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Dispersion Estimates for Third Order Equations in Two Dimensions

Matania Ben-Artzi,[1,*] Herbert Koch,[2] and Jean-Claude Saut[3]

[1]Institute of Mathematics, Hebrew University, Jerusalem, Israel
[2]Fachbereich Mathematik, Universität Dortmund, Dortmund, Germany
[3]UMR de Mathématiques, Bât, Université de Paris-Sud, Orsay, France

ABSTRACT

Two-dimensional deep water waves and some problems in nonlinear optics can be described by various third order dispersive equations, modifying and generalizing the KdV as well as nonlinear Schrödinger equations. We classify all third order polynomials up to certain transformations and study the pointwise decay for the fundamental solutions,

$$\int_{\mathbb{R}^2} e^{itp(\xi)+ix\cdot\xi}\, d\xi$$

for all third order polynomials p in two variables. We deduce the corresponding Strichartz estimates. These estimates imply global existence and uniqueness for the Shrira system.

*Correspondence: Matania Ben-Artzi, Institute of Mathematics, Hebrew University, Jerusalem 91904, Israel; E-mail: mbartzi@math.huji.ac.il.

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1. INTRODUCTION

In has been known since 1968 (Zakhrov, 1968) that the stability of finite amplitude gravity waves of deep water is governed by a nonlinear Schrödinger equation (NLS). This is found by a perturbation analysis to $O(\varepsilon^3)$ in the wave-steepness $\varepsilon = ka \ll 1$, where a is a typical wave amplitude and k the modulus of the mean wave number.

Taking perturbation analysis one step further to $O(\varepsilon^4)$, Dysthe (1979) has derived a system which improves significantly the results relating the stability of finite amplitude waves. One of the dominant new effects is the wave-induced mean flow with potential Φ. Solving the equation for Φ in terms of the complex amplitude of the wave packet allows to put the Dysthe system (in dimensionless variables) in the following form, see Ghidaglia and Saut (1993):

$$2i\left(\frac{\partial A}{\partial t} + \frac{1}{2}\frac{\partial A}{\partial x}\right) + \frac{1}{2}\frac{\partial^2 A}{\partial y^2} - \frac{1}{4}\frac{\partial^2 A}{\partial x^2} - A|A|^2$$
$$= \frac{i}{8}\left(\frac{\partial^3}{\partial x^3} - 6\frac{\partial^3}{\partial x \partial y^2}\right)A + \frac{i}{2}A\left(A\frac{\partial \bar{A}}{\partial x} - \bar{A}\frac{\partial A}{\partial x}\right) - \frac{5i}{2}|A|^2\frac{\partial A}{\partial x} + AR_1\frac{\partial}{\partial x}|A|^2 \qquad (1.1)$$

where R_1 is the Riesz transform in $\mathbb{R}^2$, that is

$$F(R_1\psi) = i\frac{\xi_1}{|\xi|}\hat{\psi}.$$

Here F denotes the Fourier transform.

The usual NLS is obtained by neglecting the right hand side of Eq. (1.1), which is of order ε^4 in the dimensional variables.

A similar derivation of the fourth order (in ε) evolution equations for the amplitude of a train of nonlinear gravity-capillary waves on the surface of an ideal fluid of infinite depth was performed by Hogan (1985). The equation reads

$$2i\left(\frac{\partial A}{\partial t} + c_g\frac{\partial A}{\partial x}\right) + p\frac{\partial^2 A}{\partial x^2} + q\frac{\partial^2 A}{\partial y^2} - \gamma|A|^2 A$$
$$= -is\frac{\partial^3 A}{\partial x \partial y^2} - ir\frac{\partial^3 A}{\partial x^3} - iu|A|^2\frac{\partial \bar{A}}{\partial x} + iv|A|^2\frac{\partial A}{\partial x} + AR_1\frac{\partial}{\partial x}|A|^2 \qquad (1.2)$$

where c_g is the group velocity and γ, p, q, s, r, u, and v are real parameters depending on the surface tension parameter. Note that q and s are strictly positive, while p can achieve both signs (in particular it is negative for purely gravity waves as in the Dysthe system and positive for purely capillary waves).

A similar system has been obtained by Lo and Mei (1985) and Hara and Mei (1994) for finite depth gravity waves. Another example of a nonlinear Schrödinger type equation involving third order derivatives has been proposed by Shrira (1981) to describe the evolution of a three dimensional packet of weakly nonlinear internal gravity waves propagating obliquely at an arbitrary angle ϕ to the vertical. If the dependence of the wave packet amplitude A on the transversal coordinate y is much slower than that on the x and z directions, one obtains the equation

$$\begin{aligned}
&i\frac{\partial A}{\partial t}+\frac{\omega_k}{2}\frac{\partial^2 A}{\partial x^2}+\frac{\omega_{nn}}{2}\frac{\partial A}{\partial z^2}+\omega_{nk}\frac{\partial^2 A}{\partial x\partial z}\\
&\quad -i\left[\frac{\omega_{kkk}}{6}\frac{\partial^3 A}{\partial x^3}+\frac{\omega_{kkn}}{2}\frac{\partial^3 A}{\partial x^2\partial z}+\frac{\omega_{knn}}{2}\frac{\partial^3 A}{\partial x\partial z^2}+\frac{\omega_{nnn}}{6}\frac{\partial^3 A}{\partial z^3}\right]\\
&\quad +i\gamma A\left(A\frac{\partial\bar{A}}{\partial s}-\bar{A}\frac{\partial A}{\partial s}\right)=0
\end{aligned}\tag{1.3}$$

Here $\partial/\partial s = c(\sin\phi(\partial/\partial x) - \cos\phi(\partial/\partial z))$. The coefficients of the linear terms in Eq. (1.3) can be computed explicitly as function of ϕ. For nonzero constants α and β we have

$$\begin{aligned}
\omega_{kkn} &= \alpha\sin\phi\cos\phi(3-5\cos^2\phi)\\
\omega_{nnn} &= \alpha\sin\phi\cos\phi(3-5\sin^2\phi)\\
\omega_{nnk} &= -\alpha(5\sin^2\phi\cos^2\phi-2/3)\\
\omega_{kkk} &= \alpha\sin^2\phi(4-5\sin^2\phi)\\
\omega_{kk} &= -3\beta\sin^2\phi\\
\omega_{kn} &= \beta\tan\phi(2-3\sin^2\phi)\\
\omega_{nn} &= -\beta(3\sin^2\phi-1)
\end{aligned}$$

see Shrira (1981) and Ghidaglia and Saut (1993, Chap. 3). In particular ω_{kkn} and ω_{knn} cannot vanish simultaneously.

Similar problems occur in nonlinear optics (see for instance Zozulya (1999)), in particular in the modeling of the dynamics of femtosecond laser pulses in a nonlinear media with normal dispersion. The evolution of the complex envelope $E(x, y, z, t)$ of the field is described by the third order NLS

$$i\frac{\partial}{\partial z}E+(1-i\varepsilon_1\partial_t)\Delta_\perp E-\frac{\partial^2 E}{\partial t^2}-i\varepsilon_2\frac{\partial^3 E}{\partial t^3}+(1+i\varepsilon_1\partial_t)g(|E|^2)E=0\tag{1.4}$$

where $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the transverse Laplacian and $\varepsilon_2 \in \mathbb{R}$, $\varepsilon_1 > 0$. As usual in nonlinear optics the evolution variable (which plays mathematically the role of time) is z. The transverse Laplacian accounts for diffraction, while the second and third time derivatives describe group velocity and third over dispersion.

Very little is known concerning the Cauchy problem for Eqs. (1.1)–(1.4). The only results available are local existence for analytic Cauchy data for Eqs. (1.1) and (1.2) by de Bouard (1993) and global existence for a very special case of Eq. (1.3), see Ghidaglia and Saut (1993).

The aim of the present article, as a first step towards solving the Cauchy problem in Sobolev classes, is to derive dispersion estimates for the linear group associated to the linearized equation at 0. Those estimates have an independent interest and seem to be new for this class of third order symbols in 2 dimensions. On the other hand it is well known that Strichartz estimates are an important tool to solve the Cauchy problem by a fixed point argument.

Thus we study linear third order equations, more precisely the decay of the fundamental solutions in x and t. This requires first a classification, second a calculation of several Fourier transforms, and third several applications of the method of stationary phases.

The short time behavior is dominated by the homogeneous part (unless it is strongly nongeneric) whereas the long time behavior is determined by the degeneracy at the strongest singularity. For generic two-dimensional third-order polynomials the long time decay is $t^{-3/4}$ and the generic short time bound is $ct^{-2/3}$.

Stationary phase relates the map $\xi \to \nabla p(\xi)$ with the decay of the fundamental solution. As side product we analyze the pointwise decay of the unique singularity (up to transformations) of codimension two (the singularity of codimension one has normal form ξ_1^3, where the counting of the codimension is different from the one in singularity theory since linear terms do not change the decay). The decay estimates imply Strichartz estimates, which in turn allow to prove global well-posedness for the Cauchy problem for the Shrira system (1.3).

The article is organized as follows. In Sec. 2 we introduce notation. Section 3 is devoted to a classification of third degree polynomials up to affine transformations of coordinates and the addition of affine terms. This extends the work of Dzhuraev and Popëlek, (Dzhuraev and Popëlek, 1989 and 1991).

In Sec. 4 we give the inverse Fourier transform of $e^{itp(\xi)}$ for those polynomials for which the inverse Fourier transform can be computed in terms of exponentials, trigonometric functions and the Airy functions. The next section, Sec. 5, introduces the Pearcey integral, which occurs for one of the phase functions, and studies one more oscillatory integral, where a change of the contour of integration allows to obtain good estimates.

All this relies on the fact that we deal with very specific phase functions.

The last two oscillatory integrals require natural but much deeper tools: We have to study the contribution from general singularities of fold type and cusp type.

In Sec. 6 we study local changes of coordinates and local normal forms for nondegenerate points, folds, and cusps. Here Mather's theory of stable maps enters crucially. The reduction to normal forms allows to establish decay estimates for compactly supported amplitudes, which is done in Sec. 7. The next section, Sec. 8 applies these results to obtain decay estimates for the inverse Fourier transforms of $e^{itp(\xi)}$ for the remaining two polynomials.

Section 9 is standard: the decay estimates imply Strichartz estimates, some of them with smoothing. Well-posedness for the Shrira system is an easy consequence. We plan to return to the study of the other systems.

It should be noted that there are at least two motivations for the problems at hand. First we establish sharp estimates for cusps—here the Pearcey integral plays a similar role as the Airy function for holds. This is a natural step in the study of oscillatory integrals. Even though the tools are available we are not aware of similar estimates in the literature, despite their importance for degenerate dispersive integrals. Secondly systems like the Shrira system have a strong motivation from physics, which makes a good analytical understanding highly desirable.

2. NOTATION

We denote the Fourier transform by F and its inverse by F^{-1}:

$$Ff(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int e^{-ix\xi} f(x)\, dx$$

and

$$F^{-1}(\phi)x = \check{\phi}(x) = 2(\pi)^{-n/2} \int e^{in\xi} \phi(\xi)\, dx.$$

For x in $\mathbb{R}^n$ we set $\tilde{x} = (x_1, \ldots, x_{n-1})$. If $A \subset \mathbb{R}^n$ is measurable we denote its volume by $|A|$. The Lebesgue spaces are denoted by L^p.

