Inversion of the Horocycle Transform on Real Hyperbolic Space via Wavelet Transforms

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Abstract

It is proved that the horocycle transform Rf on real n-dimensional hyperbolic space $\mathbf{H}$ is well-defined for $f \in L^p(\mathbf{H})$ if and only if $1 \leq p < 2$. The function f can be recovered explicitly in L^p-norm and a.e. by the formula

$$f(x) = const \times \int_0^\infty (WRf)(x,t)\frac{dt}{t^n}$$

where W is a suitable wavelet transform on the space of horocycles.

1 Introduction

Let $\mathbf{E}^{n,1}$ be $(n+1)$-dimensional pseudo-Euclidean space endowed with the inner product $[x,y] = x_{n+1}y_{n+1} - x_1y_1 - \cdots - x_ny_n$. Real n-dimensional

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hyperbolic space $\mathbf{H}$ can be regarded as the upper sheet of the two sheeted hyperboloid $\mathbf{H}=\{x \in \mathbf{E}^{n,1} \,|\, [x,x]=1,\ x_{n+1}>0\}$. Let Γ be the upper part of the light cone in $\mathbf{E}^{n,1}$, i.e. $\Gamma=\{\xi \in \mathbf{E}^{n,1} \,|\, [\xi,\xi]=0,\ \xi_{n+1}>0\}$. Geometrically, horocycles are planar sections of $\mathbf{H}$ by hyperplanes of the form $[x,\xi]=1$ ([8]). Each hyperplane of this type is parallel to a certain generatrix of Γ. The horocycle transform $Rf(\xi)$ assigns to each sufficiently nice function f on $\mathbf{H}$ the integrals of f over horocycles. For a compactly supported smooth function f one can write

$$Rf(\xi)=\int_{\mathbf{E}^{n,1}} f(x)\,\delta([x,\xi]-1)\,dx, \tag{1}$$

$$f(x)=\begin{cases} \dfrac{(-1)^m}{2(2\pi)^{2m}} \displaystyle\int_\Gamma \delta^{(2m)}([x,\xi]-1)\,Rf(\xi)\,d\xi, & \text{if } \ n=2m+1, \\[2ex] \dfrac{(-1)^m\Gamma(2m)}{(2\pi)^{2m}} \displaystyle\int_\Gamma ([x,\xi]-1)^{-2m} Rf(\xi)\,d\xi, & \text{if } \ n=2m. \end{cases} \tag{2}$$

Here δ is the Dirac delta-function and the integrals are interpreted in a suitable sense ([8], [20, p. 162]). In later works, other inversion formulas were obtained using different methods (e.g. [1], [5], [6], [10], [21]). Typically, these methods are based on the use of the dual transform $\overset{*}{R}$ (which integrates Rf over all horocycles passing through a fixed point $x \in \mathbf{H}$) and/or techniques from harmonic analysis. For functions in more general function classes, e.g. $L^p(\mathbf{H})$ or $C(\mathbf{H})$, the methods do not apply.

In the recent papers ([2], [13] – [16]) explicit inversion formulas for Radon transforms in various settings were obtained in the framework of L^p-space. The basic idea is to include R and $\overset{*}{R}$ into suitable analytic families $\{R^\alpha\}$ and $\{\overset{*}{R}{}^\alpha\}$ of fractional integrals in such a way that the inverse operator R^{-1} belongs to the family $\{\overset{*}{R}{}^\alpha\}$. The operators $\overset{*}{R}{}^\alpha$ give rise to generalized wavelet transforms. In terms of these transforms it is possible to write out the analytic continuation of $\overset{*}{R}{}^\alpha Rf$ in the form which enables us to work with L^p-functions and with continuous functions.

In the present paper we apply this method to the horocycle transform. The required families of fractional integrals were discovered by examining the proof of (2). The general idea is as follows (cf. [16]): since the delta function

$\delta(t)$ is a member of the analytic family of distributions $t_+^{\lambda-1}/\Gamma(\lambda)$ (see [9]) and these distributions generate Riemann-Liouville fractional integrals (and corresponding wavelet transforms [12]), it is natural to expect that the delta function $\delta([x,\xi]-1)$ can be associated with some fractional integrals and wavelet transforms.

The paper is organized as follows. In section 2, we give basic definitions and auxiliary results, and establish the Solmon type estimate for the horocycle transform (cf. [17], [2]). In section 3 we introduce continuous wavelet transforms associated with R and prove the inversion formula mentioned in the Abstract; the main result is given by Theorem 1. The argument explores the connection between our fractional integrals and wavelet transforms with harmonic analysis on $\mathbf{H}$.

2 Preliminaries

2.1 Algebraic and geometric notions

References to this subsection are [3] and [20]. In addition to $\mathbf{E}^{n,1}$, $\mathbf{H}$, Γ, we define the spaces $\mathbf{R}^n = \{x \in \mathbf{E}^{n,1} \,|\, x = (x_1, \dots, x_n, 0)\}$ and $\mathbf{R}^{n-1} = \{x \in \mathbf{E}^{n,1} \,|\, x = (x_1, \dots, x_{n-1}, 0, 0)\}$ with the corresponding rotation groups $K = SO(n)$ and $M = SO(n-1)$. The coordinate unit vectors are denoted by $e_1, \dots, e_{n+1}$. We write dk for the normalized Haar measure on K so that $\int_K dk = 1$. Let $\mathbf{S}^{n-1} = K/M$ be the unit sphere in $\mathbf{R}^n$; $\omega_{n-1} = |\mathbf{S}^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$.

The geodesic distance between points $x, y \in \mathbf{H}$ is defined by $\cosh d(x,y) = [x,y]$. The isometry group of $\mathbf{H}$ is $G = SO_o(n,1)$, the locally compact connected group of pseudo-rotations of $\mathbf{E}^{n,1}$ which preserve the bilinear form $[x,y]$. The subgroup K is the isotropy subgroup of the point $O = (0, \dots, 0, 1) \in \mathbf{H}$, called the origin in $\mathbf{H}$. We then have the homogeneous space identification $\mathbf{H} = G/K$.

