)^{-\frac{N(p-1)}{2r}} \||\nabla u(s)|^p\|_{r/p}\,ds = C\int_0^t (t-s)^{-\frac{N(p-1)}{2r}} \|\nabla u(s)\|_r^p\,ds\\ &\le CK^p \int_0^t (t-s)^{-\frac{N(p-1)}{2r}} s^{-p\beta} ds\\ &= CK^p t^{\frac{1}{2}-\beta} \int_0^1 (1-s)^{-\frac{N(p-1)}{2r}} s^{-p\beta} ds = CK^p t^{\frac{1}{2}-\beta},\end{aligned}$$

and that

$$\begin{aligned}\|\nabla\mathcal{G}u(t)\|_r &\le C\int_0^t (t-s)^{-\frac{N(p-1)}{2r}-\frac{1}{2}} \||\nabla u(s)|^p\|_{r/p}\,ds\\ &= C\int_0^t (t-s)^{-\frac{N(p-1)}{2r}-\frac{1}{2}} \|\nabla u(s)\|_r^p\,ds\\ &\le CK^p \int_0^t (t-s)^{-\frac{N(p-1)}{2r}-\frac{1}{2}} s^{-p\beta} ds\\ &= CK^p t^{-\beta} \int_0^1 (1-s)^{-\frac{N(p-1)}{2r}-\frac{1}{2}} s^{-p\beta} ds = CK^p t^{-\beta}.\end{aligned}$$

In particular, it follows that

$$\|\mathcal{G}u\|_W \le C_0 K^p \max(1, T^{1/2}),$$

where $C_0 = C_0(p, r, a) > 0$ (note that C_0 is independent of T). The relation (2.26) guarantees that all the integrals above are convergent.

Let

$$M_1(u_0, T) = \max\big[\sup_{(0,T)} t^\beta \|e^{t\Delta}u_0\|_r, \sup_{(0,T)} t^\beta \|\nabla e^{t\Delta}u_0\|_r\big].$$

One easily shows that $\lim_{T\to 0} M_1(u_0, T) = 0$ for all all $u_0 \in W^{1,q}$. Therefore, choosing $K \in (0, C_0^{-1/(p-1)})$, we have $K > M_1(u_0, T) + C_0K^p$ for $0 < T < 1$ sufficiently small. It follows that $\mathcal{F}$ maps $W_K(T)$ into itself. As in Lemma 2.3, an easy modification of the above calculations shows that, with C_0 perhaps replaced by a slightly larger value, $\mathcal{F}$ is indeed a strict contraction on W_K, and thus has a unique fixed point u. This fixed point is a solution of (1.1). The rest of the proof, in particular the uniqueness statement (iii) and the regularity (iv), is very similar to the corresponding proof in Theorem 2.1 and Proposition 2.3 and is thus omitted.

Proof of Theorem 2.5. We follow the ideas of the proof of Theorem 3.1 (i) in [S1] (see also [Ka1, p. 480]).

Denote $U_M = \{u_0 \in L^{q_c};\ \|u_0\|_{q_c} \le M\}$. For all $t \ge 0$, define the map $W_t : u_0 \mapsto u(t)$ from L^{q_c} into itself.

Step 1. We prove that the W_t are Lipschitz continuous on U_M for some small $M > 0$, uniformly for all $t \in [0, \infty)$. Let r and β be as in the proof of Theorem 2.1 ($q = q_c$). Let u_0, $v_0 \in U_M$ and $u(t) = W_tu_0$, $v(t) = W_tv_0$. By Remarks 2.6 (a) and (b), if M is sufficiently small, then

$$\sup_{(0,\infty)} t^{\beta+\frac12}\|\nabla u(t)\|_r \le C\|u_0\|_q, \quad \sup_{(0,\infty)} t^{\beta+\frac12}\|\nabla v(t)\|_r \le C\|v_0\|_q \tag{2.35}$$

and

$$\sup_{(0,\infty)} t^{\beta+\frac12}\|\nabla u(t) - \nabla v(t)\|_r \le C\|u_0 - v_0\|_q. \tag{2.36}$$

Now, from (2.35), (2.36) and $\frac{N}{2}(\frac{p}{r} - \frac{1}{q}) + (\beta + \frac12)p = 1$, we deduce that

$$\begin{aligned}
&\|u(t) - v(t)\|_q \\
&\quad\le \|u_0 - v_0\|_q + |a|\int_0^t (t-s)^{-\frac{N}{2}(\frac{p}{r}-\frac{1}{q})}\||\nabla u|^p - |\nabla v|^p\|_{r/p}\, ds \\
&\quad\le \|u_0 - v_0\|_q + |a|p\int_0^t (t-s)^{-\frac{N}{2}(\frac{p}{r}-\frac{1}{q})}\times \\
&\qquad\qquad \big(\|\nabla u(s)\|_r^{p-1} + |\nabla v(s)\|_r^{p-1}\big)\|\nabla u(s) - \nabla v(s)\|_r\, ds. \\
&\quad\le \|u_0 - v_0\|_q + C\big(\|u_0\|_q^{p-1} + \|v_0\|_q^{p-1}\big)\|u_0 - v_0\|_q \int_0^t (t-s)^{-\frac{N}{2}(\frac{p}{r}-\frac{1}{q})}s^{-(\beta+\frac12)p}\, ds \\
&\quad= \|u_0 - v_0\|_q + C\big(\|u_0\|_q^{p-1} + \|v_0\|_q^{p-1}\big)\|u_0 - v_0\|_q \int_0^1 (1-\tau)^{-\frac{N}{2}(\frac{p}{r}-\frac{1}{q})}\tau^{-(\beta+\frac12)p}\, d\tau \\
&\quad\le (1 + CM^{p-1})\|u_0 - v_0\|_q.
\end{aligned}$$

The claim is proved.

Step 2. We claim that $\|u(t)\|_q$ decays to 0 for $u_0 \in U_M$ and small $M > 0$ provided u_0 also belongs to L^m for $m \in (r/p, q)$.

Let $m \in (r/p, q)$, $\theta = \theta(m) = \frac{N}{2}\big(\frac{1}{m} - \frac{1}{r}\big)$ and assume $u_0 \in U_M \cap L^m$. Then, by (2.35),

$$\begin{aligned}\|\nabla u(t)\|_r &\le Ct^{-\theta-\frac12}\|u_0\|_m + |a|\int_0^t (t-s)^{-\frac{N(p-1)}{2r}-\frac12}\|\nabla u(s)\|_r^p\,ds\\ &\le Ct^{-\theta-\frac12}\|u_0\|_m + C\int_0^t (t-s)^{-\frac{N(p-1)}{2r}-\frac12}\times\\ &\qquad \|u_0\|_q^{p-1}s^{-(p-1)(\beta+\frac12)}s^{-\theta-\frac12}\big(\sup_{(0,t)}\tau^{\theta+\frac12}\|\nabla u(\tau)\|_r\big)\,ds.\end{aligned}$$

Observing that $\frac{N(p-1)}{2r} < \frac12$, $-\frac{N(p-1)}{2r} + \frac12 - (p-1)(\beta+\frac12) = 0$ and that $-(p-1)(\beta + \frac12) - \theta - \frac12 > -1$ for $m \in (r/p, q)$, it follows that

$$\begin{aligned}t^{\theta+\frac12}\|\nabla u(t)\|_r &\le C\|u_0\|_m + CM^{p-1}t^{-\frac{N(p-1)}{2r}+\frac12-(p-1)(\beta+\frac12)}\times\\ &\qquad\big(\sup_{(0,t)}\tau^{\theta+\frac12}\|\nabla u(\tau)\|_r\big)\int_0^1 (1-\tau)^{-\frac{N(p-1)}{2r}-\frac12}\tau^{-(p-1)(\beta+\frac12)-\theta-\frac12}\,d\tau\\ &= C\|u_0\|_m + CM^{p-1}\big(\sup_{(0,t)}\tau^{\theta+\frac12}\|\nabla u(\tau)\|_r\big),\end{aligned}$$

hence

$$\sup_{(0,\infty)} t^{\theta+\frac12}\|\nabla u(t)\|_r \le C\|u_0\|_m \tag{2.37}$$

(for M possibly smaller, independent of $\|u_0\|$). Next using (2.35) (2.37), we compute

$$\begin{aligned}\|u(t)\|_q &\le Ct^{-\frac{N}{2}(\frac1m-\frac1q)}\|u_0\|_m + |a|\int_0^t (t-s)^{-\frac{N}{2}(\frac{p}{r}-\frac1q)}\|\nabla u\|_r^p\,ds\\ &\le Ct^{-\frac{N}{2}(\frac1m-\frac1q)}\|u_0\|_m + C\int_0^t (t-s)^{-\frac{N}{2}(\frac{p}{r}-\frac1q)}\|u_0\|_q^{p-1}s^{-(p-1)(\beta+\frac12)}\|u_0\|_m s^{-\theta-\frac12}\,ds\end{aligned}$$

so that

$$t^{\frac{N}{2}(\frac1m-\frac1q)}\|u(t)\|_q \le C\|u_0\|_m + C\|u_0\|_q^{p-1}\|u_0\|_m.$$

The claim follows.

Step 3. Since the maps $W_t : U_M \mapsto L^q$ are Lipschitz contiuous, uniformly for $t \ge 0$, and since $W_t u_0$ decays to 0 in L^q for each u_0 in the dense subset $U_M \cap L^m$, it follows that $u(t) = W_t u_0$ decays to 0 in L^q for all $u_0 \in U_M$. The fact that $u(t)$ decays also in L^k for $q < k \le \infty$ was proved in Proposition 2.3. The proof is complete. □

3. Nonexistence and nonuniqueness results for $a > 0$, $u_0 \geq 0$

3.1. Nonexistence in L^q for $p \geq 2$

The following result shows that local existence fails in all L^q spaces ($q < \infty$) when $p \geq 2$ and $a > 0$. We have been able to discard only the existence of solutions which are classical for $t > 0$. However we note that the solutions constructed in Section A for $p < 2$ and $q \geq q_c$ are indeed classical for $t > 0$.

Proposition 3.1. *Let $p \geq 2$, $a > 0$ and $u_0 \in \mathrm{L}^1_{\mathrm{loc}}$. Assume that there exist $T > 0$ and a function $u \in \mathrm{C}^{1,2}(Q_T)$, $Q_T = \mathbb{R}^N \times (0,T)$, which is a solution of (VHJ)$_1$ in Q_T, such that* $\lim_{t\to 0} u(t) = u_0$ *in* $\mathrm{L}^1_{\mathrm{loc}}$. *Then* $\exp(au_0) \in \mathrm{L}^1_{\mathrm{loc}}$.

Proof. Assume that such T and u exist. Then u satisfies

$$u_t - \Delta u \geq a(|\nabla u|^2 - 1) \quad \text{in } Q_T.$$

Letting $v(x,t) = \exp(a(u(x,t) + t))$, we see that

$$v_t - \Delta v \geq 0 \quad \text{in } Q_T.$$

Fix $R > 0$ and $t_0 \in (0,T)$, and denote by $G_{R+1} = G_{R+1}(x,y,t)$ the heat kernel in $\mathrm{B}_{R+1}(0)$ with homogeneous Dirichlet conditions. Since $v > 0$, for all $\varepsilon \in (0, t_0/2)$, we have

$$v(t_0, 0) \geq \int_{|y|<R+1} G_{R+1}(0, y, t_0 - \varepsilon)\, v(y,\varepsilon)\, dy \geq C(t_0, R) \int_{|y|<R} v(y,\varepsilon)\, dy,$$

for some $C(t_0,R) > 0$. But the assumptions imply the existence of a sequence $\varepsilon_n \downarrow 0$ such that $u(\varepsilon_n)$ converges to u_0 a.e. Passing to the limit in the above inequality with $\varepsilon = \varepsilon_n$ and using Fatou's Lemma, we obtain

$$\int_{|x|<R} \exp(au_0(y))\, \mathrm{d}y < \infty,$$

and the conclusion follows. □

Remark 3.1. When $p = 2$, existence is true for $u_0 \in L^\infty$, as can be seen easily by using the transformation $v = e^u$ ($a = 1$). Also, existence (of a mild solution) is true for $u_0 \in W^{1,N}$, $N \geq 2$, by Proposition 2.4. (Recall that $W^{1,N} \not\subset L^\infty$ ($N \geq 2$) but that $u_0 \in W^{1,N}$ implies $e^{|u_0|} \in L^1_{\mathrm{loc}}$.) Interestingly, for $p > 2$, existence is true for $u_0 \in C_{\mathrm{b}}$ (see [GGK]) while this seems to be an open problem for $u_0 \in L^\infty$.

