A canonical arithmetic quotient for simple Lie group actions.*

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1 Introduction

The aim of this paper is to establish the existence of an essentially unique maximal arithmetic virtual quotient action for a broad class of actions of semisimple Lie groups. This includes all finite measure preserving ergodic actions of such groups with finite entropy. This entropy condition is automatically satisfied for actions on compact manifolds. In particular, our results provide arithmetic invariants of such actions, namely an algebraic $\mathbb{Q}$-group and the associated commensurability class of arithemetic subgroups.

We also discuss examples of results which ensure non-triviality of this quotient, and in particular the relationship to invariant geometric structures and representations of fundamental groups.

2 The canonical arithmetic quotient.

The aim of this section is to prove for an ergodic, finite measure preserving, finite entropy action of a semisimple Lie group with no compact factors on a compact manifold that there is an essentially unique maximal arithmetic virtual quotient. Our main result is Theorem 2.16. We begin the discussion with some general notions and notation.

Suppose a locally compact group G acts ergodically on a space X, preserving a finite measure μ. A finite extension of X is a G-space X' with a measure preserving map $X' \to X$ and finite fibers. By ergodicity of the action on X, (almost) all fibers have the same cardinality, say l. The action on X' can thus be defined by a cocycle $\alpha : G \times X \to S_l$ (the latter being the symmetric group), and the action being given on $X' \cong X \times \{1, \ldots, l\}$ (as a measure space) by $g \cdot (x, y) = (gx, \alpha(g, x)y)$. The ergodic components of X' will thus have a similar form (with a possibly smaller l.) Given two finite ergodic extensions $X_1 \to X, X_2 \to X$, one can form the fibered product $X_1 \times_X X_2 \to X$, which will be a finite, but not necessarily ergodic extension. However, any ergodic component will project surjectively onto both X_1 and X_2. Thus, we can always find a finite ergodic extension X_3 of X such that we have a commuting diagram

$$\begin{tikzcd}
 & X_2 \arrow[dr] & \\
X_3 \arrow[ur] \arrow[dr] & & X \\
 & X_1 \arrow[ur] &
\end{tikzcd}$$

and $X_3 \subset X_1 \times_X X_2$. For a detailed discussion of finite (and more generally compact homogeneous space) extensions, see [Z1].

A basic example (but not the only one of importance) is of course an action on a manifold that lifts to an ergodic action on a finite sheeted covering.

Given an ergodic G-space X, by a quotient of X we mean a G-space Y with invariant measure together with a measure preserving G-map $X \to Y$. By ergodicity of G on X, the map is essentially surjective (i.e. the image is conull) and the action on Y is ergodic. By a virtual quotient of X we mean a quotient of a finite ergodic extension of X.

Example 2.1. Suppose G acts ergodically on X and $G_0 \subset G$ is of finite index and acts ergodically on X. Suppose Y is a G-space and a G_0 quotient for the G-action on X. Then Y is a virtual quotient for the G-action on X. Namely, there is a natural quotient G-map $X \times G/G_0 = X' \to Y$.

We now define a particular class of actions. Let H be an algebraic $\mathbb{Q}$-group, $\Lambda \subset H$ an arithmetic subgroup (i.e. Λ is commensurable with $H_{\mathbb{Z}}$) and $K \subset H_{\mathbb{R}}$ a compact subgroup. Then if H has no $\mathbb{Q}$-characters (e.g., if the reductive Levi component of H is semisimple) then Λ is a lattice, so $K \backslash H_{\mathbb{R}} / \Lambda$ carries a natural finite measure which is preserved by the natural action of $Z_{H_{\mathbb{R}}}(K)$.

Definition 2.2. *By an arithmetic action of a group G we mean an ergodic action of G on a space of the form $K \backslash H_{\mathbb{R}} / \Lambda$ which is defined by a homomorphism $G \to Z_{H_{\mathbb{R}}}(K)$.*

Example 2.3. If G is a non-compact simple Lie group, $G \subset H_{\mathbb{R}}$ where $H_{\mathbb{R}}$ is semisimple with no compact factors, and Λ is an irreducible lattice, then G is ergodic on $H_{\mathbb{R}}/\Lambda$ by Moore's theorem [Z2] and hence defines an arithmetic action of G. (Precise conditions for a subgroup of a general $H_{\mathbb{R}}$ to act ergodically on $H_{\mathbb{R}}/\Lambda$ is an extensively studied situation. (c.f. [BM].)