We define L^p_w by the quasinorm

$$\|f\|_{L^p_w} = \sup_{\lambda > 0} \lambda |\{x : |f(x)| > \lambda\}|^{1/p}.$$

These spaces are Banach spaces if $1 < p < \infty$. They occur in the context of the weak Young inequality (see Stein and Weiss (1971)):

Lemma 2.1. *We have*

$$\|f * g\|_{L^r} \leq c\|f\|_{L^p} \|g\|_{L^q_w}$$

provided $1 < p, q, r < \infty$ *and*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$

In what follows, we shall derive space-time estimates for the oscillatory integral

$$I(x, t) = \int e^{itp(\xi) + ix \cdot \xi} d\xi.$$

The real polynomial p (in $\xi \in \mathbb{R}^2$) is a general cubic polynomial. However, for the estimates the constants clearly play no role while the linear terms can be dispensed with by a suitable shift x to $x + at$. Thus, we shall assume that p contains only third and second order homogeneous parts. In the following section we shall reduce it to one of a class of normal forms (modulo linear terms), by suitable linear transformations of ξ.

3. NORMAL FORMS

Homogeneous second degree polynomials are quadratic forms. There are only two invariants: rank and signature. There are three equivalence classes, represented by $\xi_1^2, \xi_1^2 + \xi_2^2$, and $\xi_1 \xi_2$.

Note let p be homogeneous of degree three. Assuming $p(1, 0) \neq 0$ (otherwise we rotate our coordinates) we can scale the ξ_1 direction to obtain

$$p(\xi_1, \xi_2) = \xi_1^3 + C\xi_1^2\xi_2 + A\xi_1\xi_2^2 + B\xi_2^3.$$

We will keep this normalization for some time. We change coordinates to $\tilde{\xi}_1 = \xi_1 + (C/3)\xi_2$ so that

$$p(\xi_1, \xi_2) = \tilde{\xi}_1^3 + \tilde{A}\tilde{\xi}_1\xi_2^2 + \tilde{B}\xi_2^3 \tag{3.1}$$

with

$$\tilde{A} = A - \frac{C^2}{3} \quad \text{and} \quad \tilde{B} = B - \frac{AC}{3} + \frac{2C^3}{27}.$$

We drop the tilde in the sequel and suppose that p is a polynomial of the form (3.1).

Given a regular matrix $D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$ we consider

$$p_D(\xi) = p(D\xi) = \lambda_1\xi_1^3 + \lambda_2\xi_1\xi_2^2 + \lambda_3\xi_1^2\xi_2 + \lambda_4\xi_2^3$$

where $\lambda_i = \lambda_i(D)$. This defines a smooth map from the space of regular matrices to $(\lambda_i)_{1\leq i\leq 4} \in \mathbb{R}^4$. An immediate computation gives

$$\begin{aligned} p_D(\eta) = {} & (d_{11}^3 + Ad_{11}d_{21}^2 + Bd_{21}^3)\eta_1^3 + (3d_{11}^2d_{12} + A(d_{12}d_{21}^2 + 2d_{11}d_{21}d_{22}) \\ & + 3Bd_{21}^2d_{22})\eta_1^2\eta_2 + (3d_{11}d_{12}^2 + A(d_{11}d_{22}^2 + 2d_{12}d_{21}d_{22}) + 3Bd_{21}d_{22}^2)\eta_1\eta_2^2 \\ & \times (d_{12}^3 + Ad_{12}d_{22}^2 + Bd_{22}^3)\eta_2^3. \end{aligned} \tag{3.2}$$

The derivative at the identity is gives by the matrix

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 2A & 0 \\ A & 0 & 3B & 2A \\ 0 & A & 0 & 3B \end{pmatrix} \tag{3.3}$$

which has determinant $\Gamma(A, B) := 3(27B^2 + 4A^3)$. This is zero if and only if $A/3 = -(B/2)^{2/3}$. Suppose that $\Gamma(A, B) \neq 0$. Using the implicit function theorem we can find a nonsingular matrix if $(\bar{A}, \bar{B})$ is close to (A, B) such that

$$p_D(\xi) = \xi_1^3 + \bar{A}\xi_1\xi_2^2 + \bar{B}\xi_2^3.$$

A simple covering argument shows that the same is true for each $(\bar{A}, \bar{B})$ in the pathcomponent of $\Gamma(A, B) \neq 0$. Obviously there are two path connected open components $\pm\Gamma(A, B) > 0$.

We choose two representative polynomials one for each component, thereby changing coefficient of ξ_1^3 : $(1/6)\xi_1^3 - (1/2)\xi_1\xi_2^2$ and $(1/6)(\xi_1^3 + \xi_2^3)$.

Now we consider the case $A/3 = -(B/2)^{2/3}$. If $A = 0$ then the polynomial is ξ_1^3. If not we may scale ξ_2 so that $A = 3$. Then $B = \pm 2$ and changing ξ_2 to $-\xi_2$ if necessary we arrive at

$$p(\xi) = \xi_1^3 - 3\xi_1\xi_2^2 + 2\xi_2^3. \tag{3.4}$$

In summary, we have four representatives for the homogeneous third degree polynomial. A short check shows that they are classified by the shape of the zero set of the determinant of the Hessian $\xi \to \det D^2 p(\xi)$, which is either the whole space, a line, the union of two transversal lines or a point. The shape is invariant under the operations we used above. This consideration allows us to use the following set of representatives:

$$\frac{1}{6}\xi_1^3, \qquad \frac{1}{2}\xi_1\xi_2^2, \qquad \frac{1}{6}(\xi_1^3+\xi_2^3) \qquad \text{and} \qquad \frac{1}{6}\xi_1^3 - \frac{1}{2}\xi_1\xi_2^2. \tag{3.5}$$

Note that the second of those four representatives is replacing the one in Eq. (3.4). For notational simplicity we introduced coefficients different from 1. In what follows we refer to the four cases in Eq. (3.5) by Cases I–IV.

We now want to classify all polynomials with cubic and quadratic terms and to find normal forms for each class. We shall use affine changes and the addition of linear and constant terms, as long as they leave the cubic terms in Eq. (3.5) invariant. As has been observed earlier, the transformed polynomials are considered modulo affine terms.

Thus, let p be a polynomial with cubic and quadratic terms. We set

$$d = \text{degree} \det(D^2 p) \qquad \text{and} \qquad S = \{\xi : \det(D^2 p(\xi)) = 0\}$$

and consider the four cases separately.

Case I. If the coefficient of ξ_2^2 is nonzero we may suppose that it is one. Then we have the polynomial

$$\frac{1}{6}\xi_1^3 + b\xi_1^2 + a\xi_1\xi_2 + \xi_2^2 = \frac{1}{6}\xi_1^3 + (a\xi_1/2 + \xi_2)^2 + (b - a^2/4)\xi_1^2. \tag{3.6}$$

We set $\eta = \xi_2 + a\xi_1/2$, plug it into the formula and rename η to ξ_2. Then

$$p(\xi) = \frac{1}{6}\xi_1^3 + \xi_2^2 + c\xi_1^2 \tag{3.7}$$

and we may assume that $c = 0$ since the term can be eliminated by a shift in ξ_1. This creates affine terms, which we neglect in our classification. It is easily seen that $d = 1$.

Now we consider the case that the coefficient of ξ_2^2 is zero and that of $\xi_1\xi_2$ is not. Then we may assume that it is one and we are left with $(1/6)\xi_1^3 + \xi_1\xi_2$. Then $d = 0$ and $\det(D^2 p)$ is constant but not zero. If both coefficients are zero then $\det(D^2 p) = 0$. We conclude that Case I leads to three normal forms:

$$\frac{1}{6}\xi_1^3, \qquad \frac{1}{6}\xi_1^3 + \frac{1}{2}\xi_2^2, \qquad \text{and} \qquad \frac{1}{6}\xi_1^3 + \xi_1\xi_2.$$

Case II. We shift ξ_1 so that the coefficient of ξ_2^2 is zero. Shifting ξ_2 we achieve the same for the coefficient of $\xi_1\xi_2$. We can normalize the coefficient of ξ_1^2 and hence obtain the two representative polynomials $(1/2)\xi_1\xi_2^2$ and $(1/2)\xi_1\xi_2^2 + (1/2)\xi_1^2$.

Then $d=2$ and S is a line or a parabola. The second case provides a normal form for a cusp. The notion of a cusp here is motivated by the fact that the image of S under the map $\xi \to \nabla p(\xi)$ is a geometric cusp. The inverse Fourier transform can be expressed by the Pearcey integral (5.1).

Case III. The same arguments show that there are only two classes with normal polynomials

$$p(\xi) = \frac{1}{6}(\xi_1^3 + \xi_2^3) \qquad \text{and} \qquad p(\xi) = \frac{1}{6}(\xi_1^3 + \xi_2^3) + \xi_1\xi_2.$$

Here $d=2$ and S consists of a hyperbola or two lines.

Case IV. There are two classes with normal forms

$$\frac{1}{6}\xi_1^3 - \frac{1}{2}\xi_1\xi_2^2 + \frac{1}{2}(\xi_1^2 + \xi_2^2), \qquad \text{and} \qquad \frac{1}{6}\xi_1^3 - \frac{1}{2}\xi_1\xi_2^2.$$

Here $d=2$ and S is either a circle or a point. There cannot be any further reduction which can be seen by checking the zero sets of the determinant of the Hessian.

This completes the classification: The third order polynomials:

$$\frac{1}{6}\xi_1^3 - \frac{1}{2}\xi_1\xi_2^2 + \frac{1}{2}\xi_1^2 + \frac{1}{2}\xi_2^2, \quad \text{(IV)} \qquad \frac{1}{6}\xi_1^3 + \frac{1}{6}\xi_2^3 + \xi_1\xi_2 \quad \text{(III)}, \tag{3.8}$$

will be discussed in Sec. 8. Their study is the most demanding and the most important part of this article. We recall that S is a circle (the first polynomial) resp. a hyperbola. The leading part of the asymptotic expansion of the oscillatory integral

$$I(x,t) = \int_{\mathbb{R}^2} e^{itp(\xi)+ix\xi}\, d\xi$$

is expressed through the Airy function near points (x,t) for which there are degenerate critical ξ_c satisfying

$$x + t\nabla p(\xi_c) = 0$$

for which the null space of $D^2p(\xi_c)$ is transversal to S—this corresponds to values of x/t lying in the image of $S \ni \to -\nabla p(\xi)$, where the image is smooth. There exist, however, three cusps ξ_0, ξ_1 and ξ_2 resp. one cusp ξ_0. The leading part of the Taylorexpansion of $tp(\xi) + x\xi$ at these values is, in suitable coordinates, $(t/2)(\xi_1^2\xi_2 + \xi_2^2) + x \cdot \xi$, which is the second polynomial in the list below. The leading part of the expansion of $I(x,t)$ close to directions

$$x/t = -\nabla p(\xi_j)$$

is expressed using the Pearcey integral. Proving this fact requires deeper notions and arguments from singularity theory. Even though the technique for getting the estimates is more or less standard it requires considerable work to carry it through for the specific case of the cusp.

The polynomials

$$\frac{1}{6}\xi_1^3 - \frac{1}{2}\xi_1\xi_2^2, \quad \text{(IV)} \qquad \text{and} \qquad \frac{1}{2}(\xi_1^2 + \xi_1\xi_2^2) \quad \text{(II)} \tag{3.9}$$

are discussed in Sec. 5. The oscillatory integral $I_{x,t}$ with the second polynomial is transformed to the Pearcey integral. It is the model for the cusp singularity.