3.2. Nonexistence in subcritical L^q spaces for $p < 2$

Theorem 3.2. *Assume $a > 0$ and $p < 2$.*

(i) Let $1 = q < q_c$ and $N \geq 2$ and set

$$u_0(x) = |x|^{-N+\delta}\ \mathbf{1}_{\{|x|<1\}}$$

with $\delta > 0$ sufficiently small (note that $u_0 \in L^1$). Then (1.1) does not admit any local pointwise mild solution, such that $u(t) \in L^1$ on a set of positive measure of t.

(ii) Let $1 < q < q_c$ and set

$$u_0(x) = |x|^{-(N/q)+\delta}\ \mathbf{1}_{\{|x|<1\}} \tag{3.1}$$

with $\delta > 0$ sufficiently small (note that $u_0 \in L^q$). Assume in addition that $N > pq$. Then (1.1) does not admit any local mild L^q solution.

Also, for initial data in Sobolev spaces, we have the following nonexistence results. Although we had to place some additional restrictions on the solution in Proposition 3.3, these results indicate that the existence part of Proposition wo is in some sense sharp (cf. property (2.8) in Proposition 2.4).

Proposition 3.3. *Assume $a > 0$ and $1 \le q < N(p-1)$. Let*

$$u_0(x) = |x|^{1-(N/q)+\delta}\ (2 - |x|)_+$$

with $\delta > 0$ sufficiently small (note that $u_0 \in W^{1,q}$). If $p > \sqrt{2}$, assume in addition that $N > (p+1)q$. Then (1.1) does not admit any pointwise mild solution satisfying (2.8).

Proposition 3.4. *Assume $a = 1$, $p \ge 2$ and $1 \le q < N$ $(= N(p-1)$ when $p = 2)$. Let*

$$u_0(x) = -N(\log |x|)\ \mathbf{1}_{\{|x|<1\}}$$

(note that $u_0 \in W^{1,q}$). Then for any $T > 0$, there exists no solution of (VHJ), classical on $\mathbb{R}^N \times (0,T)$, such that $\lim_{t\to 0} u(t) = u_0$ in $\mathrm{L}^1_{\mathrm{loc}}$.

Remarks 3.2. (a) The nonexistence result of Theorem 3.2 (ii) remains true for pointwise mild solutions satisfying the additional condition

$$|\nabla u|^p \in L^1(0,T;L^r), \quad q - \varepsilon < r < q, \qquad \text{for some } \varepsilon > 0. \tag{3.2}$$

We note that in the critical case $q = q_c$, the pointwise mild solution constructed in Theorem 2.1 does satisfy (3.2) (see (2.6)). The same remark holds for Proposition 3.3 in the critical case $q = N(p-1)$, with q^* instead of q in formula (3.2).

(b) The restrictions $N > pq$ (resp. $N > (p+1)q$) in Theorem 3.2 (resp. in Proposition 3.3) seem technical. Note that they are automatically satisfied when $q < q_c$ (resp. $q < N(p-1)$) if $p \le \sqrt{2}$.

In of the proofs of Theorem 3.2 and Proposition 3.3, we shall need the following two lemmata.

Lemma 3.1. *Let $k \in \mathbb{R}$, $\varepsilon > 0$, let $u_0(x) = |x|^{-k+\delta}\ \mathbf{1}_{\{|x|<1\}}$ and define $U(t) := e^{t\Delta}u_0$. Then for $\delta > 0$ sufficiently small, it holds*

$$\int_{\{|x|<\sqrt{t}\}} U(x,t)\,dx \ge C_\varepsilon\, t^{\frac{N-k}{2}+\varepsilon},$$

for $t>0$ small.

Proof. Let $\alpha = k-\delta$. For $|x|<\sqrt{t}$ and $t>0$ small, we have

$$U(x,t) \geq \int_{\{\sqrt{t}/2<|y|<\sqrt{t}\}} (4\pi t)^{-N/2} e^{-|x-y|^2/4t}|y|^{-\alpha}\,dy \geq Ct^{-\alpha/2}.$$

Therefore

$$\int_{\{|x|<\sqrt{t}\}} U(x,t)\,dx \geq C\,t^{(N-\alpha)/2}, \qquad \text{for } t>0 \text{ small},$$

which implies the Lemma. □

Lemma 3.2. *Let $1\leq p$, $q<\infty$, $pq<N$, $T>0$, and assume that $u:(0,T)\to L^q(\Omega)$ satisfies*

$$|\nabla u|^p \in L^1(0,T;L^q).$$

Then there exists a sequence $t_j \downarrow 0$ such that

$$\int_{\{|x|<\sqrt{t_j}\}} u(x,t_j)\,dx \leq C\,t_j^{\frac{1}{2}-\frac{1}{p}+\frac{N}{2}(1-\frac{1}{pq})}.$$

Proof. ¿From the assumption, there exists $t_j\downarrow 0$ such that

$$\|\nabla u(t_j)\|_{pq}^p = \big\||\nabla u(t_j)|^p\big\|_q \leq t_j^{-1}.$$

Therefore, by Sobolev's inequality, we have

$$\|u(t_j)\|_{(pq)^*} \leq C\|\nabla u(t_j)\|_{pq} \leq Ct_j^{-1/p},$$

with $(pq)^* = Npq/(N-pq)$. By Hölder's inequality, we deduce that

$$\int_{\{|x|<\sqrt{t_j}\}} u(x,t_j)\,dx \leq C\,t_j^{\frac{N}{2}(1-\frac{1}{(pq)^*})} 0\|u(t_j)\|_{(pq)^*} \leq C\,t_j^{\frac{1}{2}-\frac{1}{p}+\frac{N}{2}(1-\frac{1}{pq})}.$$

□

Proof of Theorem 3.2. Assume that (1.1) admits a local solution. First note that in case (i), there exists $T>0$ such that $\operatorname{ess\,lim\,inf}_{t\to T^-} <\infty$ for some $T>0$. It follows from Proposition 1.1 that $|\nabla u|^p \in L^1(0,T;L^1)$ and that u is a mild L^1 solution. We are thus reduced to proving the result in case (ii) (actually for $1\leq q<q_c$).

¿From Lemma 3.1, we see that, for $t>0$ small,

$$\int_{\{|x|<\sqrt{t}\}} u(x,t)\,dx \geq \int_{\{|x|<\sqrt{t}\}} e^{t\Delta}u_0(x)\,dx \geq C_\varepsilon\,t^{\frac{N}{2}(1-\frac{1}{q})+\varepsilon}. \tag{3.3}$$

On the other hand, by Lemma 3.2, there exists a sequence $t_j\downarrow 0$ such that

$$\int_{\{|x|<\sqrt{t_j}\}} u(x,t_j)\,dx \leq C_\varepsilon\,t_j^{\frac{1}{2}-\frac{1}{p}+\frac{N}{2}(1-\frac{1}{pq})-\varepsilon}. \tag{3.4}$$

By comparing (3.3) and (3.4) and letting $j \to \infty$ and then $\varepsilon \to 0$, we deduce that

$$\frac{N}{2}\Big(1-\frac{1}{q}\Big) \geq \frac{1}{2}-\frac{1}{p}+\frac{N}{2}\Big(1-\frac{1}{pq}\Big)$$

hence

$$\frac{N(p-1)}{q} \leq 2-p.$$

The conclusion follows. $\square$

Proof of Proposition 3.3. Let $\widetilde{u_0}(x) = |x|^{1-(k/N)+\delta}\ \mathbf{1}_{\{|x|<1\}}$. ¿From Lemma 3.1, we see that, for $t > 0$ small,

$$\begin{aligned} \int_{\{|x|<\sqrt{t}\}} u(x,t)\,dx &\geq \int_{\{|x|<\sqrt{t}\}} e^{t\Delta}u_0(x)\,dx \\ &\geq \int_{\{|x|<\sqrt{t}\}} e^{t\Delta}\widetilde{u_0}(x)\,dx \geq C_\varepsilon\, t^{\frac{N}{2}(1-\frac{1}{q})+\frac{1}{2}+\varepsilon}. \end{aligned} \tag{3.5}$$

On the other hand, the current assumptions imply $q < N$ and $pq^* = Nqp/(N-q) < N$ (this follows from $q < N(p-1)$ if $p \leq \sqrt{2}$). Assume (2.8), that is,

$$|\nabla u|^p \in L^1(0,T;L^{q^*}).$$

By Lemma 3.2, there exists a sequence $t_j \downarrow 0$ such that

$$\int_{\{|x|<\sqrt{t_j}\}} u(x,t_j)\,dx \leq C_\varepsilon\, t_j^{\frac{1}{2}-\frac{1}{p}+\frac{N}{2}(1-\frac{1}{pq^*})-\varepsilon}. \tag{3.6}$$

By comparing (3.5) and (3.6) and letting $j \to \infty$ and then $\varepsilon \to 0$, we deduce that

$$\frac{1}{2}+\frac{N}{2}\Big(1-\frac{1}{q}\Big) \geq \frac{1}{2}-\frac{1}{p}+\frac{N}{2}\Big(1-\frac{1}{pq^*}\Big)$$

hence $q \geq N(p-1)$. The conclusion follows. $\square$

Proof of Theorem 3.4. This is a direct consequence of Proposition 3.1 and the fact that for all $|x| < 1$, $e^{u_0(x)} = \frac{1}{|x|^N} \notin L^1(B_1(0))$. $\square$

3.3. Nonuniqueness in subcritical L^q spaces for $p < 2$

Theorem 3.5. *Let $a > 0$, $N \geq 1$. Assume $\frac{N+2}{N+1} < p < 2$, so that $q_c > 1$. There exists a positive self-similar solution u of (3.7) on $\mathbb{R}^N \times (0,\infty)$, of the form*

$$u(x,t) = t^{-k}U(|x|t^{-1/2}), \quad k = \frac{2-p}{2(p-1)},$$

where $U \in C_b^2$, with the following properties:

$$u \in C((0,\infty); W^{1,q}), \quad 1 \le q \le \infty,$$

$$\lim_{t\to 0} u(t) = 0 \quad \text{in } L^q \text{ for } 1 \le q < q_c.$$

In particular, the initial value problem for (3.7) in L^q, $1 \le q < q_c$, with initial data 0 has at least two solutions, the 0 solution and u.

This theorem will be proved in the subsection 3.5, together with results valid for different nonlinear terms. In particular, it will be proved that the profile U an its derivative U' both have exponential deccay (see Proposition 3.14). We have the following consequence concerning nonuniqueness in $W^{1,q}$.