The group G possesses a Cartan decomposition $G = KAK$ and an Iwasawa decomposition $G = KAN$ where A is an Abelian subgroup of the form

$$A = \left\{ a = a_t = \begin{bmatrix} I_{n-1} & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{bmatrix} \,|\, t \in \mathbf{R} \right\},$$

and N is a nilpotent subgroup of G given by

$$N = \left\{ n = n_v = \begin{bmatrix} I_{n-1} & -v^{tr} & v^{tr} \\ v & 1 - |v|^2/2 & |v|^2/2 \\ v & -|v|^2/2 & 1 + |v|^2/2 \end{bmatrix} \mid v \in \mathbf{R}^{n-1} (row\, vector) \right\}.$$

Then A normalizes N, i.e. $a_t^{-1} n_v a_t = n_{e^{-t}v}$. The Haar measure dn on N is given by Lebesgue measure dv on $\mathbf{R}^{n-1}$, so that $\int_N f(n)dn = \int_{\mathbf{R}^{n-1}} f(n_v)dv$. Any $g \in G$ has a unique expression $g = kan$ up to the centralizer M of A in K. Any $x \in \mathbf{H}$ can be written uniquely in the form $x = n_v a_t \circ O$. This leads to the horocycle coordinates on $\mathbf{H}$ given by:

$$x = n_v a_t \circ O = a_t n_{e^{-t}v} \circ O = (e^{-t}v, \sinh t + \frac{|v|^2}{2} e^{-t}, \cosh t + \frac{|v|^2}{2} e^{-t}). \quad (3)$$

In terms of the decomposition $x = n_v a_t \circ O$, the invariant Riemannian measure dx on $\mathbf{H}$ has the form $dx = e^{-2\rho t} dt dv$, where $\rho = (n-1)/2$ (the letter ρ has this meaning everywhere throughout the paper).

Horocycles in $\mathbf{H}$ can be defined as translates of the orbit $N \circ O$ under G. Any horocycle has the form $ka_t N \circ O$ for some $k \in K$ and $t \in \mathbf{R}$ (t gives the signed distance of the horocycle to the origin O). We denote the space of horocycles by Ξ. The group G is transitive on Ξ, and the subgroup MN of G leaves fixed the "basic horocycle" $N \circ O$. Hence we have the homogeneous space identification $\Xi = G/MN$. Let $\xi_0 = (0, \ldots, 0, 1, 1)$. Each horocycle $ka_t N \circ O$ is identified uniquely with the point $\xi \in \Gamma$ according to

$$\xi = ka_t \circ \xi_0 = e^t\, k \circ \xi_0 = e^t b(\omega), \quad (4)$$

where $\omega = k \circ e_n \in \mathbf{S}^{n-1}$, $b(\omega) = k \circ \xi_0 = \omega + e_{n+1} \in \Gamma$. In accordance with (4), the invariant measure on Γ is defined by $d\xi = e^{2\rho t} dt d\omega$, dt being Lebesgue measure on $\mathbf{R}$ and $d\omega$ the usual surface measure on $\mathbf{S}^{n-1}$ ([20], p. 24).

Finally notice that for each $x \in \mathbf{H}$ and each $\omega \in \mathbf{S}^{n-1}$ there is a unique horocycle passing through x and given by the point $e^t b(\omega) \in \Gamma$ with

$$t = \langle x, \omega \rangle = -\log[x, b(\omega)] \quad (5)$$

(cf. [3], p. 80). This quantity is usually called the *horocycle distance function.*

2.2 The horocycle transform and its dual

Given $\xi \in \Gamma$, let $\hat{\xi}$ be the horocycle defined by $\hat{\xi} = \{x \in \mathbf{H} \,|\, [x,\xi] = 1\}$, and, given $x \in \mathbf{H}$, let $\check{x} = \{\xi \in \Gamma \,|\, [x,\xi] = 1\}$ be the set of points of the cone Γ corresponding to all horocycles passing through x. We denote by $d_\xi x$ and $d_x \xi$ the induced Lebesgue measures on $\hat{\xi}$ and $\check{x}$ respectively. According to (1), for sufficiently nice functions $f : \mathbf{H} \to \mathbf{C}$ and $\varphi : \Gamma \to \mathbf{C}$ the horocycle transform and its dual are defined by

$$Rf(\xi) = \int_{\hat{\xi}} f(x) d_\xi x \quad \text{and} \quad \overset{*}{R}\varphi(x) = \int_{\check{x}} \varphi(\xi) d_x \xi,$$

respectively. If $\xi = e^t b(\omega) = e^t k \circ \xi_0$ and $x = g \circ O$, $g \in G$, then in group theoretic terms these transforms read as follows

$$Rf(\xi) = R_\omega f(t) = \int_N f(ka_t n \circ O)\, dn \quad \left(= \int_{\mathbf{R}^{n-1}} f(ka_t n_v \circ O)\, dv \right), \tag{6}$$

$$R^*\varphi(x) = \omega_{n-1} \int_K \varphi(gk \circ \xi_o)\, dk, \tag{7}$$

The following statement gives another representation of the dual transform.

Proposition 1 *For each $g \in G$ and each $\omega \in \mathbf{S}^{n-1}$,*

$$\int_K \varphi(gk \circ \xi_o)\, dk = \int_K e^{2\rho\langle g\circ O, k\circ\omega\rangle} \varphi(e^{\langle g\circ O, k\circ\omega\rangle} k \circ \xi_0)\, dk \tag{8}$$

provided that one of these integrals exists.