We now collect some information on issues related to arithmetic actions.

For simplicity, for the remainder of this section, for a $\mathbb{Q}$-group H we shall denote $H_{\mathbb{R}}$ simply by H and refer to $\mathbb{R}$-points of $\mathbb{R}$-groups as real algebraic groups. We also take G to be a semisimple Lie group with no compact factors.

Lemma 2.4. *Let H be a real algebraic $\mathbb{R}$-group and $L \subset H$ a connected Lie subgroup. Suppose $\Gamma \subset L$ is a lattice and that $\Gamma \subset L \cap H_{\mathbb{Z}}$. Finally, suppose $G \subset L$ and G is ergodic on L/Γ. Then:*

(i) L is a $\mathbb{Q}$-subgroup, and

(ii) $[\mathfrak{l}, \mathfrak{l}] = \mathfrak{l}$, where $\mathfrak{l}$ is the Lie algebra of L.

Proof. We have a G-map $p : L/\Gamma \to \bar{L}/\bar{\Gamma}$ where the closures are in the Zariski topology. Then $p_*(vol_{L/\Gamma})$ is a finite G-invariant ergodic measure on the variety $\bar{L}/\bar{\Gamma}$, and by Borel density [Z2] is supported on a single point. Therefore $L \subset \bar{\Gamma}$, and since $\Gamma \subset L$, we obviously have L is Zariski dense in $\bar{\Gamma}$. Thus, $[L, L]$ is Zariski dense in $[\bar{\Gamma}, \bar{\Gamma}]$. Since $[\mathfrak{l}, \mathfrak{l}]$ is an algebraic Lie algebra, it follows that $[\mathfrak{l}, \mathfrak{l}] = [\mathrm{lie}(\mathfrak{d}), \mathrm{lie}(\mathfrak{d})]$. Thus if $\mathfrak{l} \neq [\mathfrak{l}, \mathfrak{l}]$, we get a map $L/\Gamma \to \bar{\Gamma}/[\bar{\Gamma}, \bar{\Gamma}]\Gamma$ whose image has positive dimension. Since G is ergodic on L/Γ, this implies $\bar{\Gamma}/[\bar{\Gamma}, \bar{\Gamma}]\Gamma$ supports a non-trivial G-ergodic measure, which is impossible since the image of G in $\bar{\Gamma}/[\bar{\Gamma}, \bar{\Gamma}]$ is trivial. Thus, $L = [L, L]$ and is hence algebraic, and since $\Gamma \subset L \subset \bar{\Gamma}, L$ is defined over $\mathbb{Q}$.

Definition 2.5. *Let G acting on $K \backslash H/\Lambda$ be an arithmetic action. We say it is of reduced form if G is ergodic on H/Λ.*

Lemma 2.6. *Every arithmetic action of G (which we recall is semisimple with no compact factors) has a finite extension that is an arithmetic action of reduced form.*

Proof. Consider the extension of G-spaces $H/\Lambda \to K \backslash H/\Lambda$. Since the action on the base is ergodic and finite measure preserving, we may choose an ergodic component of the G-action on H/Λ that projects to the standard volume on $K\backslash H/\Lambda$. (c.f. the analysis in [Z1] of ergodic components in compact group extensions.) By Ratner's theorem [R], this measure is supported on some L'-orbit, say $L'/L' \cap h\Lambda h^{-1}$, for some Lie group L' with $G \subset L' \subset H$. Since the projection of this L'-orbit to $K\backslash H/\Lambda$ is of full measure, it follows that there is some $y \in H$ such that $KL'y\Gamma$ is of full dimension in H, and hence KL' is of full dimension. Since K is compact, $KL' = H$. Thus, we have $K \cap L'\backslash L'/L' \cap \Lambda \to K\backslash H/\Lambda$ is a surjective measure preserving map of manifolds of the same dimension and finite volume. This is therefore a finite extension. Replacing L' and its subgroups by $L = h^{-1}L'h$ and the corresponding subgroups, and applying Lemma 2.4, we deduce the desired conclusion.