Corollary 3.6. *Let $a > 0$, $N > 1$. Assume $\frac{N+1}{N} < p < 2$, so that $N(p-1) > 1$. Then the initial value problem for (3.7) in $W^{1,q}$, $1 \le q < N(p-1)$, with initial data 0 has at least two solutions in $C([0,\infty); W^{1,q}) \cap C((0,\infty); W^{1,pq})$, the 0 solution and the solution u given by Theorem 3.5.*

Remark 3.3. One easily checks that u is a mild L^q solution of (VHJ) for $q < q_c$. In particular we have $|\nabla u|^p \in L^1(0,T; L^q)$ for all $T > 0$.

Remark 3.4. For $u_0 \ge 0$ with, say, $u_0 \in L^1 \cap C_b^2$, it is easy to see that $\|u(t)\|_1$ is a nonincreasing (resp. nondecreasing) function if $a < 0$ (resp. $a > 0$). Letting $I_\infty = \lim_{t\to\infty} \|u(t)\|_1$, it was proved in [BK] (see also [AB, BL1, BGK, BLSS]) that when $a < 0$,

(i) $I_\infty = 0$ if $1 \le p \le p_0 = (N+2)/(N+1)$;

(ii) $I_\infty > 0$ if $p > p_0$.

In the case $a > 0$, the question whether I_∞ is finite or not seems to be open. For the self-similar solution constructed in Theorem 3.5 for $p_0 < p < 2$, one has $I_\infty = \infty$.

3.4. Nonuniqueness for other equations

Let us consider the following equation:

$$u_t - \Delta u = F(u, |\nabla u|), \quad x \in \mathbb{R}^N, \quad t > 0. \tag{3.7}$$

Theorem 3.7. *Let $N \ge 1$, $\frac{N+2}{N+1} < p < 2$. Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, locally Lipschitz continuous, satisfy the homogeneity condition*

$$F(\lambda^{2-p}x, \lambda y) = \lambda^p F(x,y), \quad \text{for all } \lambda,\ x,\ y \ge 0. \tag{3.8}$$

Assume that there exists $a > 0$ such that either

$$(i) \qquad N = 1 \quad \text{and} \quad F(x,y) \ge a|y|^p$$

or

$$(ii) \qquad N = 1 \quad \text{and} \quad F(x,y) \ge a|y|^r |x|^{\frac{p-r}{2-p}} \text{ for some } 1 \le r \le 2(p-1)$$

or

$$(iii) \qquad N \geq 2, \quad p < \frac{N}{N-1}, \qquad \text{and} \qquad F(x,y) \geq a(|y|^p + |x|^{\frac{p}{2-p}}).$$

Then there exists a positive self-similar solution u of (3.7) on $\mathbb{R}^N \times (0,\infty)$, of the form

$$u(x,t) = t^{-k}U(|x|t^{-1/2}), \quad k = \frac{2-p}{2(p-1)},$$

where $U \in C_b^2(\mathbb{R})$, with the following properties:

$$u \in C((0,\infty); W^{1,q}), \quad 1 \leq q < \infty,$$

$$\lim_{t\to 0} u(t) = 0 \quad \text{in } L^q \text{ for } 1 \leq q < q_c.$$

In particular, the initial value problem for (VHJ) in L^q, $1 \leq q < q_c$, with initial data 0 has at least two solutions, the 0 solution and u.

Corollary 3.8. *Assume that either*

$$(i) \qquad F(u,|\nabla u|) = a|u|^m + b|\nabla u|^{\frac{2m}{m+1}} \qquad \text{with } N \geq 1,\ \tfrac{N+2}{N} < m < \tfrac{N}{(N-2)_+},\ a,\ b > 0$$

or

$$(ii) \qquad F(u,|\nabla u|) = a|u|^m|\nabla u|^r \qquad \text{with } N = 1,\ 1 \leq r < 2,\ m \geq 1,\ mr > 1,\ a > 0.$$

Then there exists a positive self-similar solution u of (3.7) as described in Theorem 3.7 (with $k = \frac{1}{m-1}$ in case (i) and $k = \frac{2-r}{2(r+m-1)}$ in case (ii)). In particular, the initial value problem for (VHJ) in L^q, $1 \leq q < q_c$, with initial data 0 has at least two solutions, the 0 solution and u.

Remark 3.5. A similar result was obtained in [T] for $F \equiv |u|^m + b|\nabla u|^{\frac{2m}{m+1}}$ under different assumptions on b, m. We point out that the result of [T] does not apply to the equation (VHJ). On the other hand, the result of Theorem 3.7 applies e.g. to sums of nonlinearities like those in (i) or (ii) of Corollary 3.8.

3.5. Proof of nonuniqueness results: construction of forward selfsimilar solutions

Since Eqn. (3.7) involves only the values of the function $F(x,y)$ for $y \geq 0$, we may assume that F is even with respect to y (i.e., consider the function $F(x,|y|)$ instead of F). Looking for a radial self-similar solution $u(x,t) = t^{-k}U(|x|t^{-1/2})$ of (3.7), we are then reduced to the following equation for the profile U:

$$(3.9) \qquad \begin{cases} U'' + \left(\frac{N-1}{r} + \frac{r}{2}\right)U' + kU + F(U,U') = 0, & r > 0 \\ U'(0) = 0, \quad U(0) = \alpha > 0. \end{cases}$$

The basic idea, in the spirit of [HW, PTW, T] is to use a suitable shooting argument to find $\alpha > 0$ such that the solution of (3.9) is positive, defined for all $r > 0$, and has sufficiently nice decay properties as $r \to \infty$ to guarantee the belonging of $u(.,t)$ to all L^q spaces. However, due to the different nature of the nonlinearity, many of the arguments in [HW, PTW, T] do not apply and some new ideas are required (in particular for proving nonemptyness of I_- below).

Throughout Section 3.5, we assume that $k > 0$, $N \geq 1$ (not necessarily an integer) and that $F : \mathbb{R}^2 \to \mathbb{R}$ is locally Lipschitz continuous. Eqn. (3.9) is equivalent to the integral equation

$$U'(r)r^{N-1}e^{r^2/4} = -\int_0^r \big(kU(s) + F(U(s), U'(s))\big)s^{N-1}e^{s^2/4}\,ds, \qquad U(0) = \alpha. \tag{3.10}$$

It is well known that for all $\alpha > 0$, there exists a unique maximal solution $U = U(\alpha; .)$ of (3.9), defined on a maximal interval $[0, R_{\max}(\alpha))$, with $0 < R_{\max}(\alpha) \leq \infty$, and that $U \in C^2([0, R_{\max}))$. (In what follows, except when necessary, we will not emphasize the dependence of U and $R_{\max}$ upon α.)

3.5.1. Preliminary properties

Proposition 3.9. *Assume that $xF(x,0) \geq 0$ for all $x \in \mathbb{R}$. Then we have the following properties.*

(i) For all $\alpha > 0$, we have $U > 0$ and $U' < 0$ for $r > 0$ small. If $R < R_{\max}$ and $U > 0$ on $[0, R)$, then $U' < 0$ on $(0, R]$.

(ii) For all $\varepsilon > 0$, there exists $\alpha_\varepsilon > 0$ such that $R_{\max} = \infty$ for all $\alpha \in (0, \alpha_\varepsilon)$ and

$$|U(r)| + |U'(r)| < \varepsilon \quad \text{on } [0, \infty).$$

(iii) If $R_{\max} = \infty$ and $U > 0$ on $[0, \infty)$, then

$$\lim_{r\to\infty} U(r) = \lim_{r\to\infty} U'(r) = 0.$$

(iv) Assume

$$F(x, y) \leq C(x)(1 + y^2),$$

with C bounded on bounded sets. If $U > 0$ on $[0, R_{\max})$, then $R_{\max} = \infty$.

Proof. (i) The first part is clear since $U''(0) = -\frac{1}{N}(k\alpha + F(\alpha, 0)) < 0$. Next assume that $U > 0$ and $U' < 0$ on $(0, R)$ and $U'(R) = 0$. Then $U(R) > 0$ by local uniqueness and $U''(R) = -kU(R) - F(U(R), 0) < 0$: a contradiction.

(ii) Let $g(x) = F(x, 0)$ and $G(x) = \int_0^x g(s)\,ds \geq 0$. Define $h(x,y) = \frac{F(x,y)-F(x,0)}{y}$ for $y \neq 0$ (0 for $y = 0$), so that we may write $F(U, U') = g(U) + U'h(U, U')$. Since F is locally Lipschitz, h is bounded on bounded sets. Let $M = \sup_{|x|,|y|\leq 1} |h(x,y)| < \infty$. Defining

$$E_U(r) = \frac{U'^2}{2} + \frac{kU^2}{2} + G(U),$$

we have

$$E'_U(r) = -\big((\tfrac{N-1}{r} + \tfrac{r}{2}) + h(U, U')\big)U'^2.$$

Fix $\varepsilon \in (0,1)$. By continuous dependence, since $G(0)=0$, there exists $\alpha_\varepsilon > 0$ such that for all $\alpha \in (0,\alpha_\varepsilon)$, $R_{\max} > 2M$ and $E_U < \beta \equiv \frac{\varepsilon^2}{4}\min(1,k)$ on $[0,2M]$. On the other hand, for all $r \in [2M, R_{\max})$, if $E_U(r) < \beta$, then in particular $|U(r)| < 1$ and $|U'(r)| < 1$, hence $E_U'(r) \le (|h(U,U')| - M)U'^2 \le 0$. It follows that $E_U(r) < \beta$ for all $r \in [0, R_{\max})$, so that $R_{\max} = \infty$ and $|U| + |U'| < \varepsilon$ on $[0,\infty)$.

(iii) By (i), we know that $U' < 0$ on $(0,\infty]$, hence $\exists \ell \in [0,\infty)$ such that $\lim_{r\to\infty} U(r) = \ell$. Keeping the notation of (ii), for all $r > 0$, if $|U'(r)| < 1$, then $E_U(r) < K \equiv \frac{k\alpha^2+1}{2} + G(\alpha)$. Let $K' = \max(\sqrt{2K}, \sqrt{2K/k})$ and $M' = \sup_{|x|,|y|\le K'} |h(x,y)|$. Then, for all $r \ge 2M'$, $E_U(r) < K$ implies $|U|, |U'| \le K'$ hence $E_U'(r) \le 0$. But since U has a finite limit at ∞, there must exist some $r_1 \ge 2M'$ such that $|U'(r_1)| < 1$. It follows that $E_U(r) < K$ for all $r \ge r_1$. Therefore (U,U') is bounded on $[0,\infty)$.

Returning to Eqn. (3.9), we infer that

$$|U'(r)|e^{r^2/4} = |U'(1)|e^{1/4} + \int_1^r \bigl(kU + \tfrac{N-1}{s}U' + F(U,U')\bigr)e^{s^2/4}\,ds \tag{3.11}$$
$$\le C\bigl(1 + \int_1^r e^{s^2/4}\,ds\bigr) \sim \frac{C'}{r}e^{r^2/4} \quad \text{as } r \to \infty,$$

hence $\lim_{r\to\infty} U'(r) = 0$.

Finally, to prove that $\ell = 0$, we note that by (3.11), $\ell > 0$ would imply $|U'(r)| \sim (k\ell + F(\ell,0))Cr^{-1}$ as $r \to \infty$ (where $F(\ell,0) \ge 0$ and $C > 0$), contradicting the boundedness of U.

