Duality for Elliptic Curve Orientifolds and Twisted KR-Theory

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some of this joint with C. Doran and S. Mendez-Diez
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Basic Ideas of String Theory

The basic idea of string theory is to replace point particles (in conventional physics) by one-dimensional "strings." At ordinary (low) energies these strings are extremely short, on the order of the Planck length,

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.616 \times 10^{-35}\,\mathrm{m}\,.$$

A string moving in time traces out a two-dimensional surface called a worldsheet. The most basic fields in string theory are thus maps $\varphi\colon \Sigma \to X$, where Σ is a 2-manifold (the worldsheet) and X is spacetime.

String theory offers [some] hope for combining gravity with the other forces of physics and quantum mechanics.

Strings and Sigma-Models

Let Σ be a string worldsheet and X the spacetime manifold. String theory is based on the nonlinear sigma-model, where $\varphi\colon\ \Sigma \to X$ and the leading terms in the action are

$$S(\varphi) = \frac{1}{4\pi\alpha'} \int_\Sigma \|\nabla\varphi\|^2 \, d\mathrm{vol} + \int_\Sigma \varphi^*(B), \tag{1}$$

the energy of the map φ (in Euclidean signature) plus the Wess-Zumino term based on the B-field B. $1/(2\pi\alpha')$ is the string tension. B is a locally defined 2-form on X (really associated to a gerbe).

We have to add to this various gauge fields (giving rise to the fundamental particles) and a "gravity term" involving the scalar curvature of the metric on X. Usually we also require supersymmetry; this means the theory involves both bosons and fermions and there are symmetries interchanging the two.

Calabi-Yau Manifolds

It turns out that not every classical sigma-model quantizes to a consistent quantum field theory. In general one needs certain anomalies to cancel for this to happen. For superstring theories, anomaly cancellation requires $\dim X = 10$. Since ordinary (observable) spacetime is $\mathbb{R}^4$, 4-dimensional Minkowski space, usually one requires $X = \mathbb{R}^4 \times M^6$, where M^6 is a 6-manifold, often assumed compact (though this isn't necessary).

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String Theories and Mirror Symmetry

There are several variants of superstring theory. For our purposes, the most important ones are called types IIA and IIB, which involve different chirality conditions on the fermionic fields: in IIA, the left-moving and right-moving spinors have opposite handedness, and in IIB, they have the same handedness. For the case $X = \mathbb{R}^4 \times M$ with M a Calabi-Yau 3-fold, these theories emphasize different aspects of the geometry of M: the symplectic geometry of the Kähler form and the holomorphic geometry.

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D-Brane Charges and K-Theory

Physicists talk about both closed and open strings. Both kinds of strings are given by compact manifolds, but in the "open" case there is a boundary. So to get a reasonable theory one has to impose Dirichlet or Neumann boundary conditions on some submanifold Y of X where the boundary of Σ must map. These submanifolds are traditionally called D-branes, "D" for *Dirichlet* and *brane* from *membrane*.

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Orientifolds

One can construct many more string theories out of the basic Type II theories by considering orientifold theories. In these theories, the spacetime manifold X is equipped with an involution ι. The inclusion $\varphi\colon \Sigma \to X$ of a string worldsheet into X is required to be equivariant for the involution Ω on Σ given by the worldsheet parity operator. The Chan-Paton bundle on a D-brane then has to have a conjugate-linear involution compatible with ι, and so D-brane charges live in (a variant of) $KR^*(X, \iota)$, which is the K-theory of bundles with such an involution. We'll discuss this later.

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The involution ι does not have to be free. In general, its fixed set will have several components, called O-planes ("O" for orientifold). On a given O-plane, the restriction of the Chan-Paton bundle must have a real or symplectic structure, giving a class in KO^* or KSp^* of the O-plane. We refer to O^+ and O^- planes in these two cases.

Elliptic Curve Orientifolds

To get a consistent orientifold string theory on an elliptic curve, the involution must be holomorphic in type IIB, anti-holomorphic in type IIA. Since elliptic curves are algebraic, that means that a IIA elliptic curve orientifold is basically the same as a smooth (projective) elliptic curve defined over $\mathbb{R}$.