Lemma 2.7. *If $K\backslash H/\Gamma$ is an arithmetic G-space of reduced form, and $M \subset H$ is a normal $\mathbb{Q}$-subgroup containing G, then $M = H$.*

Proof. H/M is a $\mathbb{Q}$-group and we have a G-map

$$H/\Lambda \;\to\; H/M\Lambda \quad (= (H/M)/(H/M)_{\mathbb{Z}}).$$

Since $G \subset M$, the action on the target is trivial, and since G is ergodic on H/Λ (and hence on $H/M\Lambda$), we have $H = M\Lambda$. Since H is connected, $M = H$.

With these preliminaries completed, we now consider arithmetic quotients and virtual quotients of a given ergodic G-space. Supppose H is a $\mathbb{Q}$-group and $\Lambda, \Lambda' \subset H$ are arithmetic (i.e. commensurable with $H_{\mathbb{Z}}$) and hence commensurable. If $G \subset H$ and $K\backslash H/\Lambda$ is a virtual arithmetic quotient of X of reduced form, so is $K\backslash H/\Lambda'$. To see this, it suffices to consider the case $\Lambda' \subset \Lambda$ of finite index. If $X' \to K\backslash H/\Lambda$ is a G-quotient where X' is a finite ergodic extension of X, let X'' be any ergodic component of $X' \times_{K\backslash H/\Lambda} K\backslash H/\Lambda'$. Then X'' will be a finite ergodic extension of X and it has $K\backslash H/\Lambda'$ as a quotient. A similar argument shows that if Λ_1 and Λ_2 are commensurable and $X_i \to K\backslash H/\Lambda_i$ are virtual quotients of X, then there is a finite ergodic extension X' of X with quotients $K\backslash H/\Lambda_1 \cap \Lambda_2$ (and hence both $K\backslash H/\Lambda_i$.) We will speak of this situation as defining commensurable virtual quotients of X. (We remark that if one restricts attention to quotients of a fixed X, one cannot pass in this way between commensurable arithmetic groups. This is only one of the reasons for working with virtual quotients.)

A basic tool in showing the existence of a canonical virtual arithmetic quotient is that of entropy. If G acts ergodically on a space (X, μ) preserving a finite measure μ, for $g \in G$ we denote by $h(g)$ (or $h_X(g)$) the entropy with respect to μ.

Definition 2.8. *We say that G acts with finite entropy if $h(g) < \infty$ for all $g \in G$.*

Example 2.9. If G acts smoothly on a compact manifold, then G acts with finite entropy.

The following is a well-known computation.

Lemma 2.10. *Suppose $G \subset H$ and $M = K\backslash H/\Lambda$ is an arithmetic G-space. Let $A \subset H$ be the maximal $\mathbb{R}$-split torus. Then for $g \in G$,*

$$h_{H/\Lambda}(g) \;=\; h_{K\backslash H/\Lambda}(g) \;=\; \sum \{\log w(g)\}$$

where $w(g)$ is a weight of $Ad_H|G$ with respect to A and $w(g) > 1$.

Corollary 2.11. *If G is semisimple and acts ergodically on X with finite entropy, then the set of possible entropy functions $\{h(g), g \in A\}$ for virtual arithmetic quotients of X is finite.*

Proof. This follows from Lemma 2.10, standard results about the representation theory of G, and the following two facts about extensions. If $X \to Y$ is a quotient of ergodic G-spaces with finite invariant measure, then $h_X(g) \geq h_Y(g)$; and if the extension is finite $h_X(g) = h_Y(g)$.

Definition 2.12. *If $X_i \to Y_i = K_i\backslash H_i/\Lambda_i$ are virtual arithmetic quotients of X, we say that $Y_1 \succ Y_2$ if by passing to commensurable virtual arithmetic quotients $X_i' \to Y_i' = K_i\backslash H_i/\Lambda_i'$, we can find*

i) a common finite ergodic extension X' of X_i'; and

ii) a $\mathbb{Q}$-surjection $\theta : H_1 \to H_2$ such that $\theta(K_1) \subset K_2$, $\theta(\Lambda_1') \subset \Lambda_2'$, so that

$$\begin{array}{ccccc} & & X_1' & \to & Y_1' \\ & \nearrow & & & \\ X' & & & & \downarrow \theta \\ & \searrow & & & \\ & & X_2' & \to & Y_2' \end{array}$$

commutes.