(iv) By (i), we have $U' < 0$ hence $U \le \alpha$ on $(0, R_{\max})$. Assume $R_{\max} < \infty$. Using Eqn. (3.9) and the assumption on F, we obtain

$$|U'|' = -U'' \le C_1(1 + U'^2) \le C_2(1 + |U'|)^2, \quad \tfrac{R_{\max}}{2} < r < R_{\max}. \tag{3.12}$$

Since U is bounded, $U' < 0$ and $R_{\max} < \infty$, necessarily $\lim_{r\to R_{\max}} U'(r) = -\infty$. Integrating (3.12) between r and $R_{\max}$, it follows that $1 + |U'(r)| \ge C_3(R_{\max} - r)^{-1}$ as $r \to R_{\max}$, hence $U(r) \to -\infty$ as $r \to R_{\max}$: a contradiction. □

Let us now define the set

$$I_+ = \bigl\{\alpha > 0;\ R_{\max} = \infty,\ \ U > 0 \text{ and } U' < 0 \text{ on } (0,\infty)\bigr\}.$$

3.5.2. Nonemptyness of I_+

Proposition 3.10. *Assume that $0 < k < N/2$, $xF(x,0) \ge 0$ for all $x \in \mathbb{R}$, and*

$$F(x,y) = o(|x| + |y|) \quad \text{as } (x,y) \to (0,0). \tag{3.13}$$

Then there exists $\alpha_1 > 0$ such that $(0,\alpha_1) \subset I_+$.

Proof. By Proposition 3.9 (ii), we know that $R_{\max} = \infty$ for α sufficiently small. Let $r_0 > 0$ be such that $U > 0$ and $U' < 0$ on $(0, r_0)$ (see Proposition 3.9 (i)). Since $k < N/2$, we may fix γ such that $\frac{k}{N} < \gamma < \frac{1}{2}$ and define

$$z(r) = -\frac{U'}{U} > 0 \quad \text{and} \quad \phi(r) = (z(r) - \gamma r)r^{N-1}, \quad 0 < r < r_0.$$

We compute

$$z'(r) = \frac{U'^2}{U^2} - \frac{U''}{U} = z^2 + \frac{1}{U}\big[(\tfrac{N-1}{r} + \tfrac{r}{2})U' + kU + F(U,U')\big]$$
$$= z^2 - (\tfrac{N-1}{r} + \tfrac{r}{2})z + k + \tfrac{F(U,U')}{U}$$

and

$$\phi'(r) = r^{N-1}(z' + \tfrac{N-1}{r}z) - N\gamma r^{N-1}$$
$$= r^{N-1}\big(z^2 - \tfrac{r}{2}z + k - N\gamma + \tfrac{F(U,U')}{U}\big).$$

Let $\delta > 0$ to be chosen later. By Proposition 3.9 (ii) and assumption (3.13), for $\alpha < \alpha_0(\delta)$ sufficiently small, we have $|F(U,U')| \leq \delta(U + |U'|)$ on $[0,\infty)$, hence

$$\phi'(r) \leq z\phi + r^{N-1}\big[k - N\gamma + \delta + ((\gamma - \tfrac{1}{2})r + \delta)z\big].$$

By imposing $0 < \delta < N\gamma - k$, it follows in particular that

$$\phi'(r) \leq z\phi \quad \text{for all } \tfrac{\delta}{\frac{1}{2}-\gamma} \leq r < r_0. \tag{3.14}$$

On the other hand, we have, for all $r \in (0, r_0)$,

$$\phi'(r) \leq (z+\delta)\phi + r^{N-1}\big[k - N\gamma + \delta + (\gamma - \tfrac{1}{2})rz + \delta\gamma r\big]$$

hence

$$\phi'(r) \leq (z+\delta)\phi \quad \text{for all } r < r_0 \text{ such that } r \leq \tfrac{N\gamma-\delta-k}{\delta\gamma}. \tag{3.15}$$

Now, if we choose δ so small that $\frac{N\gamma-\delta-k}{\delta\gamma} > \frac{\delta}{\frac{1}{2}-\gamma}$, we deduce from (3.14) and (3.15) that

$$\forall r \in (0, r_0), \quad \phi(r) < 0 \Rightarrow \phi'(r) < 0. \tag{3.16}$$

Moreover, we observe that

$$\lim_{r\to 0} r^{-N}\phi(r) = -\frac{U''(0)}{\alpha} - \gamma = \frac{k}{N} - \gamma + \frac{F(\alpha,0)}{N\alpha}.$$

Since $F(\alpha,0) = o(\alpha)$ as $\alpha \to 0$, by taking α smaller if necessary, it follows that $\phi(r) < 0$ for $r > 0$ small, and (3.16) then implies that $\phi < 0$ on $(0, r_0)$, that is $-\frac{U'}{U} < \gamma r$. Upon integration, this yields

$$U(r) \geq \alpha e^{-\gamma r^2/2} \quad \text{on } [0, r_0). \tag{3.17}$$

By continuity, one immediately deduces that U can never vanish, and that (3.17) actually holds on $(0,\infty)$. The Proposition follows. □

Remark 3.6. The previous proof shows that $U(r) \geq \alpha e^{-kr^2/2N}$ on $[0,\infty)$ for α sufficiently small.

3.5.3. Boundedness of I_+

Proposition 3.11. *Assume that $N \geq 1$, $k > 0$, and that $F(x,y) = F(y)$ is of class C^1 and satisfies*

$$F(y) \geq a|y|^p$$

for some $a > 0$, $p > 1$. Then for all α sufficiently large, if $R_{\max}(\alpha) = \infty$, there exists $r > 0$ such that $U(r) = 0$.

To prove Proposition 3.11, we will need the following two lemmas.

Lemma 3.3. *Assume that $N > 1$, $k > 0$, and that $F(x,y) = F(y)$ is of class C^1 and satisfies $F(0) \geq 0$. If $R_{\max}(\alpha) = \infty$, then $U'' < 0$ on $[0, r_0)$, where $r_0 = \big(\frac{N-1}{k+1/2}\big)^{1/2}$.*

Proof. We have $NU''(0) = -k\alpha - F(\alpha, 0)$, so that $U'' < 0$ and $U' < 0$ for $r > 0$ small. Differentiating Eqn. (3.9) yields

$$-U''' = \left(-\tfrac{N-1}{r^2} + \tfrac{1}{2} + k\right)U' + \left(\tfrac{N-1}{r} + \tfrac{r}{2}\right)U'' + F'(U')U''. \tag{3.18}$$

Assume that there is a first $r > 0$ such that $U''(r) = 0$. Then $U'(r) < 0$ and $U'''(r) \geq 0$, and (3.18) thus implies

$$\left(k + \frac{1}{2} - \tfrac{N-1}{r^2}\right)U'(r) \leq 0$$

hence, $k + \frac{1}{2} - \frac{N-1}{r^2} \geq 0$, that is $r \geq r_0$. □

Lemma 3.4. *Under the hypotheses of Proposition 3.11, assume that $R_{\max}(\alpha) = \infty$ and $U > 0$ on $[0, \infty)$. Then we have*

$$|U'(r_2)| \geq C_0\alpha \quad \textit{for some } r_2 \in [r_1, 1],$$

where $r_1 = \min(1, r_0)$ if $N > 1$, $r_1 = 1$ if $N = 1$, and $C_0 = C_0(N, k, a) > 0$.

Proof. By Proposition 3.9, we know that $U' < 0$ on $(0, \infty)$. We consider two cases.

• If $U \geq \alpha/2$ on $[0, 1]$, it follows from Eqn. (3.10) that

$$-U' e^{r^2/4} r^{N-1} = \int_0^r (kU + F(U')) e^{s^2/4} s^{N-1}\, ds$$

hence $|U'(1)| \geq e^{-1/4}\frac{k\alpha}{2N}$ and we may take $r_2 = 1$.

• If $U(r) < \alpha/2$ for some $r \in [0, 1]$, then by the Mean Value Theorem, since $U(0) = \alpha$, there exists $r' \in [0, 1]$ such that $|U'(r')] \geq \alpha/2$.

– If $N = 1$, since $e^{r^2/4}|U'(r)|$ is nondecreasing, then $|U'(1)| \geq e^{-1/4}|U'(r_1)| \geq C\alpha$, and we take $r_2 = 1$;

– If $N > 1$ and $r' \geq r_1$, we may take $r_2 = r'$;

– If $N > 1$ and $r' < r_1 \leq r_0$, then by Lemma 3.3, we have $|U'| \geq \alpha/2$ on $[r', r_1]$, and we may take $r_2 = r_1$. □

Proof of Proposition 3.11. Fix $\alpha > 0$ and assume that $U > 0$ on $(0, \infty)$ (hence $U' < 0$). By Lemma 3.4, we have $|U'(r_2)| > C_0\alpha$. From Eqn. (3.9), we have

$$-U'' \geq a|U'|^p - \left(\tfrac{N-1}{r} + \tfrac{r}{2}\right)|U'| \quad \text{on } (0, \infty)$$

hence

$$\begin{cases} |U'|' \geq a|U'|^p - \left(\frac{N-1}{r_1} + 1\right)|U'| & \text{on } [r_1, 2] \\ |U'(r_2)| \geq C_0\alpha. \end{cases}$$

But since $r_1 \leq 1$, this would imply that $|U'|$ blows up before $r = 2$ if α is sufficiently large, which is impossible. The conclusion follows. □

Proposition 3.12. *Assume $N = 1$, $k > 0$ and*

$$F(x,y) \geq a|x|^m|y|^p \tag{3.19}$$

for some $a > 0$, $m \geq 0$, $p \geq 1$ such that $m + p > 1$. Then, for all α sufficiently large, if $R_{\max}(\alpha) = \infty$, there exists $r > 0$ such that $U(r) = 0$.

Proof. Throughout the proof, C denotes various positive constants depending only on m, p, a, k (and not on α). Fix $\alpha > 0$ and assume that $U > 0$ on $(0, \infty)$ (hence $U' < 0$ by Proposition 3.9 (i)). From Eqn. (3.10) and assumption (3.19), we have

$$|U'(r)|e^{r^2/4} \geq \int_0^r (kU + aU^m|U'|^p)e^{s^2/4}\,ds. \tag{3.20}$$

We claim that there exists $R_0 = R_0(k) \geq 2$ such that

$$U(R_0) < \frac{\alpha}{2}. \tag{3.21}$$

Indeed, if $U(R) \geq \alpha/2$ for some $R \geq 2$, then

$$|U'(r)| \geq \frac{k\alpha}{2}e^{-r^2/4}\int_0^r e^{s^2/4}\,ds \geq \frac{C(k)\alpha}{r}, \quad 2 \leq r \leq R.$$

Therefore, $\alpha > \alpha - U(R) \geq \int_1^R |U'| \geq C(k)\alpha \log R$, so that $\exists R_0(k) \geq 2$ such that $R \leq R_0(k)$. Since $U' < 0$, we thus have either $U(2) < \alpha/2$ or $U(R_0) < \alpha/2$ and the claim follows.

Writing $U^m|U'|^p = C|(U^{1+(m/p)})'|^p$, we deduce from Eqn. (3.20), Hölder's inequality and (3.21) that, for all $r \geq R_0$,

$$\begin{aligned} |U'(r)|e^{r^2/4} &\geq C\int_0^{R_0} |(U^{1+(m/p)})'|^p\,ds \\ &\geq C\Big(\int_0^{R_0} |(U^{1+(m/p)})'|\,ds\Big)^p \\ &\geq C\Big(\alpha^{1+(m/p)} - (\alpha/2)^{1+(m/p)}\Big)^p, \end{aligned} \tag{3.22}$$

Thus we have, for all $r \in [R_0, R_0 + 1]$,

$$|U'(r)| \geq e^{-(R_0+1)^2}C\alpha^{m+p},$$

hence

$$\alpha > U(R_0) - U(R_0 + 1) = \int_{R_0}^{R_0+1} |U'| \geq C\alpha^{m+p}.$$

Since $m + p > 1$, this cannot hold if α is sufficiently large. The Proposition follows. □

Remark 3.7. Similar results can be derived for the multidimensional case of Eqn. (3.9) when $p > N$, by using Sobolev type inequalities instead of (3.22). However for $N \geq 2$, this implies $p > 2$ and therefore does not enable to construct self-similar solutions. We do not know whether Proposition 3.12 holds for (some) $1 < p < 2$ when $N > 1$.