The inverse Fourier transforms of e^{itp} with p from the lists

$$\frac{1}{6}\xi_1^3 + \frac{1}{6}\xi_2^3 \quad \text{(III)}, \quad \frac{1}{2}\xi_1\xi_2^2 \quad \text{(II)}, \quad \frac{1}{6}\xi_1^3 + \xi_1\xi_2 \quad \text{(I)}, \tag{3.10}$$

$$\frac{1}{6}\xi_1^3 + \frac{1}{2}\xi_2^2, \quad \frac{1}{6}\xi_1^3 \quad \text{(I)}, \quad \xi_1\xi_2 \quad \frac{1}{2}(\xi_1^2 + \xi_2^2) \qquad \text{and} \qquad \frac{1}{2}\xi_1^2 \tag{3.11}$$

will be given in the next section.

4. SOME FOURIER TRANSFORMS

Here we give the (inverse) Fourier transforms of several functions. For simplicity we always assume $t > 0$.

(1) **Quadratic phase functions**: Let $\sqrt{\cdot}$ denote the square root in the slit domain $\mathbb{C}/\mathbb{R}^-$ and $x, \xi \in \mathbb{R}$. Then $F^{-1}(e^{(it/2)\xi^2})(x) = (1/\sqrt{it})e^{-(x^2/2t)i}$. We define

$$K(x) = \frac{1}{\sqrt{2\pi i}} e^{(i/2)x^2}.$$

(2) **The Airy function**: The Airy function is defined by

$$Ai(x) = (2\pi)^{-1/2} F^{-1}(e^{i\xi^3/3})(x).$$

It is the unique bounded solution to

$$Ai''(x) - xAi(x) = 0 \qquad Ai(0) = 3^{-1/6}\frac{\Gamma(1/3)}{2\pi}.$$

We have the estimates

$$|e^{\sqrt{-x}^3}|\big[(1 + |x|^{1/4})|Ai(x)| + (1 + |x|)^{-1/4}|Ai'(x)|\big] \le c$$

which can be seen by the WKB method or standard calculations.

(3) Let $x, \xi \in \mathbb{R}^2$. Then

$$F^{-1}(\exp(it\xi_1^3/6))(x) = (t/2)^{-1/3}\delta_0(x_2)Ai(x_1/(t/2)^{1/3}).$$

(4)

$$F^{-1}(\exp(it(\xi_1^3/6 + \xi_2^2/2)))(x) = \sqrt{2\pi i}(2t)^{-5/6}Ai(x_1/(t/2^{1/3}))K(x/\sqrt{t}).$$

(5) $F^{-1}(\exp(it(\xi_1^3 + \xi_1\xi_2)) = t^{-1}e^{i(-x(3/2)/t^2 - x_1x_2/t)}$, hence

$$F^{-1}\left(\exp\left(it\left(\frac{1}{6}\xi_1^3 + \xi_1\xi_2\right)\right)\right) = \sqrt{6t}^{-1} e^{i(-\sqrt{6}x_2^3/t^2 - 6x_1x_2/t)}.$$

(6) The inverse Fourier transform of $\exp(it(1/2)\xi_1\xi_2^2)$ is

$$F^{-1}\left(\exp\left(it\frac{1}{2}\xi_1\xi_2^2\right)\right) = \begin{cases} 0 & \text{if } x_1 \geq 0 \\ \dfrac{1}{\pi\sqrt{-tx_1/2}}\cos(x_2\sqrt{-2x_1/t}) & \text{if } x_1 < 0 \end{cases}$$

See also Fedoryuk (1977).

(7) $F^{-1}(\exp(it\frac{1}{6}(\xi_1^3+\xi_2^3))) = 2\pi(t/2)^{-2/3}Ai(x_1/(t/2)^{1/3})Ai(x_2/(t/2)^{1/3})$.

5. TWO OTHER CASES

We recall the lemma of van der Corput in the form of Stein (1993).

Proposition 5.1. *Suppose that* $k \geq 2$, $f \in C^k$ *and* $f^{(k)} \geq 1$ *on* $\mathbb{R}$. *Let* $\psi \in C_0^1(\mathbb{R})$. *Then*

$$\left|\int_{\mathbb{R}} e^{itf(\xi)}\psi(\xi)\,d\xi\right| \leq ct^{-1/k}\|\psi'\|_{L^1}.$$

This can be extended to noncompactly supported functions. In that case the integral has to be understood as suitable limit of truncated integrals.

We begin with the prototype of a phase function where we have to content ourselves with estimates for large x. Let

$$B(y,z) := \int_{\mathbb{R}} e^{i(1/24s^4+1/2ys^2+zs)}\,ds \tag{5.1}$$

for $y, z \in \mathbb{R}$. This integral was introduced by Pearcey (1946). The level lines of its modulus are shown in Arnold et al. (1999). In contrast to the previous cases there is an explicit formula for the Fourier transform. We collect estimates for B below:

Lemma 5.2. *We have*

$$|B(y,z)| \leq c(1+|z|^2+|y|^3)^{-1/18}[1+(1+|z|^2+|y|^3)^{-5/9}|(3z)^2+(2y)^3|]^{-1/4}.$$

Proof. Set

$$p(s) = \frac{1}{24}s^4 + \frac{1}{2}ys^2 + zs.$$

Since $p^{(4)} = 1$ the van der Corput lemma implies $|B(y,z)| \leq C$. It suffices therefore to take

$$\lambda := (3z)^2 + (2|y|)^3 \geq 1.$$

We normalize the integral by

$$\tilde{y} = \lambda^{-1/3}y, \qquad \tilde{z} = \lambda^{-1/2}z, \quad \text{and} \quad \tilde{s} = \lambda^{-1/6}s,$$

so that, with

$$\tilde{p}(\tilde{s}) = \frac{1}{24}\tilde{s}^4 + \frac{1}{2}\tilde{y}\tilde{s}^2 + \tilde{z}\tilde{s},$$

$$\begin{aligned} B(y,z) &= \lambda^{1/6}\int_{\mathbb{R}} e^{i\lambda^{2/3}\tilde{p}(\tilde{s})}\, d\tilde{s} \\ &= \lambda^{1/6}\left[\int \phi(\tilde{s})e^{i\lambda^{2/3}\tilde{p}(\tilde{s})}\, d\tilde{s} + \int (1-\phi(\tilde{s})e^{i\lambda^{2/3}\tilde{p}(\tilde{s})}\, d\tilde{s}\right] \\ &= \lambda^{1/6}[B_1(\tilde{y},\tilde{z}) + B_2(\tilde{y},\tilde{z}) \end{aligned}$$

where $\phi \in C_0^\infty$ is identically 1 for all arguments $|\tilde{s}| \le 10$. Then, since the derivative of $\tilde{p}$ is bounded from below outside $[-5, 5]$,

$$|B_2(\tilde{y},\tilde{z})| \le x_k\lambda^{-k} \qquad \text{forall } k \in \mathbb{N} \text{ uniformly in } \tilde{y},\tilde{z}.$$

The cubic equation

$$\tilde{p}'(\tilde{s}) = \frac{1}{6}\tilde{s}^3 + \tilde{y}\tilde{s} + \tilde{z} = 0$$

has (for given $(\tilde{y},\tilde{z}) \ne (0,0)$) either one, two or three roots, depending on whether the discriminant

$$D = (3\tilde{z})^2 + (2\tilde{y})^3$$

is positive, zero, or negative. If $D \ge \delta$ then,

$$|\tilde{p}'(\tilde{s})| + |\tilde{p}''(\tilde{s})| \ge c(\delta)$$

and, by the method of stationary phase

$$|B_1(\tilde{y},\tilde{z})| \le c\lambda^{-1/3}.$$

The third derivative of $\tilde{f}$ is uniformly bounded from below at zeroes of $\tilde{p}''$. Hence

$$|B_1(\tilde{y},\tilde{z})| \le c\lambda^{-2/9}$$

for all y and z. We need however a more precise estimate if $D < \delta$. Arguing as for fold singularities (see Hörmander (1983) and Sec. 7.5 below) we obtain

$$|B_1(\tilde{y},\tilde{z})| \le c\lambda^{-1/18}(1 + \lambda^{4/9}|D|)^{-1/4}$$

hence

$$|B(y,z)| \le \lambda^{-1/18}[1 + \lambda^{-5/9}(|3z)^2 + (2y)^3|)]^{-1/4}.$$

Remark 5.3. It is not hard to prove the estimates of Lemma 5.2 by elementary calculation and the van der Corput lemma. The key is the algebraic estimate

$$|\tilde{p}'| + |\tilde{p}''| \ge c|D|^{1/2}, \tag{5.2}$$

which implies the estimate by the van der Corput lemma. It suffices to verify Eq. (5.2) at critical points. There the second derivative of $\tilde{p}$ is of size of the square root of the discriminant, which implies Eq. (5.2)

The cusp singularity. Here we consider $p(\xi) = 1/2(\xi_1\xi_2^2 + \xi_1^2)$. The phase function

$$tp(\xi) + x\xi$$

has degenerate critical points for x/t in a cusp. Figure 1 (a) shows the set $S = \{\xi | \text{rk} D^2 p(\xi) < 2\}$ together with the kernel of the Hessian $D_\xi^2 p$ and Fig. 1 (b) shows its image and the image of close by level curves of $\det(D^2 p)$ under the map $\xi \to -\nabla p(\xi)$. F1

Let

$$I_t(x) = (2\pi)^{-1} \int_{\mathbb{R}^2} e^{i(t/2)(\xi_1\xi_2^2+\xi_1^2)+i\xi x}\, d\xi.$$

Then

$$I_t(x_1, x_2) = t^{-3/4} I_1(x_1 t^{-1/2}, x_2 t^{-1/4}) \tag{5.3}$$

and, by performing the integration with respect to ξ_1,

$$\begin{aligned}\int_{\mathbb{R}^2} e^{i1/2(\xi_1^2+\xi_1\xi_2^2)+ix\xi}\, d\xi &= 2\pi^{-1/2} \int_{\mathbb{R}} e^{i1/2(1/2\xi_2^2+x_1)^2+ix_2\xi_2}\, d\xi_2 \\ &= ce^{ix_1^2/2} B(3^{-1/2}x_1, 3^{-1/4}x_2)\end{aligned}$$

where B is the Pearcey integral (5.1). The estimate

$$\|I_t\|_{L^\infty} \le ct^{-3/4}$$

follows from Lemma 5.2.