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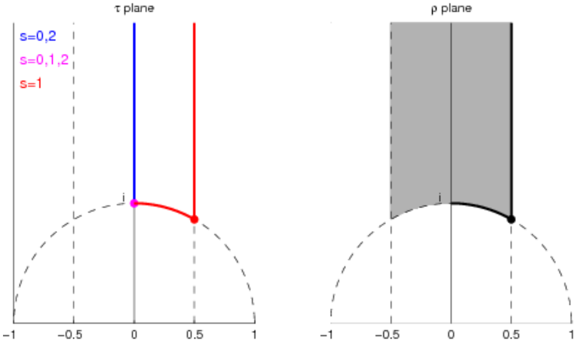
Elliptic curve Orientifolds (cont'd)



Normal Forms

Complex and algebraic geometers as far back as the 18th and 19th century gave normal forms for elliptic curves. Most familiar is the Weierstraß form

$$y^2 = x^3 + ax + b,$$

giving a parameterization $x = 4\wp(z)$, $y = 4\wp'(z)$ in terms of Weierstraß elliptic functions. However, for our purposes, it is better to work with the Jacobi/Legendre normal form

$$y^2 = \pm(1 \pm x^2)(1 \pm k^2 x^2)$$

and a parameterization in terms of Jacobi elliptic functions sn, cn, etc. Real curves of different species are obtained by varying the signs.

Dualities in String Theory

A rather amazing discovery of the "second string revolution" is that there appear to be many nontrivial dualities between different string theories, that is theories with a very different appearance, living on different spacetimes, that still predict the same physics. Mirror symmetry gives some, but not the only, examples of such dualities. Other basic examples are T-dualities, that involve replacing circles in spacetime by dual circles, and interchanging winding and momentum modes.

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- the species $s = 0, 1, 2$ in the type IIA cases;
- topology of the involution and the $+$ or $-$ charges (real or symplectic Chan-Paton bundles) of the O-planes;
- whether the B-field has value 0 or $\frac{1}{2}$ in the type IIB cases.

T-Duality Groupings

Physicists (including Agnotti, Witten, Gao-Hori) had conjectured that these various theories should break up into 3 groupings, with the theories in each group all related to one another by T-dualities. The arguments for this were based on highly non-rigorous physical intuition. We set out to determine if this is the case, and to try to find explanations for these groupings using algebraic geometry and algebraic topology. The groups listed by type and fixed set are:

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IIB, trivial inv., $B = 0$	IIB, trivial inv., $B = \frac{1}{2}$	IIB, free inv.
IIB, $\{+,+,+,+\}$	IIB, $\{+,+,+,-\}$	IIB, $\{+,+,-,-\}$
IIA, species 2	IIA, species 1	IIA, species 0
		IIA, species 2, mixed signs

Atiyah's KR-theory

A variant of K-theory, called KR or Real K-theory (with a capital R!) was introduced by Atiyah in the famous paper "K-theory and reality" in 1968. This is a theory defined on the category of Real spaces, locally compact spaces X with an involution ι (a self-homeomorphism of X with $\iota^2 = 1$). The motivating example is X the complex points of an algebraic variety defined over $\mathbb{R}$, with ι the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$.

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For (X, ι) a compact Real space, we define $KR(X)$ (usually the ι will be implicit) to be the Grothendieck group of the Real vector bundles over X, pairs (E, χ), with E a complex vector bundle over X and $\chi\colon E \to E$ a involutive conjugate-linear isomorphism compatible with ι. Note that when ι is trivial, this is equivalent to giving a real vector bundle over X, and E is just its complexification.

KR periodicity

We extend KR to a theory with compact supports on locally compact spaces. As usual it comes with a cup-product coming from the tensor product of vector bundles. Let $\mathbb{R}^{p,q}$ be $\mathbb{R}^p \oplus \mathbb{R}^q$ with the involution ι that is the identity on the first summand and -1 on the second summand. (Caution: Atiyah calls this $\mathbb{R}^{q,p}$ with p and q reversed. People seem to be divided 50/50 on the notation.) Let $S^{p,q}$ denote the unit sphere in $\mathbb{R}^{p,q}$; topologically this is S^{p+q-1}, but the involution depends on p and q. For instance it is the antipodal map in the case of $S^{0,q}$. Let $KR^{p,q}(X) = KR(X \times \mathbb{R}^{p,q})$. The Bott element β lives in $KR^{1,1}(\mathrm{pt})$.