Proposition 3.13. *Assume $N > 1$, $k > 0$ and*

$$C_1(|x|^m + |y|^p) \le F(x,y) \le C_2(|x|^m + |y|^p),$$

where $1 < p < \frac{N}{N-1}$, $1 \le m < \frac{N}{(N-2)_+}$, C_1, $C_2 > 0$. Then, for all α sufficiently large, if $R_{\max}(\alpha) = \infty$, there exists $r > 0$ such that $u(r) = 0$.

We will need the following Lemma.

Lemma 3.5. *For all large α, we have*

$$U \ge \frac{\alpha}{2} \qquad \text{on } [0, r_\alpha)$$

with $r_\alpha = C\min(\alpha^{(1-m)/2}, \alpha^{m(1-p)/p})$ and $C > 0$ depends only on p, m, k, N, C_1, C_2.

Proof. Throughout the proof, C denotes various positive constants depending only on p, m, k, N, C_1, C_2. Let $R \in (0,1]$ be such that $U > 0$ on $(0,R)$ (hence $U' < 0$ by Proposition 3.9 (i)). Define $g(r) = \sup_{[0,r]} |U'|$. By Eqn. (3.10), for all $r \in [0,R]$, we have

$$\begin{aligned} |U'(r)|r^{N-1} &\le e^{1/4} \int_0^r (k\alpha + C_2(\alpha^m + |U'|^p))s^{N-1}\,ds \\ &\le Cr^N(\alpha^m + g^p(r)). \end{aligned}$$

It follows that

$$g(r) \le Cr(\alpha^m + g^p(r)).$$

Since $g(0) = 0$ and g is continuous, if there is some (minimal) $r_0 \in (0,R]$, such that $g^p(r_0) = \alpha^m$, we will have $g(r_0) \le Cr_0 g^p(r)$ hence, $r_0 \ge Cg^{1-p}(r_0) = C\alpha^{m(1-p)/p}$. It follows that for all $r \le \min(R, C\alpha^{m(1-p)/p})$, $g^p(r) \le \alpha^m$ hence, $rg(r) \le Cr^2\alpha^m$. Therefore, putting $r_\alpha = C\min(\alpha^{m(1-p)/p}, \alpha^{(1-m)/2})$, we have $rg(r) \le \alpha/2$ for $r \le \min(R, r_\alpha)$ hence,

$$U(r) = \alpha - \int_0^r |U'|\,ds \ge \alpha - rg(r) \ge \alpha/2, \qquad 0 \le r \le \min(R, r_\alpha). \tag{3.23}$$

Now, take α large enough so that $r_\alpha < 1$. If there were a (minimal) $r \in (0, r_\alpha]$ such that $U(r) = 0$, then one could take $R = r$ in (3.23), reaching a contradiction. Therefore, $U > 0$ on $[0, r_\alpha]$ and (3.23) with $R = r_\alpha$ gives the desired conclusion. □

Proof of Proposition 3.13. We may assume $U > 0$ and $U' < 0$ on $[0,2]$, since otherwise we are done. Note that $r_\alpha < 1$ for α large enough. We first claim that for some $\theta = \theta(m,p,N) > 0$, we have

$$|U'(1)| \ge C'\alpha^\theta$$

where $C' = C'(m,p,k,N,C_1,C_2) > 0$. Indeed by Eqn. (3.10) and Lemma 3.5,

$$\begin{aligned} |U'(r_\alpha)|r_\alpha^{N-1} &\ge e^{-1/4} C_1 \int_0^{r_\alpha} U^m s^{N-1}\,ds \\ &\ge C'\alpha^m r_\alpha^N \ge C'\alpha^\theta, \end{aligned}$$

where $\theta = \min(m - N(m-1)/2, m(1 - N(p-1)/p)) > 0$. Since $|U'|r^{N-1}e^{r^2/4}$ is nondecreasing on $[0,1]$ by Eqn. (3.10), the claim follows.

Now, by (3.9), we get

$$|U'|' = -U'' \geq -(\tfrac{N-1}{r} + \tfrac{r}{2})|U'| + C_1|U'|^p \quad \text{on } (0,2].$$

Therefore, we have

$$\begin{cases} |U'|' \geq C_1|U'|^p - N|U'| & \text{on } (0,2] \\ |U'(1)| \geq C'\alpha^\theta. \end{cases} \tag{3.24}$$

But it is easily seen that (3.24) cannot hold if α is sufficiently large (for $|U'|$ would have to blow up before $r = 2$). The Proposition follows. □

3.5.4. Properties of the limiting trajectory

Proposition 3.14. *Assume that F satisfies*

$$F(x,y) = o(|x| + |y|) \quad \text{as } (x,y) \to (0,0). \tag{3.13}$$

and

$$F(x,y) \leq C(x)(1 + y^2) \tag{3.25}$$

where C is bounded on bounded sets. If $0 < \alpha_0 = \sup I_+ < \infty$, then $U_0 \equiv U(\alpha_0;.)$ satisfies the following properties.

$$R_{\max}(\alpha_0) = \infty, \quad U_0 > 0 \quad \text{and} \quad U_0' > 0 \quad \text{on } (0,\infty), \tag{i}$$

$$\forall \varepsilon > 0,\ \exists R_\varepsilon > 0,\ \forall r \geq R_\varepsilon,\ e^{-(1+\varepsilon)r^2/4} \leq U_0(r) \leq e^{-(1-\varepsilon)r^2/4}, \tag{ii}$$

$$U_0'(r) = -\frac{r}{2}U_0(r)(1 + o(1)), \quad \text{as } r \to \infty. \tag{iii}$$

Lemma 3.6. *Assume $F(0,0) = 0$. Let α, $\lambda > 0$ and define*

$$\overline{R} = \overline{R}(\alpha,\lambda,k,F) = 2\big(\lambda + \tfrac{k}{\lambda} + (1 + \lambda^{-1})L(\alpha,\lambda)\big),$$

where $L(\alpha,\lambda) = \mathrm{Lip}(F_{|[-\alpha,\alpha]\times[-\lambda\alpha,\lambda\alpha]})$. Assume that $R_{\max}(\alpha) > \overline{R}$ and that

$$U(\alpha;r) > 0, \quad U'(\alpha;r) < 0 \quad \text{on } (0,R) \text{ for some } R \in (\overline{R}, R_{\max}(\alpha)).$$

If $U'(r) + \lambda U(r) \geq 0$ for some $r \in (\overline{R}, R)$, then $U' + \lambda U > 0$ on (r,R).

Proof. Under the assumptions of the Lemma, assume that $U'(s) + \lambda U(s) = 0$ for some $s \in [r,R)$. One then has

$$\begin{aligned} U''(s) + \lambda U'(s) &= \big(\lambda - \tfrac{N-1}{s} - \tfrac{s}{2}\big)U'(s) - kU(s) - F(U(s),U'(s)) \\ &= \big(\lambda - \tfrac{N-1}{s} - \tfrac{s}{2} + \tfrac{k}{\lambda}\big)U'(s) - F(U(s),\lambda U(s)). \end{aligned}$$

Observe that $F(U(s),\lambda U(s)) \le (1+\lambda)U(s)L(\alpha,\lambda) \le -(1+\lambda^{-1})U'(s)L(\alpha,\lambda)$ hence,

$$U''(s)+\lambda U'(s) \ge \big(\lambda + \tfrac{k}{\lambda} + (1+\lambda^{-1})L(\alpha,\lambda) - \tfrac{s}{2}\big)U'(s) > 0.$$

It is easily seen that $U'+\lambda U$ must therefore remain > 0 on (r,R). The Lemma is proved. □

Lemma 3.7. *Under the assumptions of Proposition 3.14, we have* $R_{\max}(\alpha_0) = \infty$, $U_0 > 0$ *and* $U_0' < 0$ *on* $(0,\infty)$, *and for all* $\lambda > 0$,

$$U_0'(r) + \lambda U_0(r) < 0 \quad \text{for } r \text{ large enough.} \tag{3.26}$$

Proof. We know from Proposition 3.9 (iv) that $R_{\max}(\alpha_0) = \infty$. Suppose that $U_0(r) = 0$ for some (minimal) $r > 0$. Then $U_0'(r) < 0$ by local uniqueness, hence $U_0 < 0$ on $(r, r+\varepsilon]$ for some $\varepsilon > 0$ small. But this would imply that $U(\alpha; r+\varepsilon) < 0$ for α close to α_0, by continuous dependence, contradicting the definition of α_0. It follows that $U_0 > 0$ for $r > 0$, hence $U_0' < 0$ by Proposition 3.9 (i).

It remains to prove (3.26). Fix $\lambda > 0$ and suppose that $U_0'(r_0) + \lambda U(r_0) \ge 0$ for some $r_0 > \overline{R'} \equiv \overline{R}(\alpha_0+1,\lambda,k,F) \ge \overline{R}(\alpha_0,\lambda,k,F)$ (see Lemma 3.6). Then, by Lemma 3.6, we have $U_0' + \lambda U_0 > 0$ on (r_0,∞). By continuous dependence, there exists $\varepsilon \in (0,1)$ such that for all $\alpha \in (\alpha_0, \alpha_0+\varepsilon)$ we have $R_{\max}(\alpha) > r_0+1$, $U > 0$ and $U' < 0$ on $(0, r_0+1]$, and $U'(r_0+1) + \lambda U(r_0+1) > 0$. But by definition of α_0, there exists $\alpha \in (\alpha_0, \alpha_0+\varepsilon)$ and $r > r_0+1$ such that

$$U(r) = 0 \quad \text{and} \quad U > 0, \quad U' < 0 \;\text{ on } (0,r).$$

But since $U' + \lambda U > 0$ on $[r_0+1, r)$ by Lemma 3.6, we get upon integration $U(r) \ge e^{-\lambda(r-r_0-1)}U(r_0+1) > 0$, which is a contradiction. The Lemma is proved. □

Proof of Proposition 3.14. Property (i) follows from Lemma 3.7, from which we also deduce that

$$\lim_{r\to\infty} \frac{U_0'(r)}{U_0(r)} = -\infty. \tag{3.27}$$

Properties (ii) and (iii) can then be proved along the lines of [BPT, Lemmas 13, 14, 15 and Theorem 2]. (The nonlinearity there corresponds to $F \equiv -|U|^p$, $p > 1$, but once (i) and (3.27) are established, the hypothesis (3.13) alone allows one to carry over the steps of their proof.) □

Proof of Theorems 3.5 and 3.7. Under any of the assumptions (i)–(iii), the homogeneity hypothesis (3.8) implies that

$$F(x,y) \le C(|y|^p + |x|^{p/(2-p)}), \quad x,\, y \ge 0. \tag{3.28}$$

(Indeed, by taking $\lambda = \min(x^{-1/(2-p)}, y^{-1})$ for $(x,y) \ne (0,0)$ in (3.8), we obtain $F(x,y) \le \sup_{0\le a,b\le 1} F(a,b)\,\lambda^{-p} \le C\max(y^p, x^{p/(2-p)})$.) Moreover $F(0,0) = 0$.

Since we are interested in finding a positive solution of (3.7), only the values of the function $F(x,y)$ for $x,\, y \ge 0$ are involved, and we may redefine F in Theorem 3.7 by

$$\widetilde{F}(x,y) = F(|x|,|y|) + (\mathrm{sgn}(x) - 1)F(|x|,0).$$

As $F(0,0)=0$, $\widetilde{F}$ remains Lipschitz continuous, and it satisfies $x\widetilde{F}(x,0)=|x|F(|x|,0)\geq 0$, $\forall x\in\mathbb{R}$. Since (3.28) for $1<p<2$ implies (3.13), and since $0<k=\frac{2-p}{2(p-1)}<N/2$ by hypothesis, the assumptions of Proposition 3.10 are satisfied. Therefore there exists $\alpha_1>0$ such that $(0,\alpha_1)\subset I_+$.