The homogeneous integral. $p(\xi) = \dfrac{1}{6}\xi_1^3 - \dfrac{1}{2}\xi_1\xi_2^2$. Let

$$I_t(x) = \int e^{itp(\xi)+ix\xi}\, d\xi. \tag{5.4}$$

where $p(\xi) = 1/6\xi_1^3 - 1/2\xi_1\xi_2^2$. Then scaling shows

$$I_t(x) = t^{-2/3} I_1(xt^{-1/3})$$

which motivates the definition

$$C(x) = I_1(x).$$

Proposition 5.4. *The following estimates hold*:

$$|C(x)| \le c(1 + |x|)^{-1/2}$$
$$|\nabla_x C(x)| \le c.$$

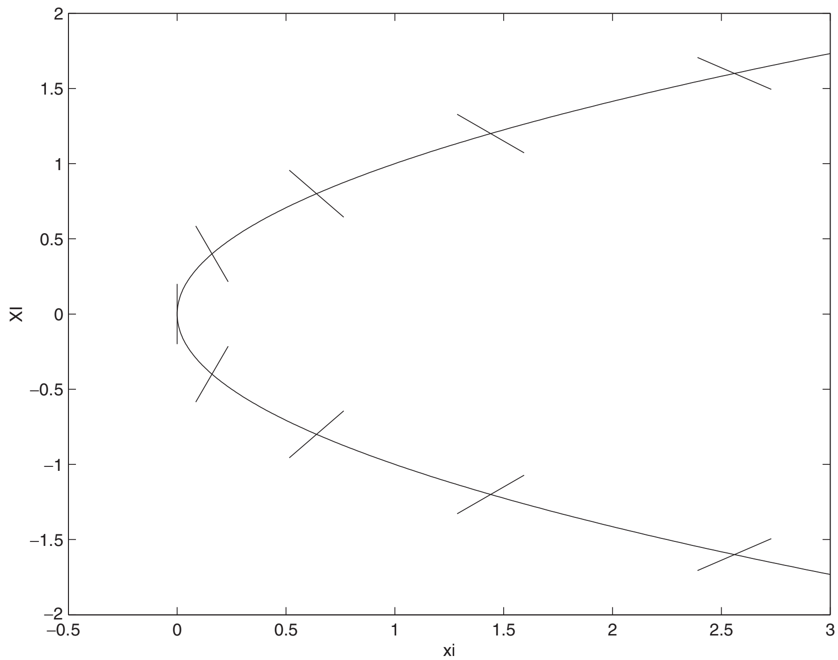

***Figure 1*(*a*).** The set S and the kernel of $D_\xi^2 p$.

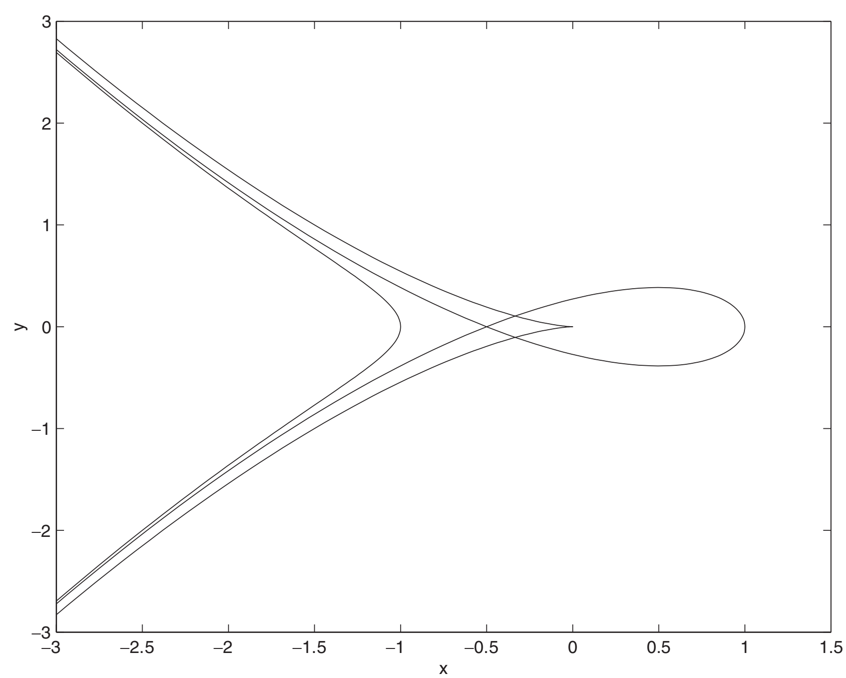

***Figure 1*(*b*).** The image of $S \ni \xi \to -\nabla p(\xi)$.

The following corollary is an immediate consequence.

Corollary 5.5. *We have*

$$|I_t(x) \leq ct^{-2/3}(1+|x|t^{-1/3})^{-1/2}$$

and

$$|\nabla_x I_t(x)| \leq ct^{-1}$$

Proof of Proposition 5.4. There are many ways to prove the estimate. We choose a change of the contour of integration, because this argument carries over to the inhomogeneous phase function and large x/t with only minor changes. For the same reason we do not make use of scaling here. Let

$$\eta(\xi) = \left(\frac{1}{2|\xi|}(\xi_1^2 - \xi_2^2),\ -\frac{1}{|\xi|}\xi_1\xi_2\right).$$

Then $|\xi|\eta = \nabla p$ and

$$|\eta|^2 = \frac{1}{|\xi|^2}\left(\frac{1}{4}(\xi_1^2 - \xi_2^2)^2 + \xi_1^2\xi_2^2\right) = \frac{1}{4}|\xi|^2$$

and

$$\operatorname{Im} p(\xi + i\eta(\xi)) = \frac{1}{|\xi|}|\nabla p|^2 - p(\eta) \geq \frac{1}{4}|\xi|^3 - \frac{1}{6}|\eta|^3 \geq \frac{1}{5}|\xi|^3.$$

We change the contour of integration:

$$C(x) = \int \exp(ip(\xi + i\eta) + ix(\xi + i\eta))\, d\xi$$

hence for all x

$$|C(x)| \geq \int e^{-1/5|\xi|^3 + 1/2|x||\xi|} d\xi < \infty.$$

We want to show that $C(x)$ decays when x is large. Let $x \neq 0$. We identify $\mathbb{R}^2$ with $\mathbb{C}$ and define

$$\eta_x(\xi) := \frac{\left(\xi_1 - i\xi_2 - \sqrt{2(x_1 + ix_2)}\right)\left(\xi_1 - i\xi_2 + \sqrt{2(x_1 + ix_2)}\right)}{2\left(\sqrt{|x|} + |\xi|\right)} = \frac{\nabla_\xi(p(\xi) - x\xi)}{\sqrt{|x|} + |\xi|}.$$

Let ξ_c be one of the two critical points of $\xi \to p(\xi) + x\xi$. Then we have $|\xi_c| = 2\sqrt{|x|}$ and

$$\operatorname{Im} p(\xi + i\eta) \geq \frac{|\xi^2 - \xi_c^2|^2}{8(\sqrt{|x|} + |\xi|)} \geq \sqrt{|x|}\min\{|\xi - \xi_c|^2, |\xi + \xi_c|^2\}.$$

Moving the domain of integration as above gives

$$|C(x)| \leq c(1 + |x|)^{-1/2}.$$

The estimate for the gradient is straightforward: there is an additional factor $2\sqrt{|x|}$.

We obtain the following result when we combine Lemma 2.1 with the previous estimates to $u(t) = S(t)u_0$ where S is the group defined by the differential equation (where we plug in $i\partial_{x_j}$ for D)

$$\partial_t u - ip(D)u = 0.$$

Theorem 5.6. *Consider the integral* (5.4), *where p is the polynomial* $1/6\xi_1^3 - 1/2\xi_1\xi_2^2$.

1. *Let* $1 \le r \le q \le \infty$. *The following estimates hold*:
$$\|u(t)\|_{L^q} \le ct^{-2/3(1/r-1/q)}\|u_0\|_{L^r}$$
if
$$\frac{1}{r} - \frac{1}{q} \ge \frac{3}{4}.$$
If the difference is 3/4 *we require that* $1 < r < q < \infty$.
2. *Moreover*
$$\|\nabla u\|_{L^\infty} \le c|t|^{-1}\|u(0)\|_{L^1}.$$

6. SOME STABLE MAPS

It remains to study the two polynomials $p_1(\xi) = 1/6\xi_1^3 - 1/2\xi_1\xi_2^2 + 1/2(\xi_1^2 + \xi_2^2)$ and $p_2(\xi) = 1/6\xi_1^3 + 1/6\xi_2^3 + \xi_1\xi_2$. Let $J_i(\xi) = \det D_\xi^2 p_i(\xi)$, $i = 1.2$. Then

$$J_1(\xi) = 1 - |\xi|^2, \qquad J_2(\xi) = \xi_1\xi_2 - 1$$

which vanish respectively on a circle or on a hyperbola. Let S_i be the set of points $x \in \mathbb{R}^2$ where thee is a degenerate critical point of $\xi \to p_i(\xi) + x\xi$. The set S_i is one dimensional. It has three cusps if $i = 1$ and it consists of two unbounded curves if $i = 2$, one of which contains one cusp. The leading part in suitable coordinates is given by the cusp singularity discussed above. It will turn out that the asymptotic behavior of the Pearcey integral (5.1) determines the asymptotic behavior of the Fourier cotransforms of $e^{itp(\xi)}$, $t > 0$, but this requires some singularity theory as well as general results about oscillatory integrals.

More generally we will be concerned with the problem of bounding oscillatory integrals of the type

$$I_{t,\psi}(x) = \int_{\mathbb{R}^n} \psi(\xi)e^{it(f(\xi)+x\cdot\xi)}\,d\xi$$

where $\psi \in C_0^\infty(\mathbb{R}^n)$ and where f is a smooth function. Suppose that

$$f(\xi) + x\cdot\xi = f_0(y,\eta) + \phi(y) \tag{6.1}$$

where $x = x(y)$ and $\xi = \xi(y, \eta)$ in the support of ψ. Then

$$I_{t,\psi}(x) = e^{it\phi(y)} \int_{\mathbb{R}^n} \psi(\xi(y,\eta)) \det\left(\frac{\partial \xi}{\partial \eta}\right) e^{itf_0(y,\eta)}\, d\eta.$$

This observation decomposes the study of oscillatory integrals into two parts: a classification of the relevant normal forms for the phase function, and an estimation of oscillatory integrals with these phase functions. In this section we classify the phase functions which are relevant. We follow the work of Duistermaat (1974) and Mather (1968, 1969). From the discussion above it is clear that we should (and do) replace $f(\xi) + x\cdot\xi$ by more general functions $f(x, \xi)$.

Definition 6.1. A function $f_0(x, \xi)$ is called stable at (x_0, ξ_0) if there exists ε, k, and δ such that for every smooth function f with

$$\| f - f_0 \|_{C^k(B_\delta(x_0,\xi_0))} \leq \varepsilon$$

there exist local diffeomorphisms $(y, n) - (x(y), \xi(y, \eta))$ (which, together with its local inverse, are defined in $B_\delta(x_0) \times B_\delta(\xi_0)$ and $\phi \in C^\infty(B_\delta(x_0))$ satisfying (6.1) in a neighborhood of (x_0, ξ_0).

Definition 6.2. We call f_0 infinitesimally stable at (x_0, ξ_0) if for every function $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ there exist smooth functions g_j, h_j, and ϕ with

$$f(x,\xi) = \sum_{j=1}^n g_j(x,\xi)\partial_{\xi j} f_0 + \sum_{j=1}^n h_j(x)\partial_{x_j} f_0 + \phi(x)$$

for x and ξ in a neighborhood of (x_0, ξ_0).