Proof. We write (8) in the equivalent form $I_1\varphi(g) = I_2\varphi(g)$, where

$$I_1\varphi(g) = \int_{\mathbf{S}^{n-1}} \varphi(g \circ b(\omega))\, d\omega,$$
$$I_2\varphi(g) = \int_{\mathbf{S}^{n-1}} e^{2\rho\langle g\circ O, \omega\rangle} \varphi(e^{\langle g\circ O, \omega\rangle} b(\omega))\, d\omega.$$

Set $g = k'a_rk''$ $(k', k'' \in K,\ a_r \in A)$. One can readily see that $I_1\varphi(g) = I_2\varphi(g)$ if and only if $I_1\varphi'(a_r) = I_2\varphi'(a_r)$, $\varphi'(\xi) = \varphi(k' \circ \xi)$. Thus it suffices to prove (8) for $g = a_r$. By passing to polar coordinates on $\mathbf{S}^{n-1}$ and taking into account the equalities

$$a_r \circ e_n = (\cosh r)e_n + (\sinh r)e_{n+1}, \quad a_r \circ e_{n+1} = (\sinh r)e_n + (\cosh r)e_{n+1},$$

we have

$$\begin{aligned} I_1\varphi(a_r) = \int_{-1}^{1} (1-\eta^2)^{(n-3)/2} \times \int_{\mathbf{S}^{n-2}} \varphi(\sqrt{1-\eta^2}\theta \\ + (\eta\cosh r + \sinh r)e_n + (\eta \sinh r + \cosh r)e_{n+1}) d\theta d\eta, \end{aligned}$$

$$I_2\varphi(a_r) = \int_{-1}^{1} \frac{(1-\tau^2)^{(n-3)/2}}{(\cosh r - \tau \sinh r)^{n-1}} \int_{\mathbf{S}^{n-2}} \varphi\Big(\frac{\sqrt{1-\tau^2}\theta + \tau e_n + e_{n+1}}{\cosh r - \tau\sinh r}\Big)\, d\theta d\tau. \quad (9)$$

The second expression can be reduced to the first one by changing the variable (put $1/(\cosh r - \tau \sinh r) = \eta \sinh r + \cosh r$). ■

Corollary 1 *For $x \in \mathbf{H}$,*

$$\overset{*}{R}\varphi(x) = \int_{\mathbf{S}^{n-1}} e^{2\rho\langle x,\omega\rangle} \varphi(e^{\langle x,\omega\rangle} b(\omega))\, d\omega. \quad (10)$$

Known properties of these transforms are the content of the following lemmas. For convenience of the reader we supply them with simple proofs.

Lemma 1 (cf. formulas (2.8) and (3.1) from [6]). *We assume that f and φ are locally integrable on $\mathbf{H}$ and Γ such that the integrals below exist a.e.*
(i) If f is a K-invariant function on $\mathbf{H}$, i.e. $f(x) = f_o(x_{n+1})$, then Rf is K-invariant, and

$$\begin{aligned} e^{\rho t} R_\omega f(t) &= 2^{\rho-1}\omega_{n-2} \int_{\cosh t}^{\infty} f_o(s)\ (s-\cosh t)^{\rho-1}\, ds & (11) \\ &= 2^{2\rho-1}\omega_{n-2} \int_{u^2}^{\infty} f_o(2\tau - 1)\ \big(\tau - u^2\big)^{\rho-1}\, du, \quad u = \cosh\frac{t}{2}. & (12) \end{aligned}$$

(ii) If φ is a K-invariant function on Γ, i.e. $\varphi(\xi) = \varphi_o(\xi_{n+1})$, then $\overset{*}{R}\varphi$ is K-invariant, and

$$\overset{*}{R}\varphi(x) = \frac{2^{\rho-1}\omega_{n-2}}{(\sinh r)^{n-2}} \int_{-r}^{r} \varphi_o(e^s)\, (\cosh r - \cosh s)^{\rho-1} e^{\rho s} ds, \ \cosh r = x_{n+1}. \quad (13)$$

Proof. (i) Since f is K-invariant, then one can ignore k in (6), and by (3) we have

$$\begin{aligned} R_\omega f(t) &= \int_{\mathbf{R}^{n-1}} f(a_t n_v \circ O)\, dv = e^{-2\rho t} \int_{\mathbf{R}^{n-1}} f(n_v a_t \circ O)\, dv \\ &= e^{-2\rho t} \int_{\mathbf{R}^{n-1}} f_o\Big(\cosh t + \frac{|v|^2}{2} e^{-t}\Big) dv \end{aligned}$$

which gives (9). The representation (10) can be obtained from (9) by putting $\cosh t = 2u^2 - 1$.

(ii) By making use of (9), we obtain

$$\begin{aligned} \overset{*}{R}\varphi(x) &= \omega_{n-2} \int_{-1}^{1} (1-\tau^2)^{\rho-1} (\cosh r - \tau \sinh r)^{-2\rho} \varphi_o\left(\frac{1}{\cosh r - \tau \sinh r}\right) d\tau \\ &= \frac{\omega_{n-2}}{(\sinh r)^{n-2}} \int_{-r}^{r} (\sinh^2 r - (\cosh r - e^{-s})^2)^{\rho-1} e^{s(2\rho-1)} \varphi_o(e^s)\, ds \end{aligned}$$

which coincides with (11). ■

Lemma 2 (Duality; cf. [10] , p. 103). *Let f and φ be functions on $\mathbf{H}$ and Γ, respectively. Then the duality relation*

$$\int_\Gamma \varphi(\xi) Rf(\xi)\, d\xi = \int_{\mathbf{H}} \overset{*}{R}\varphi(x) f(x)\, dx \tag{14}$$

holds provided that at least one of the integrals is finite for φ and f replaced by $|\varphi|$ and $|f|$, respectively.

Proof. By setting $\xi = e^t b(\omega)$, $\omega = k \circ e_n$, we write the left-hand side of (14) in the form (cf. (6))

$$\int_{\mathbf{R}} e^{2\rho t}\, dt \int_{\mathbf{S}^{n-1}} \varphi(e^t b(\omega))\, d\omega \int_{\mathbf{R}^{n-1}} f(k a_t n_v \circ O)\, dv.$$

Put $v = e^{-t}u$, $a_t n_v \circ O = n_u a_t \circ O = y$, $[ky, b(\omega)] = e^{-t}$. Then the above expression can be written as

$$\begin{aligned} &\int_{\mathbf{S}^{n-1}} d\omega \int_{\mathbf{H}} \varphi([ky, b(\omega)]^{-1} b(\omega)) f(ky) [ky, b(\omega)]^{-2\rho}\, dy \\ = &\int_{\mathbf{S}^{n-1}} d\omega \int_{\mathbf{H}} \varphi(e^{-\langle x, \omega\rangle} b(\omega)) f(x) e^{2\rho \langle x, \omega\rangle}\, dx \end{aligned}$$

which coincides with the right-hand side of (14). ■

We use the above results to characterize the behavior of R on $L^p(\mathbf{H})$ mentioned in the Abstract.