By Propositions 3.11 (in case of Theorem 3.5), 3.12 (in cases (i)(ii) of Theorem 3.7), 3.13 (in case (iii) of Theorem 3.7), we have $\sup I_+<\infty$. The results then follow immediately from Proposition 3.14. □

Remark 3.8. It can be proved that the result of Proposition 3.14 remains valid if the assumption (3.25) is replaced with $F(x,y)\geq a|y|^p-C(x)$ for some $p>1$ and C bounded on bounded sets.

4. Existence and nonexistence results for $a < 0$, $u_0 \geq 0$

4.1. Existence in all L^q_+ spaces for $p < 2$

In this section we prove

Theorem 4.1. *Let $a < 0$, $1 \leq p < 2$ and $1 \leq q < \infty$. Given $u_0 \in L^q$, $u_0 \geq 0$, there exists a (pointwise mild) solution u of (VHJ), $u \geq 0$, such that*

$$u \in C([0,\infty); L^q).$$

Moreover, u is a classical solution of (VHJ) on $\mathbb{R}^N \times (0,\infty)$.

Remark 4.1. For the equation $u_t - \Delta u + |u|^{p-1}u = 0$, it is well-known that a (unique) solution exists for any initial data $u_0 \in L^q$ and any p, $q \geq 1$. This is an easy consequence of the monotonicity of the nonlinear operator $u \mapsto \Delta u - |u|^{p-1}u \equiv Au$ (in the sense that $(Au - Av, u - v) \leq 0$ for smooth u, v). On the contrary, if u_0 is a Dirac mass δ_0, then a solution exists if and only if $p \leq (N+2)/N$ (see [BF]). Thus, in view of Theorem 4.1 and the non-existence result of [BL1] for (VHJ) when $a < 0$, $u_0 = \delta_0$ and $p > (N+1)/(N+1)$ (see also Theorem 4.4 below), we have here a similar situation for positive solutions of (VHJ) with $a < 0$. However, $\Delta u - |\nabla u|^p$ has no monotonicity property and the proof of existence that we will give now is more involved.

Remark 4.2. The basic idea of the proof is classical. One first constructs a sequence of solutions for regularized initial data (Step 1). In order to pass to the limit in the equation for $t > 0$ (Step 2), we next use some estimates from [BL1]. However, a main difficulty is then to recover the correct initial data at $t = 0$ in the limiting process. This requires some careful monotonicity arguments (see Step 3). Note that Steps 1 and 2 would work as well for measure initial data (say, $u_0 = \delta_0$). But then one would "lose" the initial data in the limiting process (cf. Theorem 4.4).

Proof. Step 1. Construction of approximate solutions.

Let $0 \leq u_0^{(k)} \uparrow u_0$ be an increasing sequence of nonnegative functions converging a.e. to u_0, and such that

$$u_0^{(k)} \in L^\infty \quad \text{and} \quad \operatorname{supp}(u_0^{(k)}) \subset\subset \mathbb{R}^N.$$

In view of Theorem 2.1, the integral equation (1.1), with $u_0^{(k)}$ replacing u_0, has a unique (mild L^r) solution $u^{(k)} \geq 0$, such that

$$u^{(k)} \in C([0,\infty); L^r) \cap C((0,\infty); W^{1,r}) \cap C((0,\infty); C^2_{\mathrm{b}}), \quad q_c < r < \infty.$$

In the following claim, we list some of the properties of the sequence $u^{(k)}$.

Claim. *The sequence $u^{(k)}$ satisfies, for some constant $C > 0$ independent of k,*

$$0 \leq u^{(k)}(t) \leq e^{t\Delta}u_0, \quad t \geq 0, \tag{4.1}$$

$$u^{(k)}(x,t) \leq C\|u_0\|_q t^{-N/2q}, \quad x \in \mathbb{R}^N, \quad t > 0, \tag{4.2}$$

(4.3) $$|\nabla u^{(k)}(x,t)| \le Ct^{-(N+2q)/2pq}, \quad x \in \mathbb{R}^N, \quad t > 0,$$

(4.4) $\{u^{(k)}(x,t)\}_{k=1}^{\infty}$ *is monotone nondecreasing for each* $(x,t) \in \mathbb{R}^N \times (0,\infty)$.

Proof of Claim. (4.1) follows immediately from $u_0^{(k)} \le u_0$, so that in view of (4.1), (4.2) is a consequence of the standard estimate for the heat kernel. Since, in addition, $u^{(k)}(t) \in C_{\mathrm{b}}^2$ for $t > 0$, we obtain, by Theorem 1 of [BL1] that, for all $t > \varepsilon > 0$

$$|\nabla u^{(k)}(x,t)| \le C_p(t-\varepsilon)^{-(N+2q)/2pq}, \quad x \in \mathbb{R}^N, \quad t > \varepsilon,$$

where $C_p > 0$ depends on p only. Letting $\varepsilon \to 0$, we get (4.3).

The monotonicity follows from the comparison principle (see e.g. [GGK, Theorem 8]). The claim is proved.

Step 2. Convergence of the approximating sequence to a solution for $t > 0$.

¿From (4.1)(4.2) and the monotone convergence theorem, it follows that $\{u^{(k)}(x,t)\}$ converges monotonically on $\mathbb{R}^N \times [0,\infty)$ to some function $u(x,t)$, and that the convergence of $u^{(k)}(.,t)$ takes place in L^q for each fixed $t \ge 0$. Moreover, from (4.1), u satisfies

$$0 \le u(t) \le e^{t\Delta}u_0, \quad t \ge 0.$$

On the other hand, from (4.3), we see that $u^{(k)}$ satisfies an equation of the form $u_t^{(k)} - \Delta u^{(k)} = g_k(x,t)$ in $\mathbb{R}^N \times (0,\infty)$, where the functions $u^{(k)}$ and g_k are bounded independently of k on every strip $\mathbb{R}^N \times (t_1,t_2)$, $0 < t_1 < t_2 < \infty$. Denote as usual by D, D^2, ... any partial derivation operator in space of order 1, 2, ... It follows from interior parabolic regularity theory (see, e.g., [Lie, chapter 7]) that for every $r \in (1,\infty)$, $R > 0$ and $0 < t_1 < t_2 < \infty$, $\partial_t u^{(k)}$ and $D^2u^{(k)}$ are bounded in $L^r(B_R \times (t_1,t_2))$ independently of k. Therefore, $Du^{(k)}$ satisfies

$$(Du^{(k)})_t - \Delta(Du^{(k)}) = h_k(x,t) \equiv p\sum_i \partial_i Du^{(k)}\partial_i u^{(k)}|\nabla u^{(k)}|^{p-2},$$

where $h_k(x,t)$ is bounded in $L^r(B_R \times (t_1,t_2))$ independently of k. A further application of parabolic regularity yields that $\partial_t Du^{(k)}$ is bounded in $L^r(B_R \times (t_1,t_2))$. Applying standard imbedding theorems for $r > 1$ sufficiently large, we obtain that $u^{(k)}$ and $Du^{(k)}$ are bounded in $C^{\alpha,\alpha/2}(B_R \times (t_1,t_2))$ for some $\alpha > 0$. By Ascoli-Arzela's Theorem and a diagonal procedure, replacing $u^{(k)}$ by a subsequence, it follows that $u^{(k)}$ and $Du^{(k)}$ converge to u and Du respectively, uniformly on compact subsets of $\mathbb{R}^N \times (0,\infty)$ and that u is C^1 in x on $\mathbb{R}^N \times (0,\infty)$.

Now, for $x \in \mathbb{R}^N$ and $t \ge \varepsilon > 0$, we write

$$u^{(k)}(x,t) = e^{(t-\varepsilon)\Delta}u^{(k)}(\varepsilon) - \int_\varepsilon^t e^{(t-s)\Delta}|\nabla u^{(k)}(s)|^p\,ds.$$

Since by (4.3), $|\nabla u^{(k)}(x,s)|^p$ is bounded independently of k on $\mathbb{R}^N \times (\varepsilon,t)$, we may pass to the limit via the dominated convergence theorem to obtain

(4.5) $$u(x,t) = e^{(t-\varepsilon)\Delta}u(\varepsilon) - \int_\varepsilon^t e^{(t-s)\Delta}|\nabla u(s)|^p\,ds, \quad x \in \mathbb{R}^N, \quad t \ge \varepsilon.$$

We claim that

$$u \in C((0,\infty); L^q). \tag{4.6}$$

First, since $u^{(k)} \in C([0,\infty); L^r)$, $r > q_c$, for each k, and since $u^{(k)}$ converges to u uniformly on compact subsets of $\mathbb{R}^N \times (0,\infty)$, we have that $u \in C((0,\infty); L^q(\{|x| \le R\}))$ for all $R > 0$. Next, we observe that for all $0 < t, t+h < \infty$,

$$\begin{aligned}\|u(t+h) - u(t)\|_{L^q(\{|x|>R\})} &\le \|e^{(t+h)\Delta}u_0\|_{L^q(\{|x|>R\})} + \|e^{t\Delta}u_0\|_{L^q(\{|x|>R\})} \\ &\le \|e^{(t+h)\Delta}u_0 - e^{t\Delta}u_0\|_{L^q(\mathbb{R}^N))} + 2\|e^{t\Delta}u_0\|_{L^q(\{|x|>R\})},\end{aligned}$$

and since $u_0 \in L^q)$, the RHS can be made arbitrily small for h small and R large. The claim (4.6) follows.

Step 3. Identification of the initial value.

It remains to identify the initial value of the constructed solution u, or in other words to show that

$$\lim_{t\to 0} u(t) = u_0 \quad \text{in } L^q.$$

Since $\|u(t)\|_q \le \|u_0\|_q$ by (4.1), $\{u(t)\}_{t>0}$ is a bounded, hence weakly precompact subset of L^q if $1 < q < \infty$. If $q = 1$, it is a weak star precompact subset of $\mathcal{M} = \mathcal{M}(\mathbb{R}^N)$, the space of bounded Borel measures. If $q > 1$, for any sequence $t_n \to 0$, there is a subsequence t'_n and a function $v_0 \in L^q$ such that

$$u(t'_n) \rightharpoonup v_0, \quad \text{weakly in } L^q.$$

(If $q = 1$, the convergence in is the weak star sense of $\mathcal{M}$ and $v_0 \in \mathcal{M}$.)

For each $k \ge 1$, from (4.1) and (4.4), we have

$$u^{(k)}(t'_n) \le u(t'_n) \le e^{t'_n\Delta}u_0$$

But on the other hand, $u^{(k)}(t'_n) \to u_0^{(k)}$ and $e^{t'_n\Delta}u_0 \to u_0$ in L^q as $n \to \infty$. It follows that for all $k \ge 1$,

$$u_0^{(k)} \le v_0 \le u_0$$

(the inequality being understood in the sense of measures if $q = 1$). Letting $k \to \infty$, we conclude that

$$v_0 = u_0.$$

Since every sequence $u(t_n)$ with $t_n \to 0$ has a subsequence converging (weakly in L^q or weak star in $\mathcal{M}$) to the same limit u_0, this means that in fact

$$u(t) \rightharpoonup u_0, \quad \text{as } t \to 0. \tag{4.7}$$

As a consequence of (4.1)(4.7), note that

$$0 \le e^{t\Delta}u_0 - u(t) \rightharpoonup 0 \quad \text{weakly in } L^q\ (q > 1) \text{ or weak star in } \mathcal{M}\ (q = 1), \text{ as } t \to 0. \tag{4.8}$$

We now proceed to show that the convergence in (4.7) is actually in the norm sense of L^q.