We observe that $Y_1 \succ Y_2$ amd $Y_2 \succ Y_1$ if and only if Y_i are commensurable virtual quotients. Furthermore, if Y_i and Z_i are commensurable, then $Y_1 \succ Y_2$ if and only if $Z_1 \succ Z_2$.

For the remainder of this section G is a semisimple Lie group with no compact factors, and X is an ergodic G-space of finite entropy.

Lemma 2.13. *There is a virtual arithmetic quotient action of X, $K\backslash H/\Lambda$, in reduced form such that*

i) H is semisimple; and

ii) $K\backslash H/\Lambda$ is maximal (up to commensurability) among all virtual semisimple arithmetic quotients (i.e. those of the form $C\backslash L/\Gamma$ with L semisimple) of reduced form.

Proof. Choose a semisimple virtual arithmetic quotient of reduced form, $K'\backslash H'/\Lambda'$ which has a maximal entropy function among all such virtual quotients. This exists by Corollary 2.11. Consider the algebraically simply connected algebraic covering group of H' defined over $\mathbb{Q}$, say $q : H \to H'$, such that the inclusion $G \hookrightarrow H'$ lifts to a smooth homomorphism $G \hookrightarrow H$ and such that $K\backslash H/\Lambda$ is a virtual arithmetic quotient for some $K \subset q^{-1}(K')$ and $\Lambda \subset q^{-1}(\Lambda')$. By the descending chain condition on compact subgroups we can further assume K is minimal with this property. We now claim that $K\backslash H/\Lambda$ satisfies assertion (ii).

If not, then we have $X' \to C\backslash L/\Gamma \to K\backslash H/\Lambda$ for some finite extension X' of X, where $C\backslash L/\Gamma \to K\backslash H/\Lambda$ is induced by a $\mathbb{Q}$-surjection $\theta : L \to H$. Let $N = \ker(\theta)$. Then we can write $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{n}$ to be a direct sum of $\mathbb{Q}$-ideals. The projection of $\mathfrak{g}$ to $\mathfrak{n}$ must be trivial for otherwise the maximality of the entropy function of $K\backslash H/\Lambda$ would be contradicted by Lemma 2.10. However, if we have $\mathfrak{g} \subset \mathfrak{h}$, and $C\backslash L/\Gamma$ is of reduced form, it follows from Lemma 2.7 and the choice of H that $L = H$, i.e. θ is a $\mathbb{Q}$-isomorphism. Finally, by minimality of K, $\theta(C) = K$.

Lemma 2.14. *There is a virtual arithmetic quotient in reduced form that is maximal (up to commensurability).*

Proof. Fix $K\backslash H/\Lambda$ to be a virtual semisimple arithmetic quotient satisfying the conclusion of Lemma 2.13. Now consider all larger virtual arithmetic quotients of reduced form, and choose one of maximal entropy function among all such, say $C\backslash M/\Gamma \to K\backslash H/\Lambda$. Finally consider all virtual arithmetic quotients of reduced form $E\backslash P/\Delta \succ C\backslash M/\Gamma$. To prove the Lemma, it suffices to see that the possible values of $\dim(P)$ is bounded.

Let $N = \ker\,(P \to M)$. Then N is a $\mathbb{Q}$-group. Suppose N is not unipotent. Then letting $\rho : P \to P/\mathrm{Rad}_u(P)$, we see that $\rho(E)\backslash\rho(P)/\rho(\Delta)$ is a virtual arithmetic quotient in reduced form strictly larger than $K\backslash H/\Lambda$. Furthermore, it is of reduced form (since $E\backslash P/\Delta$ is) and since G is semisimple, $P = [P,P]$ by Lemma 2.7. In other words, $\rho(P)$ is also semisimple, which contradicts the choice of H. Thus, we may assume N is unipotent. Since $C\backslash M/\Gamma$ has a maximal entropy function, Lemma 2.10 implies that G centralizes N. Thus, $Z_P(N)$ is a normal $\mathbb{Q}$-subgroup of P, and we have $G \subset Z_P(N)$. By Lemma 2.7 again $Z_P(N) = P$, i.e. N is central in P.