Proposition 2 *If $f \in L^p(\mathbf{H})$, $1 \le p < 2$, then $Rf(\xi)$ exists a.e. and*

$$\int_\Gamma |\xi|^{\beta-\rho} \, ||\xi|-1|^{-2\beta} \, |Rf(\xi)| \, d\xi \le C_{n,\beta} \, \|f\|_p \,, \tag{15}$$

provided $1 + (n-1)(1/2 - 1/p) < \beta < \min(1, 1/2 + n/p')$, $1/p + 1/p' = 1$. *Further, there exists an $\widetilde{f} \in L^p(\mathbf{H})$ such that $Rf(\xi) \equiv \infty$, for every $p \ge 2$.*

Proof. Let $\varphi(\xi) = e^{-\rho t}(\cosh t - 1)^{-\beta}$, where $\xi = e^t b(\omega)$. By (11),

$$\overset{*}{R}\varphi(x) = const \, \times (\cosh r - 1)^{1/2-\beta}(\cosh r + 1)^{1-n/2}, \ \cosh r = x_{n+1}.$$

By the conditions on β, for $f \in L^p(\mathbf{H})$, the right-hand side of (14) is an absolutely convergent integral. Consequently (14) is valid and it follows that Rf is defined a.e. Substituting into (14) and applying Hölder's inequality gives the estimate. The examples in the case when $p \ge 2$ can be constructed easily by considering K-invariant functions and using (11). ■

Remark 1 *Conceivably, an analog of the above result can be obtained for general rank one symmetric spaces of non-compact type by making use of the formulas (2.8) and (3.1) from [6]) together with the corresponding duality relation. Undoubtably, the result can be extended to symmetric spaces of any rank, however the computational aspects of the above proof do not seem to generalize easily.*

2.3 Approximate identities on H

Approximate identities have been introduced in the symmetric space setting by various authors, e.g. [18] or [4]. Here we introduce a modification appropriate for our needs. Given an integrable function $k_0 : (0, \infty) \to \mathbf{C}$, consider the convolution operator

$$K_\varepsilon f(x) = \int_{\mathbf{H}} k_\varepsilon([x, y]) \, f(y) \, dy, \tag{16}$$

where $\varepsilon > 0$, and the kernel is given by

$$k_\varepsilon(\tau) = \frac{(\tau^2-1)^{1-n/2}}{\varepsilon} k_0\left(\frac{2(\tau-1)}{\varepsilon}\right).$$

Convolutions of this form arise naturally in the inversion procedure for the horocycle transform given in the next section.

Lemma 3 *Let f be a measurable function on* $\mathbf{H}$.
(i) If k_0 has a decreasing integrable majorant, then

$$K^* f \leq const \times f^*, \tag{17}$$

where $f^*(x)$ is the Hardy-Littlewood maximal function on $\mathbf{H}$ *defined by*

$$f^*(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy,$$

$B(x,r)$ is a geodesic ball of radius r centered at x.
(ii) Let $f \in L^p(\mathbf{H})$ for some $1 \leq p < \infty$. Then

$$\lim_{\varepsilon\to 0} K_\varepsilon f = c_0 f, \qquad c_0 = \frac{\omega_{n-1}}{2}\int_0^\infty k_0(s)\, ds, \tag{18}$$

in the L^p-norm. Further, if k_0 has decreasing integrable majorant, then the limit holds a.e.
(iii) If $f \in C_0(\mathbf{H}) = \{f \in C(\mathbf{H}) \mid f(x) \to 0$ as $d(O,x) \to \infty\}$, then the limit holds uniformly on $\mathbf{H}$.

Proof. (i) By passing to polar coordinates on $\mathbf{H}$, we get

$$\begin{aligned} K_\varepsilon f(x) &= \frac{\omega_{n-1}}{\varepsilon}\int_0^\infty k_0\Big(\frac{2(\cosh r - 1)}{\varepsilon}\Big) M_r f(x)\, d(\cosh r) \\ &= \int_0^\infty k_0(s) u(2+\varepsilon s)\, ds \end{aligned} \tag{19}$$

where $M_r f(x)$ is the mean value of f over the geodesic sphere of radius r centered at x, $u(t) = 2^{-1}\omega_{n-1} M_{\mathrm{arcosh}(t/2)} f$, $t \geq 2$. Properties of $M_r f$ for $f \in L^p(\mathbf{H})$ and $f \in C_0(\mathbf{H})$ were studied, e.g., in [11], [2]. Without loss of generality one can assume $f \geq 0$. Since k_0 has a decreasing integrable majorant, then

$$|K_\varepsilon f| \le c \sup_{h\in\mathbf{R}} \frac{1}{2h}\int_{2-h}^{2+h} u(t)\,dt = \frac{c}{2}\sup_{h>0}\gamma(h) \tag{20}$$

(for $t < 0$, $u(t)$ is defined by zero), where $\gamma(h) = h^{-1}\int_1^{1+h} M_{\operatorname{arcosh}\tau} f\,d\tau$, $c = \text{const}$. Consider the function

$$\psi(t) = \omega_{n-1}\int_1^t (s^2-1)^{n/2-1} M_{\operatorname{arcosh} s} f\,ds = \int_{B(x,\ \operatorname{arcosh} t)} f(y)\,dy.$$

Let $\nu(t) = |B(x, \operatorname{arcosh} t)|$. Since

$$\nu(t) = \int_{[x,y]<t} dy = c\int_1^t (\tau^2-1)^{n/2-1} d\tau,$$

then $\nu(t) = O((t^2-1)^{-n/2})$ for $t \le 2$ and $\nu(t) = O(t^{n-1})$ for $t > 2$. Hence $\psi(t) \le \nu(t) f^*$, and integration by parts yields

$$\begin{aligned}\gamma(h) &= \frac{\omega_{n-1}^{-1}}{h}\int_1^{1+h}(t^2-1)^{1-n/2} d\psi(t)\\ &\le \frac{cf^*}{h}\left[((1+h)^2-1)^{1-n/2}\nu(1+h) + \int_1^{1+h}\nu(t)(t^2-1)^{-n/2} t dt\right] \le Af^*,\end{aligned}$$

A being independent of h. This estimate together with (20) implies (17).