For each $K \subset\subset \mathbb{R}^N$, fix some continuous function φ with compact support, such that $0 \le \varphi \le 1$ and $\varphi = 1$ on K. Formula (4.8) implies that

$$\|e^{t\Delta}u_0 - u(t)\|_{L^1(K)} \le \int_{\mathbb{R}^N} (e^{t\Delta}u_0 - u(t))\varphi\, dx \to 0, \quad \text{as } t \to 0.$$

In other words, $e^{t\Delta}u_0 - u(t) \to 0$ in $L^1_{\text{loc}}(\mathbb{R}^N)$, hence $u(t) \to u_0$ in $L^1_{\text{loc}}(\mathbb{R}^N)$, as $t \to 0$. By diagonal procedure, it follows that for each sequence $t_n \to 0$, there exists a subsequence t'_n such that $u(t'_n) \to u_0$ a.e. in $\mathbb{R}^N$. But since $0 \le u(t'_n) \le e^{t'_n\Delta}u_0$ and $e^{t'_n\Delta}u_0 \to u_0$ in L^q, the dominated convergence theorem implies that

$$u(t'_n) \to u_0 \quad \text{in } L^q.$$

Since there is no other limit, this means that

$$u(t) \to u_0 \quad \text{in } L^q, \text{ as } t \to 0.$$

Finally, by letting $\varepsilon \to 0$ in (4.5), we obtain, by the monotone convergence theorem, that

$$u(x,t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}|\nabla u(s)|^p\, ds, \quad x \in \mathbb{R}^N, \quad t \ge 0.$$

The proof is complete. □

4.2. Existence-uniqueness in all L^q_+ spaces for $p = 2$

Theorem 4.2. *Let $a < 0$, $p = 2$ and $1 \le q < \infty$. Given $u_0 \in L^q$, $u_0 \ge 0$, there exists a classical solution u of (VHJ), $u \ge 0$, such that*

$$u \in C([0,\infty); L^q), \quad u(0) = u_0. \tag{4.9}$$

Moreover, for all $T > 0$, u is the unique function such that $u \in C^{2,1}(Q_T)$, $u \ge 0$, u satisfies (VHJ)$_1$ in Q_T and $u \in C([0,\infty); L^q)$ with $u(0) = u_0$.

Proof. (i) *Existence.* Assume $a = -1$ without loss of generality and put $v_0 = 1 - e^{-u_0}$. Since $0 \le 1 - e^{-s} \le s$ for $s \ge 0$, it follows that $0 \le v_0 \in L^q$. Next define $v(t) = e^{t\Delta}v_0$. Clearly, $v \in C([0,\infty); L^q) \cap C^\infty(Q)$, where $Q = \mathbb{R}^N \times (0,\infty)$. Since $u_0 \in L^q$, then $v_0 = 1 - e^{-u_0} < 1$ a.e., so that $v < 1$ in Q.

Now define $u = -\log(1 - v) \ge 0$. Since $v(t) = e^{t\Delta}v_0$, it is well-known that $v(t), \partial_i v(t), \partial_{ij}v(t) \in C_0(\mathbb{R}^N)$ for each $t > 0$ $(i,j = 1,\dots,N)$. In particular, it follows that $u(t) \in C_0(\mathbb{R}^N)$ for each $t > 0$. Moreover, since $v = 1 - e^{-u}$, we have $\partial_i u(t) = e^u \partial_i v \in C_0$ and $\partial_{ij}u(t) = e^u(\partial_{ij}v + e^u\partial_i v \partial_j v) \in C_0$, so that in particular

$$u(t) \in C^2_{\mathrm{b}}, \quad t > 0.$$

A straightforward calculation shows that

$$u_t = \Delta u - |\nabla u|^2, \quad (x,t) \in Q.$$

It remains to verify (4.9). Noting that $e^{-u(t)} = e^{t\Delta}e^{-u_0}$ and that $s \mapsto e^{-s}$ is convex, Jensen's inequality entails that $e^{-u(t)} \geq \exp(-e^{t\Delta}u_0)$ hence,

$$0 \leq u(t) \leq e^{t\Delta}u_0, \quad t \geq 0. \tag{4.10}$$

Fix $t_0 \geq 0$. Since $v(t) \to v(t_0)$ in L^q as $t \to t_0$, for each sequence $t_n \to t_0$, there is a subsequence t'_n such that $v(t'_n) \to v(t_0)$ a.e., hence $u(t'_n) \to u(t_0)$ a.e. In view of (4.10), and since $e^{t\Delta}u_0 \to e^{t_0\Delta}u_0$ in L^q as $t \to t_0$, it follows from Lebesgue's dominated convergence theorem that $u(t'_n) \to u(t_0)$ in L^q. This implies (4.9).

(ii) *Uniqueness.* Let u be a solution with the stated properties, and let $v := 1 - e^{-u}$. Then v satisfies

$$v_t = \Delta v, \quad (x,t) \in Q. \tag{4.11}$$

We claim that

$$v \in C([0,\infty); L^q), \quad v(0) = v_0 := 1 - e^{-u_0} \in L^q. \tag{4.12}$$

For $0 \leq t, t+h < T$, we note that

$$v(t+h) - v(t) = e^{-u(t)}\big(1 - e^{-(u(t+h)-u(t))}\big)$$

and

$$v(t+h) - v(t) = -e^{-u(t+h)}\big(1 - e^{-(u(t)-u(t+h))}\big).$$

Since $1 - e^{-s} \leq s$, $s \in \mathbb{R}$, it follows that

$$-e^{-u(t+h)}(u(t) - u(t+h)) \leq v(t+h)) - v(t) \leq e^{-u(t)}(u(t+h) - u(t))$$

Using $u \geq 0$, we get

$$|v(t+h)) - v(t)| \leq |u(t+h) - u(t)|.$$

By (4.9), this proves the claim (4.12). Now, it is well-known that (4.11)(4.12) has a unique solution, namely $v(t) = e^{t\Delta}v_0$. The uniqueness of u follows. □

4.3. Existence in all classes $L^q_{+,\mathrm{approx}}$ for $p > 2$

Theorems 4.1 and 4.2 yield the existence of (at least) a positive solution of (VHJ) for $a < 0$ when $p \leq 2$ and $0 \leq u_0 \in L^q$, $q \geq 1$. Define $L^q_{+,\mathrm{approx}}$ to be the space of those functions $0 \leq u_0 \in L^q$ which can be approximated pointwise by a monotonically nondecreasing sequence of nonnegative continuous functions. For $p > 2$, we then have the following partial extension of Theorem 4.1.

Theorem 4.3. *Let $a < 0$, $p > 2$ and $1 \leq q < \infty$. Given $u_0 \in L^q_{+,\mathrm{approx}}$, there exists a (pointwise mild) solution u of (VHJ), $u \geq 0$, such that*

$$u \in C([0,\infty); L^q).$$

Furthermore, the solution is classical for $t > 0$, satisfying the regularity property

$$u \in C((0,\infty); C^2_{\mathrm{b}}).$$

Proof. Let $u_0^{(k)}$ be a nondecreasing sequence of nonnegative continuous functions which converge pointwise to u_0. By a truncation procedure we can assume each $u_0^{(k)}$ to be compactly suupported. By [GGK, Theorems 2 and 7], since $u_0^{(k)} \in C_{\mathrm{b}}$, there exists a unique classical solution of (VHJ) with initial data $u_0^{(k)}$. The rest of the proof then follows along the lines of the proof of Theorem 4.1. □

Remark 4.3. The space $L^q_{+,\mathrm{approx}}$ contains in particular all the functions $\phi \in L^q$, $\phi \geq 0$, which are radially symmetric and radially nonincreasing (with a possible singularity at 0).

4.4. Nonexistence for p-atomic measures, $p_0 < p < 2$

Let $N \geq 2$, $p_0 = \frac{N+2}{N+1} < p < N$ and $p^* = \frac{Np}{N-p}$. Let $\mu \geq 0$ be a Borel measure on $\mathbb{R}^N$.

Definition 4.1. *We say that μ is p–atomic if there exist constants $C > 0$, $0 < \delta < 1$, such that the following is satisfied: for every $0 < t < 1$ there exist sequences $\{x_k\}_{k=1}^\infty \subset \mathbb{R}^N$, $\{r_k\}_{k=1}^\infty \subset (0,\infty)$, such that*

$$\text{(i)} \qquad \mathrm{supp}(\mu) \subset \bigcup_{k=1}^\infty B(x_k, r_k) \qquad (B(y,r) = \{x; |x-y| < r\}),$$

$$\text{(ii)} \qquad \sum_{k=1}^\infty r_k^{N(1-(1/p^*))} \leq C t^{1/p},$$

$$\text{(iii)} \qquad \sum_{k=1}^\infty \mu(B(x_k, r_k)) e^{-(1-\delta) r_k t^{-1/2}} \to 0, \ \text{as } t \to 0.$$

Note that any atomic measure (i.e., supported on countably many points $x_1, x_2, \ldots$) is p–atomic for $p > p_0$. Indeed one just takes $r_k = 2^{-k} t^{1/(p+N(p-1))}$.

Theorem 4.4. *Let $a < 0$, $N \geq 2$, $p_0 = \frac{N+2}{N+1} < p < N$, and $\mu \geq 0$, $\mu \not\equiv 0$, be a p–atomic measure. Then there is no local pointwise mild nonnegative solution of (VHJ) such that*

$$u(.,t) \to \mu \quad \text{weak star in } \mathcal{M} \text{ as } t \to 0. \tag{4.13}$$

Proof. Assume that there exists such a solution u. First, by the argument of proof of Proposition 1.1, we have

$$|\nabla u|^p \in L^1((0,T);L^1).$$

Therefore, for a given $\varepsilon > 0$, there exists a sequence $t_j \downarrow 0$ such that

$$\int |\nabla u(x,t_j)|^p \, dx \leq \varepsilon t_j^{-1},$$

which implies, by the Sobolev inequality,

$$\int u(x,t_j)^{p^*} \, dx \leq (C\varepsilon t_j^{-1})^{p^*/p}. \tag{4.14}$$

Observe that $u \leq \tilde{u}$ where $\tilde{u}(t) = e^{t\Delta}\mu$. Now, for any t_j, let $\{x_{j,k}\}_{k=1}^{\infty}$, $\{r_{j,k}\}_{k=1}^{\infty}$ be sequences guaranteed by the fact that μ is p–atomic. Apply (4.14) and Hölder's inequality to get,

$$\int_{|x-x_{j,k}|\leq 2r_{j,k}} u(x,t_j)\, dx \leq (C\varepsilon t_j^{-1})^{1/p}(\omega_N 2^N r_{j,k}^N)^{1-(1/p^*)}. \tag{4.15}$$

Now denote $K_j = \cup_{k=1}^{\infty} B(x_{j,k}, 2r_{j,k})$. Summing in (4.15) over $k = 1, 2, \ldots$ and using (ii), we have

$$\int_{K_j} u(x,t_j)\, dx \leq C\varepsilon^{1/p}. \tag{4.16}$$

Consider the estimate for u over $\mathbb{R}^N \setminus K_j$. Since $0 \leq u \leq \tilde{u}$, it suffices to estimate $\tilde{u}$. Recall that $\operatorname{supp}(\mu) \subset \cup_{k=1}^{\infty} B(x_{j,k}, r_{j,k})$ so that if $x \in \mathbb{R}^N \setminus K_j$,

$$\tilde{u}(x,t_j) \leq \sum_{k=1}^{\infty} \int_{B(x_{j,k},r_{j,k})} G(x-y,t_j)\, d\mu(y),$$

with $G(z,t) = (4\pi t)^{-N/2} e^{-z^2/4t}$, and where $|x-y| \geq r_{j,k}$. We obtain

$$\begin{aligned}\int_{\mathbb{R}^N\setminus K_j} \tilde{u}(x,t_j)\, dx &\leq \sum_{k=1}^{\infty} \mu(B(x_{j,k},r_{j,k})) \int_{|z|\geq r_{j,k}} G(z,t_j)\, dz \\ &= C\sum_{k=1}^{\infty} \mu(B(x_{j,k},r_{j,k})) \int_{|\xi|\geq r_{j,k}t_j^{-1/2}} e^{-|\xi|^2}\, d\xi.\end{aligned}$$

Clearly,

$$\int_{|\xi|\geq r_{j,k}t_j^{-1/2}} e^{-|\xi|^2}\, d\xi \leq C_{\delta,N} e^{-(1-\delta)r_{j,k}t_j^{-1/2}},$$

so that, in view of (iii), we get

$$\lim_{j\to\infty} \int_{\mathbb{R}^N\setminus K_j} \tilde{u}(x,t_j)\, dx = 0.$$

Combining (4.16) and (4.13), we have finally, for all $\varphi \in C_0(\mathbb{R}^N)$,

$$\mu(\varphi) = \lim_{t\to 0} \int_{\mathbb{R}^N} u(x,t)\varphi(x)\, dx = 0,$$

hence $\mu \equiv 0$. The theorem is proved. □

REFERENCES

[Al] N. Alaa, *Solutions faibles d'équations paraboliques quasi-linéaires avec données initiales mesures*, Ann. Math. Blaise-Pascal 3 (1996) 1–15.