Given a function f we define the set

$$S_{f_0} = \{(x,\xi) | \partial_{\xi_j} f(x,\xi) = 0 \text{ for } 1 \leq j \leq n\}$$

Proposition 6.3. (*Theorem* 2.1.5 *of Duistermaat* (1974)). *Suppose f_0 is smooth near (x_0, ξ_0) and that the projection $S_\phi \ni (x, \xi) \to x$ is proper. Then f_0 is stable at (x_0, ξ_0) if and only if it is infinitesimally stable at (x_0, ξ_0).*

Proof. The implication stable implies infinitesimally stable is obvious. The other direction has been proven (in much greter generality) by Mather (1968 and 1969), and closer in the form needed here, by Duistermaat (1974). □

It is instructive to look at examples. Let A be a symmetric nondegenerate real matrix. Then

$$f_0(x,\xi) = \frac{1}{2}\xi^t A\xi + x\cdot\xi$$

is stable for every point (x, ξ). To see this we assume that f is close to f_0 and we have to find $g = (g_1, \ldots, g_n)$, $h = (h_1, \ldots, h_n)$ and ϕ

$$f(x,\xi) = g^t(x,\xi)(A\xi + x) + h^t\xi + \phi(x)$$

in a neighborhood of (x_0, ξ_0). This is trivial if

$$A\xi_0 + x_0 \neq 0$$

and we restrict ourselves to $A\xi_0 = -x_0$ in the sequel. Let $\eta = \xi - \xi_0$ and $y = x - x_0$. Then the problem reduces to the same problem near (0, 0). For simplicity we assume that A is the identity (otherwise we first change coordinates so that A is a diagonal with entries ± 1 in the diagonal). Let

$$\tilde{f}(x, \eta) = f(x, \eta - x).$$

Then it suffices to fine $\tilde{g}(x, \eta)$ and ϕ such that

$$\tilde{f}(x, \eta) = \tilde{g}^t(x, \eta) \cdot \eta + \phi(x).$$

We set

$$\phi(x) = \tilde{f}(x, 0)$$

and

$$\tilde{g}(x, \eta) = \int_0^1 \nabla_\xi f(x, t\eta)\, dt.$$

The same arguments show that $f(\xi) + x \cdot \xi$ is stable at (x_0, ξ_0) if $D^2 f(\xi_0))_{ij}$ is nondegenerate.

Now let A be a symmetric nonsingular real $(n-1) \times (n-1)$ matrix. Let $\tilde{\xi} = (\xi_1, \ldots, \xi_{n-1})$. We claim that the normal form for a fold

$$f_0(x, \xi) = \frac{1}{6}\xi_n^3 + \frac{1}{2}\tilde{\xi}^t A \tilde{\xi} + x \cdot \xi$$

is stable at (0, 0). We again restrict ourselves to the case $A = id_{\mathbb{R}^{n-1}}$.

We want to verify infinitesimal stability which amounts to finding g, h, and ϕ so that (with f a given function)

$$f(x, \xi) = \sum_{j=1}^{n-1} g_j(x, \xi)(\xi_j + x_j) + g_n(x, \xi)\left(\frac{1}{2}\xi_n^2 + x_n\right) + h^t(x) \cdot \xi + \phi(x).$$

We begin with the one-dimensional case. Given $\tilde{f}$ we want to find $\tilde{g}$, $\tilde{h}$, and $\tilde{\phi}$ with

$$\tilde{f}(s, t) = \tilde{g}(s, t)\left(\frac{1}{2}t^2 + s\right) + \tilde{h}(s)t + \tilde{\phi}(s)$$

near (0, 0). This cane be done by the Malgrange preparation theorem, which is the appropriate tool in general. Here we may also use a more elementary construction. It is not hard to see that

$$\tilde{f}\left(-\frac{1}{2}t^2, t\right) = \tilde{h}\left(-\frac{1}{2}t^2\right)t + \tilde{\phi}\left(-\frac{1}{2}t^2\right)$$

hence

$$\tilde{\phi}\left(-\frac{1}{2}t^2\right) = \frac{1}{2}\left(\tilde{f}\left(-\frac{1}{2}t^2, t\right) + \tilde{f}\left(-\frac{1}{2}t^2, -t\right)\right)$$

and

$$\tilde{h}\left(-\frac{1}{2}t^2\right) = \frac{1}{2t}\left(\tilde{f}\left(-\frac{1}{2}t^2, t\right) - \tilde{f}\left(-\frac{1}{2}t^2, -t\right)\right).$$

We extend both functions smoothly to positive arguments. The definition of $\tilde{g}$ is straight forward.

We can do the same construction in the higher dimensional case as long as we restrict ourselves to the subspace where $\xi_1 \ldots \xi_{n-1} = 0$. Then we repeat the construction for quadratic phase functions above to obtain the remaining functions.

It is clear that the same arguments imply stability for (with $3 < k \leq n+1$)

$$f_0(x, \xi) = \frac{1}{2}\tilde{\xi}^t A\tilde{\xi} + \xi_n^k + x \cdot \xi + \sum_{j=1}^{k-3} x_j \xi_n^{j+1}.$$

This is a normal form for a cusp if $k = 4$.

Moreover, if f is stable at (x_0, ξ_0) and we add a nondegenerate phase function in additional variables, then we obtain a stable singularity.

Finally we consider

$$f_0(x, \xi) = \frac{1}{2}\xi_1\xi_2^2 + \frac{1}{2}\xi_1^2 + x\xi.$$

We claim that f_0 is stable at $(0, 0)$. To see this we change variables: let

$$\eta_1 = \xi_1 + \frac{1}{2}\xi_2^2 \quad \text{and} \quad \eta_2 = \xi_2.$$

Then

$$f_0(x, \xi) = \tilde{f}(x, \eta) = \frac{1}{2}\eta_1^2 - \frac{1}{8}\eta_2^4 + x_1\eta_1 - \frac{1}{2}x_1\eta_2^2 + x_2\eta_2.$$

This function is infinitesimally stable near (0, 0). The properness assumption is clearly satisfied. Hence f_0 is stable.

We collect the results:

Lemma 6.4. *The following phase functions are stable at* $(0, 0)$:

(1) $f(x, \xi) = \xi^t a\xi + x \cdot \xi$ (*if A is symmetric and invertible*),
(2) $f(x, \xi) = 1/6\xi_1^3 + \bar{\xi}^t A\xi + x \cdot \xi$ (*where A is* $a(n-1) \times (n-1)$ *matrix, symmetric and invertible*),
(3) $f(x, \xi) = \xi_1^4 \pm \xi_2^2 + x_2\xi_1^2 + x \cdot \xi$ *and*
(4) $f(x, \xi) = 1/2\xi_1^2\xi_2 + 1/2\xi_2^2 + x \cdot \xi$.

7. ESTIMATES FOR OSCILLATORY INTEGRALS

In this section we prove estimates for oscillatory integrals with degenerate phase functions (not necessarily polynomials).

We introduce some notation. Let $U \subset \mathbb{R}^n$ be open and $f \in C^\infty(U)$.

Definition 7.1. A (Nondegenerate). We say that f is nondegenerate at ξ_0 if $D^2 f$ is nondegenerate at ξ_0, or, equivalently, if the map

$$\xi \to \nabla f(\xi)$$

is nondegenerate near ξ_0. Let

$$J = \det(D^2 f(\xi)).$$

Then f is nondegenerate at ξ_0 iff $J(\xi_0) \neq 0$.

At a degenerate point ξ_0 we assume that J has a simple zero. As a consequence, its zero level-set is a smooth hypersurface S, and the null-space of $D^2 f$ is one-dimensional at all points $\xi \in S$. Let $X \in \ker(D^2 f)$ be a nontrivial vectorfield, defined along S i.e., at every point in S, but not necessarily in TS.

B (Fold). We say that f has a fold at the degenerate point ξ_0 if X is transversal to S at ξ_0., i.e., if $X \cdot \nabla J \neq 0$ at ξ_0.

C (Cusp). We say that f has a cusp at the degenerate point ξ_0 if

(1) $X \cdot \nabla J$ has a simple zero at ξ_0 (as a function on S) and:
(2) Let S_1 be the zero level-set of $X \cdot \nabla J$ in S. Then X is transversal to S_1.

D (Global). We say that f is nondegenerate if it nondegenerate at every point. It has at most folds if at every point it is either nondegenerate or it has a fold. It has at most cusps if at every point it is either nondegenerate, or it has a fold or a cusp.

Remark 7.2. Nondegeneracy at ξ_0 is a condition on the Taylor expansion up to second order, existence of a fold at ξ_0 is decided by the Taylor expansion up to order 3 and the conditions for a cusp require a Taylor expansions up to order 4. Nondegeneracy implies that the map

$$\xi \to \nabla f(\xi)$$

is invertible in a neighborhood of ξ_0. If ξ_0 is a fold then

$$S \ni \xi \to \nabla f(\xi)$$

parameterizes a smooth hypersurface (since the derivative of $(\nabla f)|S$ has full rank) and the image of

$$\xi \to \nabla f(\xi)$$

lies locally on one side of S—which is a consequence of the results below. If ξ_0 is a cusp then

$$S_1 \ni \xi \to \nabla f(\xi)$$

parameterizes a smooth submanifold of codimension two and the image of $S \ni \xi \to \nabla f(\xi)$ is a geometric cusp near $\nabla f(\xi_0)$, which is again a consequence of the considerations below.

A fold corresponds to the singularity A_2 and a cusp to A_3 in the notation of Arnol'd et al. (1985), Sec. 11.1.

Remark 7.3. We will apply the estimates below only to functions with two dimensional ξ variable. In that case S_1 will consist of isolated point.

Remark 7.4. We shall see that the functions above are closely related to the ones discussed in the last section.

Theorem 7.5. *Let* $\psi \in C_0^\infty(U)$ *and*

$$I_{t,\psi}(x) = \int_U \psi(\xi) e^{itf(\xi)+ix\xi}\, d\xi.$$

1. *Suppose that f is non-degenerate in the support of* ψ. *Then*
$$\|I_{t,\psi}\|_{L^\infty(\mathbb{R}^n)} \le ct^{-n/2} \qquad \text{and} \qquad \|I_{t,\psi}\|_{L^1(\mathbb{R}^n)} \le ct^{n/2}.$$
2. *Suppose that f has at most folds in the support of* ψ. *Then*
$$\|I_{t,\psi}\|_{L^\infty(\mathbb{R}^n)} \le ct^{-n/2+1/6}, \quad \|I_{t,\psi}\|_{L^4_w(\mathbb{R}^n)} \le ct^{-n/4}, \quad \text{and} \quad \|I_{t,\psi}\|_{L^1(\mathbb{R}^n)} \le ct^{n/2}.$$
3. *Suppose that f has at most cusps in the support of* ψ. *Then*
$$\|I_{t,\psi}\|_{L^\infty(\mathbb{R}^n)} \le ct^{-n/2+1/4}, \quad \|I_{t,\psi}\|_{L^4_w(\mathbb{R}^n)} \le ct^{-n/4}, \quad \text{and} \quad \|I_{t,\psi}\|_{L^1(\mathbb{R}^n)} \le ct^{n/2}.$$

Remark 7.6. Estimates for other values of p follow by interpolation. Then for $p < 4$ the estimate for $\|I_{t,\psi}\|_{L^p}$ are the same in all cases.

Proof. We begin with the observation that we have for every integer

$$|I_{t,\psi}(x)| \le c_N(1 + |t| + |x|)^{-N} \tag{7.1}$$

for x with $\mathrm{dist}(-x/t, D) \ge 1$ where $D = \{\nabla f(\xi) : \xi \in \mathrm{supp}\psi\}$.

A standard application of stationary phase shows that for the nondegenerate phase function

$$|I_{t,\psi}(x)| \le ct^{-n/2}.$$

This implies the first statement.

Using a suitable partition of unity it suffices to prove the statements for amplitudes supported in a small neighbourhood of a given point. The proof of the second statement will be achieved by a (\,finite) sequence of lemmas.

Step 2 folds. Let ξ_0 be a point where f has a fold. We change coordinates so that $\xi_0 = 0$, $S = \{\xi : J(\xi) = 0\}$ is tangential to $\mathbb{R}^{n-1}$ and $(0_{\mathbb{R}^{n-1}}, 1)$ is in the null space of $D^2 f(0)$. We may and do assume that ψ is supported in $B_\varepsilon(0)$ for a small number ε and $f(0) = \Delta f(0) = 0$. Then, as above,

$$|I_{t,\psi}(x)| \le c_{N,\varepsilon}(1 + |t| + |x|)^{-N}$$

if $|x| \ge 2\varepsilon|t|$.