(ii) Let us prove the limit relation (18). By (19),

$$K_\varepsilon f - c_0 f = \frac{\omega_{n-1}}{2}\int_0^\infty k_0(s)\left[M_{\operatorname{arcosh}(1+\varepsilon s/2)} f - f\right]\,ds \to 0$$

as $\varepsilon \to 0$ in the L^p-norm (and uniformly for $f \in C_0(\mathbf{H})$) owing to the properties of $M_r f$ (see Lemma 2.1 from [2]). The a.e. conversgence is then a consequence of (17) due to the estimate $\|f^*\|_p \le \|f\|_p$ (for $p = 1$ the corresponding weak estimate holds; see [7], [19]). ■

3 Inversion of the Horocycle Transform

With the horocycle transform and its dual we associate the following fractional integral operators

$$R^\alpha f(\xi) = c_{n,\alpha}\int_{\mathbf{H}} f(x)\,h_\alpha(x,\xi)\,dx, \tag{21}$$

$$\overset{*}{R}{}^{\alpha}\varphi(x) = c_{n,\alpha}\int_{\Gamma}\varphi(\xi)\,h_{\alpha}(x,\xi)\,d\xi. \tag{22}$$

Here $\operatorname{Re}\alpha > 0$, $\quad \alpha \neq 1, 3, \dots,$ $\quad c_{n,\alpha} = 2^{-\alpha}\pi^{-1/2}\Gamma((1-\alpha)/2)/\Gamma(\alpha/2)$,

$$h_{\alpha}(x,\xi) = |[x,\xi]-1|^{\alpha-1}\,[x,\xi]^{-\rho-\alpha/2} = \left|[x,\xi]^{1/2} - [x,\xi]^{-1/2}\right|^{\alpha-1}[x,\xi]^{-\rho-1/2}.$$

These operators share properties with the horocycle transform and its dual, namely, duality

$$\int_{\mathbf{H}} f(x)\,\overset{*}{R}{}^{\alpha}\varphi(x)\,dx = \int_{\Gamma} R^{\alpha}f(\xi)\,\varphi(\xi)\,d\xi.$$

Furthermore, they intertwine the action of G on $\mathbf{H}$ and on Γ, i.e. $R^{\alpha}f(g\circ\xi) = R^{\alpha}f_g(\xi)$ and $\overset{*}{R}{}^{\alpha}\varphi(g\circ x) = \overset{*}{R}{}^{\alpha}\varphi_g(x)$, where $f_g(x) = f(g\circ x)$ and similarly for φ_g.

The following lemma links these fractional integrals with the horocycle transform and its dual.

Lemma 4 *Let f and φ be smooth compactly supported functions on $\mathbf{H}$ and Γ respectively, and let $r_{\alpha}(t) = 2^{\alpha-1}c_{n,\alpha}e^{-t(\rho+1/2)}\,|\sinh(t/2)|^{\alpha-1}$. Then*

$$R^{\alpha}f(\xi) = \int_{\mathbf{R}} r_{\alpha}(t)\,R_{\omega}f(s-t)\,dt, \quad \xi = e^{s}b(\omega), \tag{23}$$

$$\overset{*}{R}{}^{\alpha}\varphi(x) = \int_{\mathbf{R}} r_{\alpha}(s)\Big[\int_{\mathbf{S}^{n-1}} e^{2\rho(s+\langle x,\omega\rangle)}\varphi(e^{s+\langle x,\omega\rangle}b(\omega))\,d\omega\Big]ds, \tag{24}$$

and the following relations hold

$$\lim_{\alpha\to 0^+} R^{\alpha}f = Rf, \qquad \lim_{\alpha\to 0^+}\overset{*}{R}{}^{\alpha}\varphi = \overset{*}{R}\varphi. \tag{25}$$

Proof. In order to prove (23) we write $\xi = e^{s}k\circ\xi_0$. Then $[x,\xi] = e^{s}[k^{-1}\circ x,\xi_0)]$ and from (21) we have

$$R^{\alpha}f(\xi) = c_{n,\alpha}\int_{\mathbf{H}} f(k\circ x)\left|e^{s/2}[x,\xi_0)]^{1/2} - e^{-s/2}[x,\xi_0)]^{-1/2}\right|^{\alpha-1}$$
$$\times\, e^{-(\rho+1/2)s}\,[x,\xi_0)]^{-\rho-1/2}dx.$$

We now express the integral using horocycle coordinates (3) in the form $x = a_t n \circ O$. Then $[x, \xi_0)] = x_{n+1} - x_n = e^{-t}$, and therefore

$$R^\alpha f(\xi) = c_{n,\alpha} \int_{\mathbf{R}} \int_N f(ka_t n \circ O) \left| e^{(s-t)/2} - e^{-(s-t)/2} \right|^{\alpha-1} e^{-(\rho+1/2)(s-t)} dn\, dt.$$

This is equivalent to (23) by (6). The equality (24) can be derived from (22) by putting $\xi = e^t b(\omega)$, $d\xi = e^{2\rho t} dt d\omega$, $[x, \xi] = \exp(t - \langle x, \omega \rangle)$ (cf. (5)), and further, $t = s + \langle x, \omega \rangle$. The limit relations (25) follow from (23) and (24) due to normalization (cf. [9], Ch. I, Sec. 3.5). ■

In view of (25), the horocycle transform and its dual can be regarded as members of the analytic families of operators $\{R^\alpha\}$ and $\{\overset{*}{R}{}^\alpha\}$, respectively. This is the key observation in our approach. The form of the kernel h_α in (21) and (22) was discovered taking into account calculations in [20, p. 164].