[AW] L. Alfonsi, F. B. Weissler, *Blow up in $\mathbb{R}^N$ for a parabolic equation with a damping nonlinear gradient term,* Progress in nonlinear differential equations (1992), N. G. Lloyd *et al.* Editors, Birkhäuser.

[AB] L. Amour, M. Ben-Artzi, *Global existence and decay for viscous Hamilton–Jacobi equations*, Nonlin. Anal. TMA. 31 (1998) 621–628.

[AR1] L. Amour, Th. Raoux, *L^1 decay properties for a semilinear parabolic system*, Israel J. Math. 123 (2001), 157–177.

[AR2] L. Amour, Th. Raoux, *The Cauchy problem for a coupled semilinear parabolic problem*, preprint (2000).

[An] D. Andreucci, *Degenerate parabolic equations with initial data measures*, Trans. Amer. Math. Soc. 349 (1997), 3911–3923.

[AQR] J. Arrieta, P. Quittner, A. Rodrìguez-Bernal, *Parabolic problems with nonlinear dynamical boundary conditions and singular initial data*, Differ. Integral Equations 14 (2001), 1487–1510.

[Ba] P. Baras, *Non-unicité des solutions d'une équation d'évolution non-linéaire,* Ann. Fac. Sc. Toulouse 5 (1983), 287–302.

[BP] P. Baras, M. Pierre, *Critère d'existence de solutions positives pour des équations semi-linéaires non monotones*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 2 (1985), 185-212.

[B1] M. Ben-Artzi, *Global existence and decay for a nonlinear parabolic equation,* Nonlinear Anal. TMA 19 (1992), 763–768.

[B2] M. Ben-Artzi, *Global solutions of two-dimensional Navier–Stokes and Euler equations*, Arch. Rational Mech. Anal. 128 (1994) 329–358.

[B3] M. Ben-Artzi, *Global properties of some nonlinear parabolic equations*, Lecture Notes of the Collège de France Seminar, to appear

[BGL] M. Ben-Artzi, J. Goodman, A. Levy, *Remarks on a nonlinear parabolic equation*, Trans. Amer. Math. Soc. 352 (2000), 731–751.

[BK] M. Ben-Artzi, H. Koch, *Decay of mass for a semilinear parabolic equation*, Commun. Partial Differ. Equations 24 (1999) 869–881.

[BSW] M. Ben-Artzi, Ph. Souplet, F. B. Weissler, *Sur la non-existence et la non-unicité des solutions du problème de Cauchy pour une équation parabolique semi-linéaire*, C. R. Acad. Sci. Paris Sér. I Math. 329 (1999), 371–376.

[BL1] S. Benachour, Ph. Laurençot, *Global solutions to viscous Hamilton–Jacobi equation with irregular data*, Comm. Partial Differ. Equations 24 (1999), 1999–2021.

[BL2] S. Benachour, Ph. Laurençot, *Solutions très singulières d'une équation parabolique non linéaire avec absorption*, C. R. Acad. Sci. Paris Sér. I Math. 328 (1999), 215–220. 371–376.

[BL3] S. Benachour, Ph. Laurençot, *Very singular solutions to a nonlinear parabolic equation with absorption. I – existence*, Proc. Roy. Soc. Edinburgh Sect. A 131 (2001), 27–44.

[BLS] S. Benachour, Ph. Laurençot, D. Schmitt, *Extinction and decay estimates for viscous Hamilton-jacobi equations in* $\mathbb{R}^n$, Proc. Amer. Math. Soc, to appear.

[BLSS] S. Benachour, Ph. Laurençot, D. Schmitt, Ph. Souplet, *Extinction and non-extinction for viscous Hamilton-Jacobi equations in* $\mathbb{R}^n$, preprint # 2001-7, Institut Elie Cartan, Université Nancy 1.

[BRV1] S. Benachour, B. Roynette, P. Vallois, *Solutions fondamentales de* $u_t - \frac{1}{2}u_{xx} = \pm|u_x|$, Astérisque, 236 (1996), 41–71.

[BRV2] S. Benachour, B. Roynette, P. Vallois, *Asymptotic estimates of solutions of* $u_t - \frac{1}{2}\Delta u = -|\nabla u|$ *in* $\mathbb{R}_+ \times \mathbb{R}^d$, $d \geq 2$, J. Funct. Anal. 144 (1997), 301-324.

[BGK] P. Biler, M. Guedda, G. Karch, *Asymptotic properties of solutions of the viscous Hamilton-Jacobi equation*, preprint (2000).

[BMP] L. Boccardo, F. Murat, J.-P. Puel, *Existence results for some quasilinear parabolic equations*, Nonlinear Anal., 13 (1989) 373–392.

[Br] H. Brézis, *Remarks on the preceding paper by M. Ben-Artzi "Global solutions of two-dimensional Navier–Stokes and Euler equations"*, Arch. Rational Mech. Anal. 128 (1994) 359–360.

[BC] H. Brézis, T. Cazenave, *A nonlinear heat equation with singular initial data*, J. Anal. Math. 68 (1996) 877–894.

[BF] H. Brézis, A. Friedman, *Nonlinear parabolic equations involving measures as initial data*, J. Math. Pures Appl. 62 (1983) 73–97.

[BPT] H. Brézis H., L. Peletier, D. Terman, *A very singular solution of the heat equation with absorption*, Arch. Rational Mech. Anal. 95 (1986) 185–209.

[CaW] T. Cazenave, F. B. Weissler, *Asymptotically self-similar global solutions of the nonlinear Schrödinger and heat equations*, Math. Z. 228 (1998), 83–120.

[ChW] M. Chipot, F. B. Weissler, *Some blow up results for a nonlinear parabolic problem with a gradient term*, SIAM J. Math. Anal. 20 (1989), 886–907.

[FL] M. Fila, G. Lieberman, *Derivative blow-up and beyond for quasilinear parabolic equations*, Differ. Integral Equations 7 (1994), 811–821.

[GMO] Y. Giga, T. Miyakawa and H. Osada, *Two-dimensional Navier-Stokes flow with measures as initial vorticity*, Arch. Rational Mech. Anal. 104 (1988), 223–250.

[GGK] B. Gilding, M. Guedda, R. Kersner, *The Cauchy problem for* $u_t = \Delta u + |\nabla u|^q$, preprint LAMFA 28, Université de Picardie, 1998.

[HW] A. Haraux A., F. B. Weissler, *Non-uniqueness for a semilinear initial value problem*, Indiana Univ. Math. J. 31 (1982), 167–189.

[KPZ] M. Kardar, G. Parisi, Y. C. Zhang, *Dynamic scaling of growing interfaces*, Phys. Rev. Lett. 56 (1986), 889–892.

[Ka1] T. Kato, *Strong* L^p*–solutions of the Navier-Stokes equation in* $\mathbb{R}^m$*, with applications to weak solutions*, Math. Z. 87 (1984), 471–480.

[Ka2] T. Kato, *The Navier-Stokes equation for an incompressible fluid in* $\mathbb{R}^2$ *with a measure as the initial vorticity*, Differ. Integral Equations 7 (1994), 949–966.

[KF] T. Kato, H. Fujita, *On the nonstationary Navier-Stokes system*, Rend. Sem. Mat. Univ. Padova 32 (1962), 243–260.

[KS] J. Krug, H. Spohn, *Universality classes for deterministic surface growth*, Phys. Rev. A. 38 (1988), 4271–4283.

[La] Ph. Laurençot, *Behaviour of solutions to $u_t - \Delta u + |\nabla u|^p = 0$ as $p \to +\infty$*, Banach Center Publ. 52 (2000), 153–161.

[Lie] G. Lieberman, Second Order Parabolic Differential Equations, World Scientific, Singapore, 1996.

[Lio] P. L. Lions, Generalized solutions of Hamilton-Jacobi Equations, Pitman Research Notes in Math. 62, 1982.

[NS] W-M. Ni, P. Sacks, *Singular behavior in nonlinear parabolic equations*, Trans. Amer. Math. Soc. 287 (1985), 657–671.

[P] R. Pinsky, *Decay of mass for the equation $u_t = \Delta u - a(x)u^p|\nabla u|^q$*, J. Differ. Equations 165 (2000), 1–23.

[SnTW1] S. Snoussi, S. Tayachi, F. B. Weissler, *Asymptotically self-similar global solutions of semilinear parabolic equations with nonlinear gradient terms*, Proc. Roy. Soc. Edinburgh Sect. A 129 (1999), 1291–1307.

[SnTW2] S. Snoussi, S. Tayachi, F. B. Weissler, *Asymptotically self-similar global solutions of a general semilinear heat equation*, Math. Ann. 321 (2001), 131–155.

[S1] Ph. Souplet, *Geometry of unbounded domains, Poincaré inequalities and stability in semilinear parabolic equations*, Commun. Partial Differ. Equations. 24 (1999), 951–973.

[S2] Ph. Souplet, *Recent results and open problems on parabolic equations with gradient nonlinearities*, Electronic J. Differ. Equations Vol. 2001 (2001), No. 20, 1–19.

[S3] Ph. Souplet, *Gradient blow-up for multidimensional nonlinear parabolic equations with general boundary conditions*, Differ. Integral Equations 15 (2002), 237–256.

[T] S. Tayachi, *Forward self-similar solutions of a semilinear parabolic equation with a nonlinear gradient term*, Differ. Integral Equations, 9 (1996) 1107–1117.

[W1] F. B. Weissler, *Semilinear evolution equations in Banach spaces*, J. Funct. Anal. 32 (1979), 277-296.

[W2] F. B. Weissler, *Local existence and nonexistence for semilinear parabolic equations in L^p*, Indiana Univ. Math. J. 29 (1980), 79-102.

[W3] F. B. Weissler, *Existence and nonexistence of global solutions for a semilinear heat equation*, Israel J. Math. 38 (1981), 29-40.

[W4] F. B. Weissler, *L^p-energy and blow-up for a semilinear heat equatio*, Proc. Sympos. Pure Math. 45 (1986), 545–551.