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Theorem (Atiyah)

Cup-product with β is an isomorphism $KR^{p,q}(X) \to KR^{p+1,q+1}(X)$ for any X and any p, q. Thus $KR^{p,q}(X)$ only depends on $p - q$, and it's periodic with period 8 in this index.

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There are natural isomorphisms $KR(X \times S^{0,1}) \cong K(X)$ *and* $KR(X \times S^{0,2}) \cong KSC(X)$ *(self-conjugate* K*-theory of Anderson and Green).* $KR(X \times S^{0,4})$ *is* 8*-periodic. For* $p \geq 3$ *there are short exact sequences*

$$0 \to KR^{-q}(X) \to KR^{-q}(X \times S^{0,p}) \to KR^{p+1-q}(X) \to 0.$$

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If the involution ι *on* X *is free, then* $KR^{-q}(X)$ *is* 4*-periodic.*

This "explains" the 4-periodicity of KSC. But the theorem is false ☹ in general; it contradicts the 8-periodicity of $KR^{-q}(S^{0,4})$.

KR for free involutions

In the case X compact and ι free, what happens more precisely is this. Locally, $X \cong Y \times S^{0,1}$ (where $S^{0,1}$ is two points, interchanged by the involution), and $KR^*(X) \cong K^*(Y)$. However, this is not true globally. However, there is a spectral sequence, the analogue of the Atiyah-Hirzebruch spectral sequence, $H^p(X/\iota, \utilde{KR^q(S^{0,1})}) \Rightarrow KR^{p+q}(X)$. Here $\utilde{KR^q(S^{0,1})}$ is a sheaf locally isomorphic to $\mathbb{Z}$ for q even, and is 0 for q odd.

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An example of the spectral sequence for KR

Recall that $KSC^* = KR^*(S^{0,2})$. So take $X = S^{0,2}$, $X \to X/\iota$ a 2-to-1 covering map. We have $E_2^{p,q} = 0$ unless $p = 0$ or 1 and q is even. For $q \equiv 0 \pmod 4$, we have $E_2^{p,q} = H^p(S^1, \mathbb{Z}) = \mathbb{Z}$ for $p = 0$, 1. For $q \equiv 2 \pmod 4$, we have $E_2^{p,q} = H^p(S^1, \underset{\sim}{\mathbb{Z}})$. This cohomology with local coefficients is the same as $H^p_{\text{group}}(\mathbb{Z}, \underline{\mathbb{Z}})$, where $\underline{\mathbb{Z}}$ is the $\mathbb{Z}$-module isomorphic to $\mathbb{Z}$ as an abelian group, but on which 1 (the generator of the group) acts by -1. The spectral sequence looks like:

$$
\begin{array}{rc|cc}
 & q & p = 0 & p = 1 \\
 & 4 & \mathbb{Z} & \mathbb{Z} \\
 & 3 & 0 & 0 \\
\text{periodicity } \cong & 2 & 0 & \mathbb{Z}/2 \\
 & 1 & 0 & 0 \\
 & 0 & \mathbb{Z} & \mathbb{Z} \dashrightarrow p.
\end{array}
$$

We see that KSC^* is 4-periodic with groups $\mathbb{Z}$, $\mathbb{Z}$, 0, $\mathbb{Z}/2$.

Connection with noncommutative geometry

All the standard variants of K-theory — K, KO, KSp, KSC, and KR — can be unified by thinking of them as topological K-theory for various Banach algebras (in fact, C^*-algebras) over $\mathbb{R}$. For X locally compact, we have

$$\begin{cases} K^{-q}(X) = K_q(C_0(X)), \\ KO^{-q}(X) = KO_q(C_0^{\mathbb{R}}(X)), \\ KSp^{-q}(X) = KO_q(C_0^{\mathbb{H}}(X)), \\ KSC^{-q}(X) = KO_q(C_0^{\mathbb{R}}(X) \otimes T), \end{cases}$$

where $T = \{f \in C([0,1]) \mid f(0) = \overline{f(1)}\}$. In addition, if (X, ι) is a Real space, then $KR^{-q}(X) = KO_q(C_0(X, \iota))$, where $C_0(X, \iota) =_{\text{def}} \{f \in C_0(X) \mid f(x) = \overline{f(\iota x)}\}$.