Following the philosophy given in [2, 14, 16], it is natural to expect that the fractional integral R^α can be inverted by the dual operator via the formula $(R^\alpha)^{-1} = \overset{*}{R}{}^{1-n-\alpha}$. Consequently, the inversion formula for the horocycle transform would take the form ($\alpha = 0$)

$$f = \overset{*}{R}{}^{1-n} R f. \tag{26}$$

However, the integral defining $\overset{*}{R}{}^{1-n}$ is divergent in general, hence the right-hand side of the above formula must be interpreted via analytic continuation in a suitable sense. In what follows, this analytic continuation is carried out via a wavelet-like transform.

Specifically, fix a complex-valued function $w \in L^1(0, \infty)$. The *wavelet transform* of a function φ living on Γ is given for $x \in \mathbf{H}$, and $t > 0$ by

$$W\varphi(x,t) = \frac{1}{t} \int_\Gamma \varphi(\xi)\, w\left(\frac{\left| [x,\xi]^{1/2} - [x,\xi]^{-1/2} \right|}{t} \right) [x,\xi]^{-n/2} d\xi. \tag{27}$$

The name wavelet transform is appropriate as the function w (called a wavelet) will be required to satisfy certain growth and moment conditions (specified later). The structure of (27) is motivated by the formula

$$\overset{*}{R}{}^\alpha \varphi(x) = \text{const } \times \int_0^\infty W\varphi(x,t) \frac{dt}{t^{1-\alpha}}, \quad 0 < \operatorname{Re} \alpha < 1, \tag{28}$$

which can be checked by interchanging the order of integration in the right-hand side. This formula can be used to give the analytic continuation of $\overset{*}{R}^{\alpha}$ for $\operatorname{Re}\alpha \leq 0$.

¿From (26) and (28) one expects inversion of the horocycle transform via

$$f(x) = c_w \int_0^\infty WRf(x,t)\frac{dt}{t^n},$$

for suitable wavelets w. The following makes precise the above formula.

Theorem 1 *Let $f \in L^p(\mathbf{H})$ for some $1 \leq p < 2$. Let $w : (0,\infty) \to \mathbf{C}$ satisfy*

$$ess\sup_{s>0}(1+s)^\mu\,|w(s)| < \infty, \text{ for some } \mu > n; \tag{29}$$

$$\int_0^\infty s^j w(s)\,ds = 0, \text{ for } j = 0, 2, \ldots 2[\rho], \qquad \rho = \frac{n-1}{2}. \tag{30}$$

Then

$$\lim_{\varepsilon\to 0}\int_\varepsilon^\infty WRf(x,t)\frac{dt}{t^n} = c_n f(x), \tag{31}$$

where the limit is understood in the L^p-norm and a.e., and

$$c_n = \frac{\omega_{n-1}\omega_{n-2}}{2}\begin{cases}\Gamma(-\rho)\displaystyle\int_0^\infty s^{2\rho}w(s)\,ds, & \text{if } n \text{ is even,}\\[2ex] \dfrac{(-1)^{\rho+1}}{2\rho!}\displaystyle\int_0^\infty s^{2\rho}w(s)\log s\,ds, & \text{if } n \text{ is odd.}\end{cases} \tag{32}$$

Moreover, if $f \in C_0(\mathbf{H}) \cap L^p(\mathbf{H})$ for some $1 \leq p < 2$, then (31) holds uniformly.

The proof will depend on several preliminaries concerning the wavelet transform (27). The growth condition (29) and moment conditions (30) are the reason we call (27) a wavelet transform.

For functions φ defined on Γ, consider the operator

$$B\varphi(x) = \int_\Gamma \varphi(\xi)\,b([x,\xi])\,d\xi. \tag{33}$$

The wavelet transform given by (27) has this structure and we need to determine conditions on the kernel b so that $B\varphi$ is defined when $\varphi = Rf$ for $f \in L^p(H)$ and some $1 \le p < 2$. For this it is useful to express the operator group theoretically as follows. Let $x = g \circ O$ and write $\xi = gka_t \circ \xi_0$. Then

$$\begin{aligned} B\varphi(x) &= \omega_{n-1} \int_{-\infty}^{\infty} \Big[\int_K \varphi(gka_t \circ \xi_0)\, dk \Big] b([O, a_t \circ \xi_0])\, e^{2\rho t} dt \\ &= \omega_{n-1} \int_{-\infty}^{\infty} \overset{*}{M}{}^t \varphi(x)\, b(e^t)\, e^{2\rho t} dt, \end{aligned} \tag{34}$$

where $\overset{*}{M}{}^t\varphi(x)$ is an averaging operator in brackets. The last formula is justified provided one of the integrals is finite with φ and b replaced by $|\varphi|$ and $|b|$, respectively.

Lemma 5 *Let $0 \le \beta \le \rho$. Assume that $b : [0, \infty) \to \mathbf{C}$ and $\varphi : \Gamma \to \mathbf{C}$ are non-negative and measurable with*

$$\gamma_\beta = \operatorname*{ess\,sup}_{s>0} s^{\rho-\beta}(1+s)^{2\beta} b(s) < \infty.$$

Then

$$B\varphi(x) \le c_\beta\, \gamma_\beta\, x_{n+1}^{\rho+\beta} \int_\Gamma |\xi|^{\beta-\rho}\, \big||\xi| - 1\big|^{-2\beta}\, \varphi(\xi)\, d\xi \;\; a.e., \tag{35}$$

where c_β is independent of b.