The Lie algebra $\mathfrak{p}$ is defined over $\mathbb{Q}$ and we choose a subspace $\mathfrak{n}' \subset \mathfrak{p}$ such that $\mathfrak{n}'$ is defined over $\mathbb{Q}$ and $\mathfrak{n} \oplus \mathfrak{n}' = \mathfrak{p}$. (We note that $\mathfrak{n} \subset \mathfrak{p}$ is a central ideal, but $\mathfrak{n}'$ is only a linear subspace defined over $\mathbb{Q}$.) Let $B : \mathfrak{n}' \times \mathfrak{n}' \to \mathfrak{n}$ be given by $B(x,y) = \mathrm{proj}_{\mathfrak{n}}([x,y])$. Since $\mathfrak{n} \subset \mathfrak{z}(\mathfrak{p})$, $I = \mathfrak{n}' \oplus \mathfrak{B}(\mathfrak{n}' \times \mathfrak{n}')$ is an ideal in $\mathfrak{p}$ and I is defined over $\mathbb{Q}$. We have $\dim(\mathfrak{n}') \leq \dim(M)$, so if $\dim P > \dim M + (\dim M)^2$, then $I \subset \mathfrak{p}$ is a proper $\mathbb{Q}$-ideal. Thus $J = [I, I]$ is a proper $\mathbb{Q}$-ideal that is also an algebraic Lie subalgebra. Furthermore, since $\mathfrak{n}$ is unipotent we can choose $\mathfrak{n}'$ such that some conjugate of $\mathfrak{g}$ is contained in $\mathfrak{n}'$, hence in I, and since $\mathfrak{g}$ is semisimple, in J. It follows from Lemma 2.7 that some conjugate of G does not act ergodically on P/Δ, and hence G does not either. This contradicts the fact that $E\backslash P/\Delta$ is in reduced form. Therefore, we must have $\dim P \leq \dim M + (\dim M)^2$, and this proves the lemma.

To discuss uniqueness of maximal virtual arithmetic quotients, we first remark that if H is a $\mathbb{Q}$-group, Λ is an arithmetic subgroup, and G acts on H/Λ via $\pi : G \to H$, then G will also act via $g \mapsto h\pi(g)h^{-1}$ for any $h \in H$. If $h \notin H_{\mathbb{Q}}$, the resulting actions will not a priori be commensurable. More generally, we consider the following situation.

Definition 2.15. *Let $K\backslash H/\Lambda$ be an arithmetic G-space defined via a homomorphism $\pi_1 : G \to Z_H(K)$ Let $z \in Z_H(\pi_1(G))$ and $h \in H$. Then G also acts on $hzKz^{-1}h^{-1}\backslash H/\Lambda$ via $\pi_2 : G \to Z_H(hzKz^{-1}h^{-1})$, where $\pi_2(g) = h\pi_1(g)h^{-1}$. We call two such arithmetic actions $\mathbb{R}$-conjugate.*

Clearly, any two $\mathbb{R}$-conjugate arithmetic actions are isomorphic in the category of G-actions, and in particular, one will be a quotient of an ergodic G-space if and only if the other is. We also remark that one is in reduced form if and only if the other is.

We can now state the main result of this section.

Theorem 2.16. *Let G be a semisimple Lie group with no compact factors. Let X be an ergodic G-space with finite invariant measure and finite entropy. Then:*

(1) There is a maximal (up to commensurability) virtual arithmetic quotient of reduced form, say $A(X)$.

(2) $A(X)$ is unique up to $\mathbb{R}$-conjugacy. More precisely, if Y is another such virtual quotient, then $A(X)$ and Y have $\mathbb{R}$-conjugate commensurable virtual arithmetic quotients.

(3) If Y is any virtual arithmetic quotient of X of reduced form, then $Y \prec Z$, where Z is an $\mathbb{R}$-conjugate of $A(X)$.