After a linear change of coordinates there exists a symmetric nondegenerate $(n-1)\times(n-1)$ matrix A such that

$$f(\xi) = \frac{1}{2}\tilde{\xi}^t A\tilde{\xi} + \frac{1}{6}\xi_n^3 + O(|\tilde{\xi}|^2|\xi| + |\xi_n|^4).$$

Here $\tilde{\xi}$ denotes that first $n-1$ components of the vector. Now we define $\xi_\lambda = (\lambda^{1/2}\tilde{\xi}, \lambda^{1/3}\xi_n)$ and $f_\lambda(\xi) = \lambda^{-1} f_\lambda(\xi_\lambda)$. Then

$$\| f_\lambda(\xi) - \frac{1}{2}\tilde{\xi}^t A\tilde{\xi} - \frac{1}{6}\xi_n^3\|_{C^N(B_1(0))} \leq c\lambda^{1/3}.$$

for all $N \in \mathbb{N}$. The function $f_0 = 1/2\tilde{\xi}^t A\tilde{\xi} + 1/6\xi_n^3 + x\xi$ is stable. Thus, if λ is small we can change coordinates according to the last section so that we have the phase function f_0 and an amplitude depending on x/t and ξ. Then estimate follows from Lemma 7.7 below.

Lemma 7.7. *Let A be a nondegenerate symmetric $(n-1)\times(n-1)$ matrix and let $\psi \in C_0^\infty(B_1(0)\times B_\varepsilon(0))$ be a smooth compactly supported function. Then*

$$\begin{aligned} I_{t,\psi}(x) :&= \int_{\mathbb{R}^n} \psi(x/t,\xi) e^{it(1/2\tilde{\xi}^t A\tilde{\xi} + 1/6\xi_n^3) + ix\xi}\, d\xi \\ &= t^{-n/2+1/6}\psi(x/t, -[A^{-1}\tilde{x}/t, 0]) e^{(\tilde{x}^t A^{-1}\tilde{x})/2ti} \sqrt{\det iA}^{-1} Ai(x_n/(2t)^{1/3}) \\ &\quad + O(t^{-n/2}). \end{aligned}$$

where $[A^{-1}\tilde{x}/t, 0]$ denotes the obvious vector in $\mathbb{R}^n$.

Proof. We may assume that x/t is small. We have

$$\begin{aligned} &\int_{\mathbb{R}^{n-1}} \psi(x/t,\xi) e^{it(1/2)\tilde{\xi}^t A\tilde{\xi} + i\tilde{x}\cdot\tilde{\xi}}\, d\tilde{\xi} \\ &\quad = (4\pi t)^{(1-n)/2}(\det(iA))^{-1/2}\psi(x/t, -\tilde{x}/t, \xi_n) e^{i(\tilde{x}^t A^{-1}\tilde{x})/2t} \\ &\qquad + \int [\psi(x/t,\xi) - \psi(x/t, -\tilde{x}/t, \xi_n)] e^{it/(1/2)\tilde{\xi}^t A\tilde{\xi} + i\tilde{x}\tilde{\xi}} d\tilde{\xi}. \end{aligned}$$

The second term in the right hand side is $O(t^{-n/2})$. This calculation reduces the problem to the one dimensional one:

Lemma 7.8.

$$\int_{\mathbb{R}} \psi(s,\sigma) e^{it(1/6s^3 + s\sigma)}\, ds = \psi(0,0) t^{-1/3} Ai(\sigma t^{2/3}) + O(t^{-2/3})$$

if $\sigma \geq 0$ and, for $\sigma < 0$,

$$\int_{\mathbb{R}} \psi(s,\sigma) e^{it(1/6s^3 + s\sigma)}\, ds = \frac{\psi(\sqrt{-\sigma/3}, 0 + \psi(-\sqrt{-\sigma/3}, 0)}{2} t^{-1/3} Ai(\sigma t^{2/3}) + O(t^{-2/3}).$$

The proof of this estimate is straight forward. See also Theorem 7.7.18 (Hörmander, 1983) for related estimates. □

Now, if $ct^{-n/2} \le \lambda \le ct^{-n/2+1/6}$

$$\{x : |I_{t,\psi}(x)| > \lambda\} \subset \{x : |x| \le ct, |x_n| \le c\lambda^{-4} t^{1-2n}\}$$

hence, since $|I_{t,\psi}(x)| \le ct^{-n/2+1/6}$ and since the estimate is obvious for smaller λ,

$$|\{x : |I_{t,\psi}(x)| > \lambda\}| \le c\lambda^{-4} t^{-n}$$

and

$$\|I_{t,\psi}\|^4_{L^4_w} = \sup \lambda^4 |\{x : |I_{t,\psi}(x)| > \lambda\}| \le ct^{-n}.$$

This is the L^4_w estimate. The L^1 estimate is much simpler and we omit it.

Step 3, cusps. We suppose that f has a cusp at ξ_0. We shift ξ_0 to zero. Without loss of generality we assume that $f(0) = \nabla f(0) = 0$. By assumption J has a simple zero at 0. We make a linear change of coordinates so that $X = e_n$ at $\xi = 0$ and $\nabla J(0)$ is a multiple of e_1. Thus the tangent space of S at $\xi = 0$ is orthogonal to e_1. Let $A = D^2 f(0)$ with components a^{ij} and $\tilde{A} = (a^{ij})_{1\le i,j\le n-1}$. The Taylor expansion is

$$f(\xi) = \frac{1}{2} a^{ij}\xi_i\xi_j + \frac{1}{6} c^{ijk}\xi_i\xi_j\xi_k + \frac{1}{24} d^{ijkl}\xi_i\xi_j\xi_k\xi_l + O(|\xi|^5)$$

where c^{ijk} and d^{ijkl} are symmetric in all coefficients, the summation convention is used, and, by assumption, $a^{in} = a^{ni} = 0$ for $1 \le i \le n$ and $\tilde{A}$ is invertible. Then

$$J = c^{nni}\xi_i \det \tilde{A} + o(|\xi|)$$

and hence $c^{nni} = 0$ for $2 \le i \le n$ and $c^{nni} \ne 0$. After scaling ξ_1 we may and do assume that $c^{nn1} = 1$. Then

$$(D^2 f)^{ij} = a^{ij} + c^{ijk}\xi_k + \frac{1}{2} d^{ijkl}\xi_k\xi_l + O(|\xi|^3)$$

and, since by definition on S

$$0 = (D^2 f)^{ij} X_j = a^{ij} X_j + c^{ijk}\xi_k X_j + O(|\xi|^2)$$

and since we may set $X_n = 1$ we have (on S) for $1 \le i \le n-1$

$$X_i = -\tilde{A}^{-1}_{ij} c^{njk}\xi_k + O(|\xi|^2).$$

We compute

$$J = \det(\tilde{A})\left(\xi_1 + \frac{1}{2} d^{mnij}\xi_i\xi_j - \tilde{A}^{-1}_{ij} c^{nik}\xi_k c^{njm}\xi_m + \tilde{A}^{-1}_{ij}\xi_k\xi_1\right) + O(|\xi|^3)$$

where the sum runs from 1 to $n-1$ if $\tilde{A}$ is involved. Thus

$$\nabla J \cdot X = \det(\tilde{A})(d^{nnni}\xi_i - 3\tilde{A}^{-1}_{1j} c^{njk}\xi_k) + O(|\xi|^2).$$

The singularity is a cusp provided

$$d^{nnnn} \neq 3\tilde{A}_{11}^{-1}.$$

We set

$$\eta_i = \xi_i + \frac{1}{2}\tilde{A}_{1i}^{-1}\xi_n^2, \qquad \eta_n = \xi_n$$

which gives

$$f(\xi) = \frac{1}{2}a^{ij}\eta_i\eta_j + \frac{1}{24}(d^{nnnn} - 3\tilde{A}_{11}^{-1})\eta_n^4 + O(|\eta|^5 + |\tilde{\eta}|^2|\eta|)$$

and

$$f(\xi) + \frac{x}{t}\cdot\xi = \frac{1}{2}a^{ij}\eta_i\eta_j + \frac{1}{24}(d^{nnnn} - 3\tilde{A}_{11}^{-1})\eta_n^4 + \frac{x}{t}\cdot\eta - \frac{1}{4}\eta_n^2\tilde{A}_{1i}^{-1}\frac{x_i}{t} + O(|\eta|^5 + |\tilde{\eta}|^2|\eta|).$$

For notational simplicity we restrict to $\tilde{A} = 1_{\mathbb{R}^{n-1}}$ and $(d^{nnnn} - 3A_{11}^{-1}) = 1$. The leading part is stable by Lemma 6.4. Scaling $\tilde{\eta}_i = \lambda^{1/2}\eta_i$ for $1 \leq i \leq n-1$ and $\tilde{\eta}_n = \lambda^{1/4}\eta_n$ we see as above that, at the expense of having an amplitude ϕ depending on $y = \Phi(x/t)$ as well, we may suppose that we have the phase function

$$f(\xi) + x/t\cdot\xi = \frac{1}{2}|\tilde{\eta}|^2 + \frac{1}{24}\eta_n^4 + \frac{1}{2}y_1\eta_n^2 + y\cdot\eta.$$

We integrate over $\tilde{\eta}$ and obtain for a suitable function ϕ

$$I_{t,\psi}(x) = (it)^{-(n-1)/2}e^{it|\tilde{y}|^2/2}\int_{\mathbb{R}}\phi(y,[-\tilde{y},s])e^{it(1/24s^4+1/2y_1s^2+y_ns)}\,ds + O(t^{-n/2}).$$

We continue with the proof of Theorem 7.5, Step 3. Let

$$M_{t,\lambda} = \{x : |I_{t,\psi}| > \lambda\}.$$

By Eq. (7.1) we have for $n \geq 2$, $1 \leq p \leq 8$, and some $c > 0$

$$\int^{(ct)^{-1}} \lambda^{p-1}|M_{t,\lambda} \cap (\mathbb{R}^n \backslash B_{ct}(0)|c\lambda \leq c_N t^{-N}.$$

where $|.|$ denotes the volume. Clearly $M_{t,\lambda}$ is empty if $\lambda >> t^{-n/2+1/4}$.

The L_w^4 estimate follows from

$$|M_{t,\lambda} \cap B_t(0)| \leq ct^{-n}\lambda^{-4}. \tag{7.2}$$

This estimate is trivial for $\lambda \leq t^{-n/2}$ and it remains verify Eq. (7.2) for

$$c^{-1}t^{-n/2} \leq \lambda \leq ct^{-n/2+1/4}. \tag{7.3}$$

According to Duistermaat (1974) there exists a smooth function ρ with compact support and new coordinates $y = \phi(x/t)$ with

$$\det\left(\frac{\partial y}{\partial x}\right) \sim t^{-n}$$

such that locally

$$I_{t,\psi}(x) = t^{-n/2+1/4}\rho(\tilde{x}/t)e^{it\tilde{y}^t\tilde{A}\tilde{y}}B(t^{1/2}y_1, t^{3/4}y_n) + O(t^{-n/2}).$$

where B is the Pearcey integral (5.1). The $O(t^{-n/2})$ term is easily dominated.