Proof. We start from (34) to obtain

$$B\varphi(x) \le \gamma_\beta \omega_{n-1} \int_{-\infty}^{\infty} \overset{*}{M}{}^t \varphi(x)\, e^{(\beta-\rho)t} (e^t + 1)^{-2\beta} e^{2\rho t} dt,$$

or as an integral over Γ,

$$B\varphi(x) \le \gamma_\beta \omega_{n-1} \int_\Gamma \varphi(g \circ \xi)\; \Big[\xi_{n+1}^{\beta-\rho} (1 + \xi_{n+1}^{-2\beta}) \Big]\, d\xi, \qquad x = g \circ O.$$

Changing the variable we see it suffices to estimate the quantity in brackets where ξ is replaced by $g^{-1} \circ \xi$. For this in the case $|\xi| > 1$ we have

$$\begin{aligned} \Omega(x, \xi) &\equiv \frac{(g^{-1} \circ \xi)_{n+1}^{\beta-\rho} \left((g^{-1} \circ \xi)_{n+1} + 1\right)^{-2\beta}}{|\xi|^{\beta-\rho}\, \big||\xi| - 1\big|^{-2\beta}} = \frac{|\xi|^{\rho-\beta}\, \big||\xi| - 1\big|^{2\beta}}{(g^{-1} \circ \xi)_{n+1}^{\rho-\beta} \left((g^{-1} \circ \xi)_{n+1} + 1\right)^{2\beta}} \\ &\le \frac{|\xi|^{\rho-\beta}\, \big||\xi| + 1\big|^{2\beta}}{(g^{-1} \circ \xi)_{n+1}^{\rho+\beta}}. \end{aligned}$$

Now $(g^{-1} \circ \xi)_{n+1} = [x, \xi]$ and if we write this in coordinate form ($x = (\cosh s, \sinh s\, \omega)$, $\xi = e^t b(\omega')$), then $(g^{-1} \circ \xi)_{n+1} = e^t(\cosh s - \sinh s\, (\omega \cdot \omega'))$, the dot representing ordinary Euclidean inner product. Hence, $(g^{-1} \circ \xi)_{n+1} \geq e^{t-s} = e^{t-d(x,O)}$. The rest of the estimation is straightforward. If $|\xi| < 1$, then $\Omega(x, \xi) \leq (|\xi|/(g^{-1} \circ \xi)_{n+1})^{\rho-\beta}$ which does not exceed $x_{n+1}^{\rho-\beta} \quad (\leq x_{n+1}^{\rho+\beta})$ up to a constant multiple. ■

For the next lemma we introduce the notation

$$w_1(s) = s^{-1/2} w(s^{1/2})$$

and the associated fractional integral operator

$$I_+^\mu w_1(u) = \frac{1}{\Gamma(\mu)} \int_0^u (u - s)^{\mu-1} w_1(s)\, ds. \tag{36}$$

Lemma 6 *Let $f \in L^p(\mathbf{H})$ for some $1 \leq p < 2$ and let $w \in L^1(0, \infty)$ satisfy*

$$ess \sup_{s>0} (1 + s)^{2\beta+1} |w(s)| < \infty \tag{37}$$

for some $\beta \in [0, \min(1, \rho, 1/2 + n/p')]$.

(i) If f and w are nonnegative, then for each $t > \varepsilon > 0$ and $x \in \mathbf{H}$,

$$WRf(x, t) \leq c_{\beta,\varepsilon} t^{2\beta} x_{n+1}^{\rho+\beta} \|f\|_p \tag{38}$$

with some constant $c_{\beta,\varepsilon}$, independent of f.

(ii) Let $f_x(\cosh r) = M^r f(x)$. Then

$$WRf(x, t) = 2\Gamma(\rho)\omega_{n-1}\omega_{n-2} t^{n-3} \int_1^\infty f_x(2\tau - 1)\, I_+^\rho w_1\left(\frac{4(\tau - 1)}{t^2}\right) d\tau \tag{39}$$

where $f_x(\cosh r) = M^r f(x)$.

Proof. Set $\varphi = Rf$. Then $W\varphi$ has the form of the operator (33) with

$$b(s) = t^{-1} s^{-n/2} w(\left|s^{1/2} - s^{-1/2}\right| / t).$$

We will first demonstrate that $W\varphi$ is well-defined for $f \in L^p(\mathbf{H})$ via the previous lemma. For simplicity assume that w and f are non-negative. By Lemma 5 and Proposition 2 we need to show that for $t > \varepsilon$,

$$t^{-1} s^{-\beta-1/2} (1 + s)^{2\beta} w(\left|s^{1/2} - s^{-1/2}\right| / t) \leq c_{\beta,\varepsilon} t^{2\beta}$$

provided that β satisfies $0 \le \beta \le \rho$ and the hypothesis of Proposition 2 (i.e. $\beta \in [0, \min(1, \rho, 1/2 + n/p')]$). This is a straightforward, albeit tedious calculation as follows. Let $s = e^{2u}$ and $v = |\sinh u|$, then

$$\begin{aligned} e^u &= \cosh u + \sinh u = \begin{cases} \sqrt{1+v^2} + v, & \text{if } u > 0 \\ \sqrt{1+v^2} - v, & \text{if } u < 0 \end{cases} \\ &\ge \sqrt{1+v^2} - v = \frac{1}{\sqrt{1+v^2}+v}. \end{aligned}$$

Consequently, for $t > \varepsilon$,

$$\begin{aligned} t^{-1}s^{-\beta-1/2}(1+s)^{2\beta}w(\left|s^{1/2} - s^{-1/2}\right|/t) &= \frac{(2\cosh u)^{2\beta}}{te^u} w\Big(\frac{2\left|\sinh u\right|}{t}\Big) \\ &\le 4^\beta t^{-1}(1+v^2)^\beta \left(\sqrt{1+v^2}+v\right) w\Big(\frac{2v}{t}\Big) \\ &\le c_\beta t^{-1}(1+v)^{2\beta+1} w\Big(\frac{2v}{t}\Big) \\ &\le \frac{c_\beta}{t} \sup_{s>0} \left(\frac{1+st/2}{1+s}\right)^{2\beta+1} \le c_{\beta,\varepsilon} t^{2\beta}. \end{aligned}$$

Now we will verify (39). From (34) we have the formula

$$W\varphi(x,t) = t^{-1}\omega_{n-1} \int_{-\infty}^{\infty} \overset{*}{M}{}^s \varphi(x)\, w\Big(\frac{2\left|\sinh(s/2)\right|}{t}\Big)\, e^{(\rho-1/2)s} ds.$$