(4) Any virtual arithmetic quotient of X is a quotient of some finite ergodic extension of $A(X)$.

Proof. Let $K\backslash H/\Lambda$ satisfy the conclusions of Lemma 2.14. Let $C\backslash L/\Gamma$ be any other virtual arithmetic quotient of X in reduced form. Passing to commensurable actions, we can assume there is a finite ergodic extension X' of X and measure preserving G-maps $\psi : X' \to K\backslash H/\Lambda$ and $\Psi : X' \to C\backslash L/\Gamma$. Letting μ be the relevant measure on X', then $\nu = (\psi, \Psi)_*\mu$ is a finite G-invariant ergodic measure for the diagonal action on $K\backslash H/\Lambda \times C\backslash L/\Gamma$, that projects to the standard measure on each factor. Then it is not difficult to see that we can lift ν to an ergodic measure ν' on $H/\Lambda \times L/\Gamma$ that projects to the standard measure on both factors and projects to ν. By Ratner's theorem, this measure is supported on the orbit of a Lie group J, $\mathrm{diag}(G) \subset J \subset H \times L$. It follows that J projects surjectively to both H and L. Thus, we can write the J orbit supporting ν' as J/Δ where $\Delta = (\Lambda \times l\Gamma l^{-1}) \cap J$ for some $l \in L$. Give L the $\mathbb{Q}$-structure obtained by conjugating the given structure by l. Then by Lemma 2.4, J is a $\mathbb{Q}$-group, $\Delta \subset J$ is an arithmetic subgroup, and the projection $J \to H$ is a $\mathbb{Q}$-surjection. Furthermore the projection of J/Δ to $K\backslash H/\Lambda \times C\backslash L/\Gamma = K \times C\backslash H \times L/\Lambda \times \Gamma$ is $(K \times C)\backslash(K \times C)J/\Delta \cong (K \times C) \cap J\backslash J/\Delta$. Thus, letting $D = (K \times C) \cap J$, we have that $D\backslash J/\Delta$ is an arithmetic G-space which is a virtual arithmetic quotient of X in reduced form, and for which we have, via the projection, $D\backslash J/\Delta \succ K\backslash H/\Lambda$. This shows that they must be commensurable. Via a conjugate $\mathbb{Q}$-structure on J, we see also that $D\backslash J/\Delta \succ C\backslash L/\Gamma$. Assertions (1) and (2) then follow directly, and (3), (4) follow from these and earlier results in this section.

3 Examples

If we write $A(X) = K\backslash H/H_{\mathbb{Z}}$, then (the commensurability class of) $H_{\mathbb{Z}}$ is canonically attached to the action of G on X. There are a number of results that relate this discrete group to $\pi_1(X)$.

Theorem 3.1 [Z2] *Let M be a compact real analytic manifold. Supose G is a simple Lie group with $\mathbb{R}$-rank $(G) \geq 2$, and that G acts on M preserving an ergodic volume density and a real analytic connection. Assume further that $\pi_1(M)$ is (abstractly) isomorphic to a subgroup of some arithmetic group. Then there is a $\mathbb{Q}$-group L and a local embedding $G \to L$ such that*

(i) $A(M) \succ C\backslash L/L_{\mathbb{Z}}$.

(ii) $L_{\mathbb{Z}} \subset \pi_1(M)$.

The proof of this result (for which we refer the reader to [Z2]) uses a combinaton of superrigidity for cocycles, Ratner's Theorem, and Gromov's work on rigid transformation groups.

In [LZ], we obtianed much sharper results with engaging hypotheses (and without the geometric hypotheses.) We briefly recall some definitions, referring the reader to [LZ] for a more detailed discussion.

Let M be a compact space for which covering space theory holds. Assume we have a continuous action of G on M where G is a connected Lie group. Then $\widetilde{G}$ acts on any covering space M' of M. Suppose there is an ergodic finite invariant measure μ for the G action on M.

Definition 3.2

(i) The G action on M is called engaging if $\widetilde{G}$ is ergodic on M' for every finite covering $M' \to M$.

(ii) The G action on M is called totally engaging if for every non-trivial convering $p : M' \to M$, there is no $\widetilde{G}$-equivariant measurable section of p.