After localization and a change of coordinates it suffices to show for

$$\tilde{M}_{t,\lambda} = \{ y \in \mathbb{R}^2 \| y| \leq 1, |B(t^{1/2}y_1, t^{3/4}y_2|) > t^{2n-1/4}\lambda\}.$$

with

$$t^{-n/2} \leq \lambda \leq t^{-n/2+1/4}$$

the estimate

$$|\tilde{M}_{t,\lambda}| \leq ct^{-2n}\lambda^{-4}. \tag{7.4}$$

We set $R = z^2 + |y|^3$ and $T = z^2 + y^3$. By Lemma 5.2 it suffices to prove

$$\|\{(y,z) \in B_1(0)|R^{-1/18}[t^{-2/3} + R^{-5/9}|T|]^{-1/4} > \lambda\}| \leq c\lambda^{-4}.$$

for

$$1 << \lambda \leq t^{1/4}.$$

This set is contained in

$$A_1 = \{(y,z)\|z^2 + y^3| \leq \lambda^{-4}R^{1/3}\}, \text{ and in } A_2 = \{(y,z)|(z^2 + |y|^3) < \lambda^{-24/5}\}.$$

The second set has size

$$|A_2| \sim \lambda^{-4}$$

and, if $(y,z) \in A_1$, for fixed $|z| \geq 1/2\lambda^{-2}$, then y lies in an interval of size

$$\Delta y \sim \lambda^{-4}z^{-2/3}.$$

This implies Eq. (7.4), and hence the L^4_w estimate. The L^1 and L^8 estimate are simple consequences. □

8. THE REMAINING CASES

The classification of third order polynomials has led to the list (3.8)–(3.11). We have seen that there are explicit formulas for the oscillatory integrals of Sec. 2 for p as in Eqs. (3.10) and (3.11). The polynomial in list (3.9) have been studied above. It remains to study the oscillatory integrals with p in the list (3.8).

A. The case $p(\xi) = 1/6\xi_1^3 - 1/2\xi_1\xi_2^2 + 1/2(\xi_1^2 + \xi_2^2)$. This is a second order perturbation of Eq. (5.4). The second order term changes the long term asymptotics.

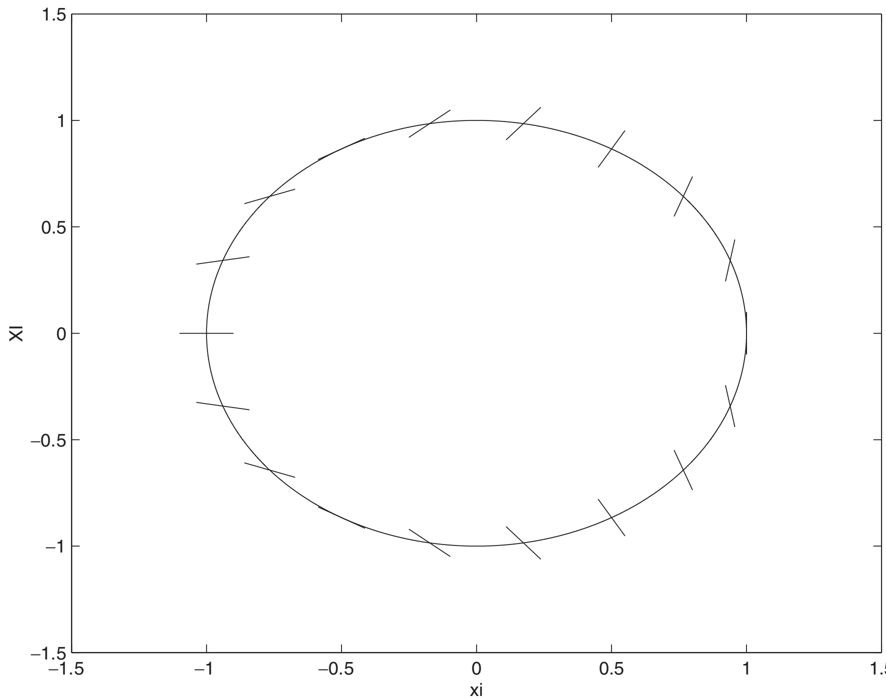

Figure 1(c). The set S and the kernel of $D^2_\xi p$.

The degenerate singularity of the previous example is broke up into three cusps, see Fig. 1 (c) for S and the null space $D^2_\xi p$ and (d) for the image of $-\nabla p|s$ and closeby lines.

We begin with a discussion of the mapping properties of $\xi \to \nabla p$ and of degenerate points. We have $\det(D^2_\xi p) = 1 - |\xi|^2$, which vanishes on the circle of radius 1. There are three points where the kernal of $D^2_\xi p$ is tangent to that curve. Those are the most degenerate sets. They are mapped to three points $(-1, 0)$, $(1/2, \sqrt{3}/2)$, and $(1/2, -\sqrt{3}/2)$by $\xi \to -\nabla p$ in the x space, which are connected through the image of the arcs of the circle. Every point outside has two preimages (by mapping degree arguments) and every point inside has four preimages. The map of the circle to that curve is a homeomorphism. This is discussed in detail in Arnol'd et al. (1985).

Suppose that $|x|/t \leq 3$. Let $\phi \in C_0^\infty(\mathbb{R}^2)$, $\phi(\xi) = 1$ for $|\xi| \leq 10$ and

$$I_{t,\phi}(x) = \int \phi(\xi) e^{itp(\xi) + x\cdot\xi}\, d\xi.$$

Then by Theorem 7.5

$$\|I_{t,\phi}\|_{L^\infty} \leq c(1+t)^{-3/4}$$

and

$$\|I_{t,\phi}\|_{L^4_\omega} \leq c(1+t)^{-1/2}.$$

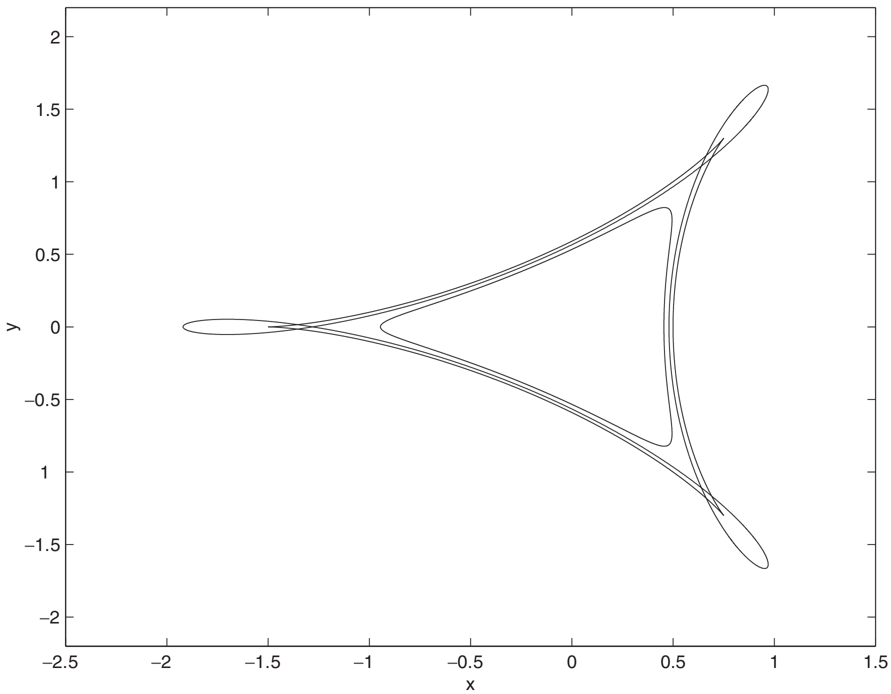

Figure 1(d). The image under $S \ni \xi \to -\nabla p(\xi)$.

We claim that for the points x under consideration

$$\left| \int (1 - \phi(\xi)) e^{itp(\xi) + x \cdot \xi} \, d\xi \right| \leq ct^{-N}.$$

Indeed, let $\psi(\xi)$ be supported in $B(0, 20)$ and $\psi(\xi) = 1$ for $|\xi| \leq 10$. We define

$$\eta = (1 - \psi(\xi)) \frac{\nabla p(\xi)}{1 + |\xi|}$$

and deform the contour of integration. We decompose the integral into one where we multiply the integrand by $\psi(./2)$ and one where we multiply it with $1 - \psi(./2)$. There is no critical point of the phase function in the support of the amplitude. Hence this integral decays fast. The other integrand is bounded by

$$ce^{-t(|\xi|+1)}$$

which decays exponentially in t.

If $|x|/ \geq 3$ we use the same arguments as in the previous section, replacing $\pm\sqrt{x_1 + ix_2}$ by the roots of $\nabla p(\xi) = -x/t$. Here we mix real and complex notation. We collect all estimates:

Proposition 8.1. *The following estimates hold*:

$$|I_t(x)| \leq c \min\{t^{-3/4}, t^{-2/3}\},$$
$$|\nabla_x I_t(x)| \leq c \max[t^{-1}, t^{-3/4}\}.$$

If $|x|/t \geq 3$ *then*

$$|I_t(x)| \leq ct^{-1/2}|x|^{-1/2}.$$

We have

$$\|I_t\|_{L^4_w} \leq ct^{-(1/2)}.$$

Theorem 8.2. *Part* 1 *of Theorem* 5.6 *holds here. Moreover*

$$\|\nabla u(t)\|_{L^\infty} \leq c(|t|^{-1} + |t|^{-3/4})\|u(0)\|_{L^1}.$$

B. The polynomial $(1/6)(\xi_1^3 + \xi_2^3) + \xi_1\xi_2$. This is a second order perturbation of $(1/6)\xi_1^3 + (1/6)\xi_2^2$. The second order terms lead to a cusp, see Fig. 1(e) and 1(f).

$$I_t(x) = \frac{1}{2\pi}\int e^{it((1/6)\xi_1^3+(1/6)\xi_2^3+\xi_1\xi_2)+ix\xi}\, d\xi.$$

The map $\xi \to \nabla p(\xi)$ is degenerate at the hyperbola, $\xi_1\xi_2 = 1$. The kernel of $D_\xi^2 p$ is tangential to the hyperbola only at (1, 1). The hyperbola is mapped to two curves in the x space with a singularity at $(-1.5, -1.5)$. We denote these curves by $\gamma_\pm$.

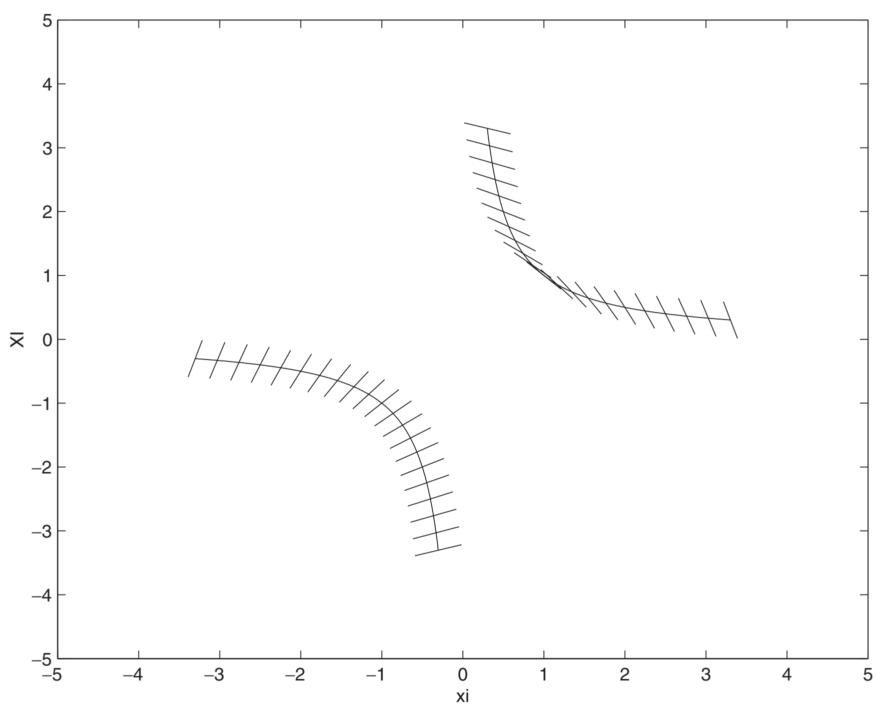

Figure 1(e). The set S and the kernel of $D_\xi^2 p$.