With $\varphi = Rf$, a simple group theoretic argument shows that

$$\overset{*}{M}{}^s \varphi(x) = Rf_x(s).$$

The latter may be computed using (12):

$$Rf_x(s) = 2^{2\rho-1}\omega_{n-2} e^{-\rho s} \int_{u^2}^{\infty} f_x(2\tau - 1)\,(\tau - u^2)^{\rho-1} d\tau,$$

where $u = \cosh(s/2)$. Substituting in the formula for $W\varphi$ we have

$$\begin{aligned} W\varphi(x,t) &= \frac{2^{2\rho-1}\omega_{n-1}\omega_{n-2}}{t} \int_{-\infty}^{\infty} \Big[\int_{\cosh^2(s/2)}^{\infty} f_x(2\tau-1)\,(\tau-\cosh^2(s/2))^{\rho-1} d\tau \Big] \\ &\qquad \times w\Big(\frac{2\,|\sinh s/2|}{t}\Big)\, e^{-s/2} ds \\ &= \frac{2^{2\rho}\omega_{n-1}\omega_{n-2}}{t} \int_{0}^{\infty} \Big[\int_{\cosh^2(s/2)}^{\infty} f_x(2\tau-1)\,(\tau-\cosh^2(s/2))^{\rho-1} d\tau \Big] \\ &\qquad \times w\Big(\frac{2\sinh(s/2)}{t}\Big)\, \cosh(s/2)\, ds. \end{aligned}$$

Making the change of variable $u = \cosh^2(s/2)$ and interchanging the order of integration, we get the following

$$W\varphi(x,t) = \frac{2^{2\rho}\omega_{n-1}\omega_{n-2}}{t} \int_{1}^{\infty} f_x(2\tau-1) \int_{1}^{\tau} w\left(\frac{2\sqrt{u-1}}{t}\right) \frac{(\tau-u)^{\rho-1}}{\sqrt{u-1}} du\, d\tau.$$

Now the change of variable $\eta = 2\sqrt{u-1}/t$ yields:

$$W\varphi(x,t) = 2^{2\rho}\omega_{n-1}\omega_{n-2} \int_{1}^{\infty} f_x(2\tau-1) \int_{0}^{2\sqrt{\tau-1}/t} w(\eta) \left(\tau - 1 - \frac{t^2\eta^2}{4}\right)^{\rho-1} d\eta\, d\tau.$$

The last formula coincides with (39). Please note the use of Fubini's theorem is justified by (i). ■

The following Corollary relates the integral in the left-hand side of (31) to a structural form similar to an approximate identity. The proof of Theorem 1 will then follow by verifying that indeed we have an approximate identity.

Corollary 2 *Let f and w satisfy the conditions of Lemma 6, and let $g(s) = s^{-1} I_+^{\rho+1} w_1(s)$. Then*

$$\int_{\varepsilon}^{\infty} \frac{WRf(x,t)}{t^n} dt = \int_{\mathbf{H}} k_\varepsilon([x,y])\, f(y)\, dy, \tag{40}$$

$$k_\varepsilon(\tau) = \frac{\omega_{n-2}\Gamma(\rho)}{2} \frac{(\tau^2-1)^{1-n/2}}{\varepsilon^2} g\left(\frac{2(\tau-1)}{\varepsilon^2}\right). \tag{41}$$

Proof. Using (39) and interchanging the order of integration we have

$$\int_{\varepsilon}^{\infty} \frac{WRf(x,t)}{t^n} dt = 2\Gamma(\rho)\omega_{n-1}\omega_{n-2} \int_1^{\infty} f_x(2\tau - 1) \int_{\varepsilon}^{\infty} I_+^{\rho} w_1\left(\frac{4(\tau-1)}{t^2}\right) \frac{dt}{t^3}\, d\tau$$

$$= \Gamma(\rho)\omega_{n-1}\omega_{n-2} \int_1^{\infty} f_x(2\tau-1) \int_0^{4(\tau-1)/\varepsilon^2} I_+^{\rho} w_1(z)\, dz \frac{d\tau}{4(\tau-1)}$$

$$= \frac{\Gamma(\rho)\omega_{n-1}\omega_{n-2}}{\varepsilon^2} \int_1^{\infty} f_x(2\tau-1)\, g\Big(\frac{4(\tau-1)}{\varepsilon^2}\Big)\, d\tau$$

$$= \frac{\Gamma(\rho)\omega_{n-1}\omega_{n-2}}{2\varepsilon^2} \int_0^{\infty} M^r f(x)\, g\Big(\frac{2(\cosh r - 1)}{\varepsilon^2}\Big) \sinh r\, dr.$$

The last expression is equivalent to (40). Application of Fubini's theorem was possible due to Lemma 6(i). ■

Finally, we will obtain the proof of Theorem 1 by verifying that k_ε defines an approximate identity. This is accomplished via the following statement which is a consequence of Lemma 2.4 from [14].

Lemma 7 *Let $w \in L^1(0,\infty)$ satisfy (29) and (30). Set $\gamma = \rho + (\mu - n)/4$. Then $\int_0^\infty s^\gamma |w_1(s)|\, ds < \infty$ and the function g from (41) enjoys the following properties:*

(i)

$$g(s) = \begin{cases} O(s^{\rho-1}), & \text{if } 0 < s \le 1, \\ O(s^{\delta-1}), & \text{if } s > 1, \end{cases} \qquad \delta = \rho - \min(1+[\rho], \gamma) < 0; \tag{42}$$

(ii)

$$\int_0^{\infty} g(s)\, ds = \begin{cases} \Gamma(-\rho) \displaystyle\int_0^\infty s^\rho w_1(s)\, ds, & \text{if } \rho \notin \mathbf{N}, \\[2ex] \dfrac{(-1)^{\rho+1}}{\rho!} \displaystyle\int_0^\infty s^\rho w_1(s) \log s\, ds, & \text{if } \rho \in \mathbf{N}. \end{cases} \tag{43}$$

The above Lemma shows that the kernel (41) satisfies the conditions of Lemma 3. Hence we deduce that

$$\lim_{\varepsilon \to 0} \int_\varepsilon^\infty \frac{WRf(x,t)}{t^n} dt = c_n f(x), \quad c_n = \frac{\omega_{n-1}\omega_{n-2}\Gamma(\rho)}{4} \int_0^\infty g(s)\, ds,$$

with the limit interpreted in the required senses. One can easily check that the constant c_n above coincides with that in (32).

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