Remark 3.3

(i) Totally engaging implies engaging.

(ii) Arithmetic actions are totally engaging. (See [LZ].)

To state the main results of [LZ] and their relation to $A(M)$, we need one more definition from [LZ].

Definition 3.4. ([LZ, Definition 4.1]) *Suppose H_i are algebraic k-groups, $i = 1, 2, \ldots$, and $H_i = L_i \ltimes U_i$ are Levi decompositions defined over k. We call H_1 and H_2 k-isotopic if there is a k-isomorphism $L_1 \to L_2$, such that under this isomorphism $\mathfrak{u}_i (=$ Lie algebra of $U_i)$ are k-isomorphic L_i modules.*

The main results of [LZ] exhibit the relationship of $A(M)$ to representations of $\pi_1(M)$. Roughly, they assert that with engaging hypotheses, a representation of $\pi_1(M)$ yields an arithmetic quotient of M and hence of $A(M)$.

Theorem 3.5 ([LZ, Theorem 5.1]) *Let G be a connected simple Lie group with $\mathbb{R}$-rank $(G) \geq 2$, and suppose G acts on a compact M, preserving a finite measure and engaging. Let $\sigma : \pi_1(M) \to GL(n, \mathbb{R})$ be any linear representation, with image $\Gamma = \sigma(\pi_1(M))$ an infinite group, and Zariski closure $\overline{\Gamma} \subset GL(n, \mathbb{R})$. Then Γ is $\mathfrak{s}$-arithmetic. More precisely, there is a real algebraic $\mathbb{Q}$-group H, an embedding $\Gamma \hookrightarrow H_{\mathbb{Q}}$ (and hence necessarily in $H_{\mathbb{Z}_{\mathbb{S}}}$ for some finite set of primes S) and subgroups $\Gamma_\infty \subset \Gamma_0 \subset \Gamma$ such that:*

(i) *H contains a group $\mathbb{R}$-isotopic to $\overline{\Gamma}$*

(ii) $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$.

(iii) $[\Gamma : \Gamma_0] < \infty$

(iv) *Γ_∞ is profinitely dense in Γ_0*

(v) *Γ_∞ is commensurable with $H_{\mathbb{Z}}$ and is a lattice in H.*

Furthermore, (perhaps by passing to a finite cover of G), there is a local embedding $G \to H$ such that $C \setminus H/\Gamma_\infty$ is a virtual arithmetic quotient of M. In particular, $C \setminus H/\Gamma_\infty \prec A(M)$, where $A(M)$ is the canonical maximal arithmetic quotient.

Theorem 3.6 ([LZ, Theorem 5.2]) *With the hypotheses of Theorem 3.5, and the additional hypothesis that the action is totally engaging, we may take $\Gamma_\infty = \Gamma_0$. In particular, Γ is arithmetic.*

One of the motivating questions for the developments in the study of actions of simple groups of higher rank can be formulated as follows:

Question 3.7 If $\mathbb{R}$-rank $(G) \geq 2$, when does $A(M) = M$?

Constructions of "non-standard" actions on manifolds (see [B]) have involved constructions along sets of zero measure. Thus, at the level of measure theory, there are still no known examples in higher rank for which equality does not hold. Without the higher rank assumption, however, $A(M)$ may be a very proper quotient of M. This can be seen via the examples of Furman-Weiss [FW] (which they discussed for different purposes.) Namely, let $G = O(1,n)$ and $\Gamma \subset G$ be a cocompact lattice with a surjective homomorphism $h : \Gamma \to \mathbb{Z}$. Let Y be a compact $\mathbb{Z}$-space with ergodic invariant measure and positive entropy. Let Γ act on Y via h and let $X = (G \times Y)/\Gamma$ be the induced G space. Then X is a compact G-space, with finite entropy. Furthermmore, by continously varying the entropy of the $\mathbb{Z}$-actions on Y, one can continuously (and non-trivially) vary the entropy for elements $g \in G$ acting on X. Since the entropy for arithmetic actions is controlled by Lemma 2.10, this shows that for most such X we will have $A(X) \neq X$, as they cannot have equal entropies. In fact, one can have such examples with $A(X) = G/\Gamma$.

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