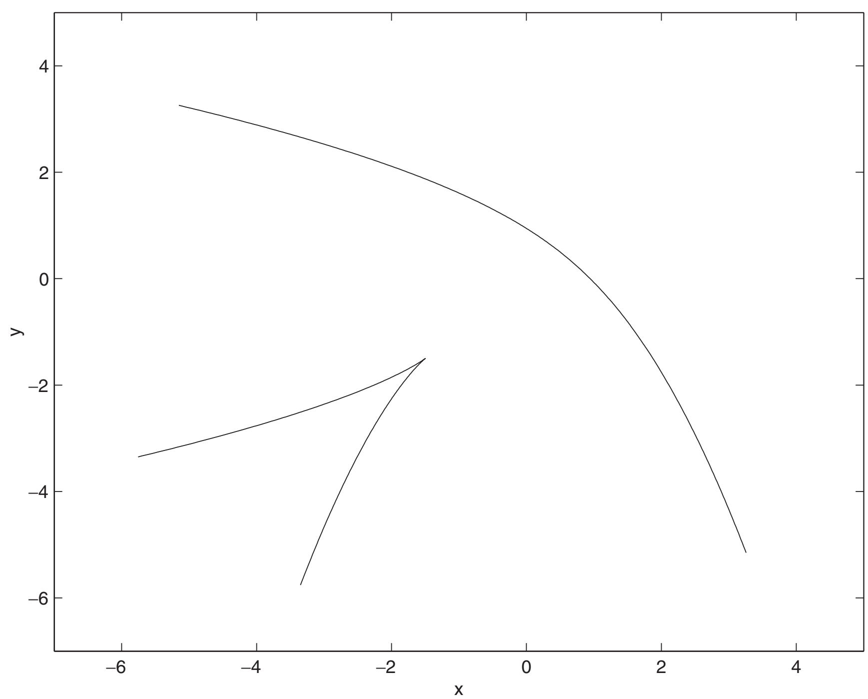

Figure 1(f). The image under $S \ni \xi \to -\nabla p(\xi)$.

Proposition 8.3. *The following estimates hold*:

$$\|I_t\|_{L^\infty} \le c \min\{t^{-2/3}, t^{-3/4}\}.$$

Moreover, for all $p > 4$

$$\|I_t\|_{L^p} \le ct^{-(2/3)+2/3p}.$$

Proof. We consider first the case $|x|/t \le 5$. Then

$$I_{t,\phi}(x) = \frac{1}{2\pi}\int \phi(\xi) e^{it((1/6)\xi_1^3+(1/6)\xi_2^3+\xi_1\xi_2)+ix\xi}\, d\xi$$

is controlled by Theorem 7.5. Now

$$\frac{1}{2\pi}\int (1-\phi(\xi)) e^{it((1/6)\xi_1^3+(1/6)\xi_2^3+\xi_1\xi_2)+ix\xi} d\xi$$
$$= -\frac{1}{2\pi t}\int e^{it((1/6)\xi_1^3+(1/6)\xi_2^3+\xi_1\xi_2)+ix\xi}\nabla \cdot \frac{(1-\phi(\xi))(\xi_1^2,\xi_2^2)}{\frac{1}{2}\xi_1^4+\frac{1}{2}\xi_2^4+\xi_1\xi_2^2=\xi_2\xi_1^2+x_1\xi_1^1+x_2\xi_2^2}\, d\xi.$$

It is not hard to see that the integrand is integrable, which yields the decay t^{-1}. The argument could be iterated to yield $c_N t^{-N}$. For $t \le 1$ we apply a similar argument, but with cutoff function

$$\phi(\xi) = \phi(\xi t^{1/3}).$$

Then the argument gives

$$|I| \le ct^{-2/3}.$$

It remains to study the decay if $|x| \ge 5t$. We may assume that $|x_1| \ge |x_2|$.

Let $R = |x|/t$, $y = x/R$, $\eta = \xi/\sqrt{R}$. Then it suffices to consider

$$I_{t,\phi}(x) = R \int \phi(\eta) e^{itR^{3/2}(\eta_1^3 + \eta_2^2 + R^{-1/2}\eta_1\eta_2 + y\cdot\eta)}\, d\eta$$

since the integral with amplitude $1 - \phi(\eta)$ can be estimated by $c|x|^{-1/2}t^{-1/2}$ by the same arguments as above. The estimate follows now from Theorem 7.5 by a tedious computation. □

Theorem 8.4. *The estimate*

$$\|u(t)\|_{L^q} \le ct^{-(2/3)(1/p-1/q)}\|u(0)\|_{L^p}$$

holds for p and q satisfying the strict inequalities of Theorem 5.6.

9. STRICHARTZ ESTIMATES AND CONSEQUENCES

The (local) Strichartz estimates follow by standard arguments (complex interpolation, TT^* arguments and the Hardy-Littlewood-Sobolev inequality) from the local estimate

$$|I_t(x)| \le ct^{-\alpha} \quad \text{for} \quad x \in \mathbb{R}^2, \quad t \in (0, T]. \tag{9.1}$$

A pair (q, r) is admissible if (for $\mathbb{R}^n$)

$$2 \le r < \infty, \qquad \frac{1}{q} = \alpha\left(\frac{1}{2} - \frac{1}{r}\right).$$

If Λ denotes the operator

$$\Lambda(f)(t) = \int_0^t S(t-s)f(s)\,ds \quad \text{for} \quad 0 < t \le T$$

then for any admissible pairs (q, r) and $(\bar{q}, \bar{r})$ one has

$$\sup_{0<t\le T} \|\Lambda f(t)\|_{L^2} + \|\Lambda f\|_{L^q([0,T];L^r(\mathbb{R}^2))} \le c\|f\|_{L^{\bar{q}'}([0,T];L^{\bar{r}'}(\mathbb{R}^2))}$$

where $\bar{q}'$ and $\bar{r}'$ are the Hölder conjugate exponents of $\bar{q}$ and $\bar{r}$, and

$$\|S(\cdot)\phi\|_{L^q([0,T];L^r(\mathbb{R}^2)))} \le c\|\phi\|_{L^2(\mathbb{R}^2)}.$$

For instance estimate (9.1) holds with $\alpha = 2/3$ if $p(\xi)$ is one of the following polynomials

$$\frac{1}{6}\xi_1^3 - \frac{1}{2}\xi_1\xi_2^2, \qquad \frac{1}{6}(\xi_1^3 + \xi_2^3), \qquad \frac{1}{6}\xi_1^3 - \frac{1}{2}\xi_1\xi_2^2 + \frac{1}{2}(\xi_1^2 + \xi_2^2),$$
$$\frac{1}{6}(\xi_1^3 + \xi_2^3) + \xi_1\xi_2. \tag{9.2}$$

It holds with $\alpha = 3/4$ if $p(\xi) = (1/2)(\xi_1\xi_2^2 + \xi_1^2)$, with $\alpha = 5/6$ if $p(\xi) = \xi_1^3/6 + \xi_2^2$ and with $\alpha = 1$ if $p(\xi) = (1/6)\xi_1^3 + \xi_1\xi_2$. For these polynomials we obtain the admissible pairs and the corresponding Strichartz estimates by the procedure described above.

We now give a simple application of the Strichartz estimates to the Cauchy problem for the Shrira system (1.3). A partial result has been obtained in Ghidaglia and Saut (1993). The idea was to use the conservation laws (see Proposition 3.6 in Ghidaglia and Saut (1993))

$$\frac{d}{dt}\int_{\mathbb{R}^2} |A|^2\, dx\, dz = 0 \qquad \text{and} \qquad \frac{d}{dt}\int_{\mathbb{R}^2} |A_s|^2\, dx\, dz = 0$$

together with the Strichartz estimates.

In Ghidaglia and Saut (1993) only two very special cases were considered, corresponding to

$$p(\xi) = \xi_1^3 + \xi_2^3, \qquad \text{and} \qquad p(\xi) = \frac{1}{6}\xi_1^3 + \frac{1}{2}\xi_2^2.$$

We obtain

Theorem 9.1. *Let*

$$\begin{aligned} L = {} & \frac{\omega_{kk}}{2}\frac{\partial^2}{\partial x^2} + \omega_{nn}\frac{\partial^2}{\partial z^2} + \frac{\omega_{nk}}{2}\frac{\partial^2}{\partial x \partial z} \\ & - i\left[\frac{\omega_{kkk}}{6}\frac{\partial^3}{\partial x^3} + \frac{\omega_{kkn}}{2}\frac{\partial^3}{\partial x^2 \partial z} + \frac{\omega_{knn}}{2}\frac{\partial^3}{\partial x \partial z^2} + \frac{\omega_{nnn}}{6}\frac{\partial^3}{\partial z^3}\right] \end{aligned}$$

and assume that the symbol is equivalent to a phase p with $\alpha \geq 2/3$.

Then, if $A_0 \in L^2(\mathbb{R}^2)$ *and* $\partial_s A_0 \in L^2(\mathbb{R}^2)$ *there exists a unique solution of Eq.* (1.3) *with initial data* A_0 *linear part L such that*

$$\begin{aligned} & A \in C(\mathbb{R}_+; L^2(\mathbb{R}^2)) \cap L^{\lambda}_{\text{loc}}(\mathbb{R}^+, L^6(\mathbb{R}^2)); \\ & \frac{\partial A}{\partial s} \in C(\mathbb{R}_+; L^2(\mathbb{R}^2)), \end{aligned}$$

where

$$\gamma = \frac{9}{2} \text{ if } \alpha = \frac{2}{3}, \quad \gamma = \frac{18}{5} \text{ if } \alpha = \frac{5}{6}, \quad \gamma = 4 \text{ if } \alpha = \frac{3}{4}, \quad \gamma = 3 \text{ if } \alpha = 1. \tag{9.3}$$

Proof. It results immediately from the proof of Theorem 3.8 in Ghidaglia and Saut (1993) by using the Strichartz estimates explained above. □

Remark 9.2. A straight forward calculation shows that the second order part of J is

$$(\omega_{kkk}\omega_{knn} - \omega_{kkn}^2)\xi_1^2 + (\omega_{kkk}\omega_{nnn} - \omega_{kkn}\omega_{knn})\xi_1\xi_2 + (\omega_{knn}\omega_{nnn} - \omega_{knn}^2)\xi_2^2.$$

This is a quadratic form with determinant

$$D = (\omega_{kkk}\omega_{knn} - \omega_{kkn}^2)(\omega_{knn}\omega_{nnn} - \omega_{knn}^2) - (\omega_{kkk}\omega_{nnn} - \omega_{kkn}\omega_{knn}).$$

The number D is a function of ϕ. If $D \neq 0$ then the polynomial is equivalent to one of the four polynomial with $\alpha = 2/3$. Numerically one sees that there are two angles ($<\pi/2$) where D vanishes: $\phi = 0$, and $\phi \sim 7.405$. In both cases the normal form for the polynomial is

$$\frac{1}{2}(\xi_1^2 + \xi_1\xi_2^2).$$

Thus we obtain global existence for all cases of the Shrira system.

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