Twistings from noncommutative geometry

All of the K-groups K, KO, KSp, KSC, and KR have twisted versions that are special cases of the K-theory of real continuous-trace (CT) algebras. I originally studied these back in the 1980's for purely operator-algebraic reasons, but they also arise in modern physics.

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All of the K-groups K, KO, KSp, KSC, and KR have twisted versions that are special cases of the K-theory of real continuous-trace (CT) algebras. I originally studied these back in the 1980's for purely operator-algebraic reasons, but they also arise in modern physics. A complex C^*-algebra A is said to have continuous trace if $\widehat{A}$ is Hausdorff and if the continuous-trace elements

$$\{a \in A_+ \mid \operatorname{Tr} \pi(a) < \infty \, \forall \pi \in \widehat{A}, \text{ and } \pi \mapsto \operatorname{Tr} \pi(a) \text{ continuous on } \widehat{A}\}$$

are dense in A_+. A real C^*-algebra A is said to have continuous trace if its complexification does. Note that commutative real C^*-algebras automatically have continuous trace.

Structure theory of CT algebras

A structure theory for (complex) continuous-trace algebras was developed by Dixmier and Douady in the 1960's. They showed that if X is locally compact and second countable, and if A is a separable (complex) CT algebra with spectrum X, then A is determined up to stable isomorphism (or Morita equivalence) by a Dixmier-Douady class $\delta \in H^3(X, \mathbb{Z})$. This class classifies a principal PU-bundle over X, and since $PU(\mathcal{H}) = \operatorname{Aut} \mathcal{K}(\mathcal{H})$, there is an associated bundle of algebras $\mathcal{A}$ over X with fibers $\mathcal{K}$, and $A \otimes \mathcal{K} \cong \Gamma_0(X, \mathcal{A})$.

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The real case is more complicated. A real CT algebra is built out of three pieces of real, quaternionic, and complex type, respectively. These are locally isomorphic to $C_0^{\mathbb{R}}(X) \otimes \mathcal{K}(\mathcal{H}_{\mathbb{R}})$, $C_0^{\mathbb{R}}(X) \otimes \mathcal{K}(\mathcal{H}_{\mathbb{H}})$, and $C_0(X) \otimes \mathcal{K}(\mathcal{H}_{\mathbb{C}})$, respectively.

Twisted K-theory

Twisted (complex) K-theory of X with twisting $\delta \in H^3(X, \mathbb{Z})$ can be defined simply to be $K_*(A)$, where A is a CT algebra with spectrum X and Dixmier-Douady class δ. When $\delta = 0$, A is Morita equivalent to $C_0(X)$, and we get back $K^{-*}(X)$.

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Twisted (complex) K-theory of X with twisting $\delta \in H^3(X, \mathbb{Z})$ can be defined simply to be $K_*(A)$, where A is a CT algebra with spectrum X and Dixmier-Douady class δ. When $\delta = 0$, A is Morita equivalent to $C_0(X)$, and we get back $K^{-*}(X)$. In a similar fashion, since $\operatorname{Aut} \mathcal{K}(\mathcal{H}_{\mathbb{R}}) = PO$, which is a $K(\mathbb{Z}/2, 1)$ space, algebras locally Morita equivalent to $C_0^{\mathbb{R}}(X)$ are classified by an invariant $w \in H^2(X, \mathbb{Z}/2)$, which one can think of as a Stiefel-Whitney class or the real analogue of the Dixmier-Douady class, and one gets twisted KO-groups $KO^{-j}(X, w) = KO_j(CT^{\mathbb{R}}(X, w))$, which appear, for example, in the Poincaré duality theorem for KO of non-spin manifolds. And since $\operatorname{Aut} \mathcal{K}(\mathcal{H}_{\mathbb{H}}) = PSp$, which is also a $K(\mathbb{Z}/2, 1)$ space, we also get groups $KSp^{-j}(X, w)$ for $w \in H^2(X, \mathbb{Z}/2)$.

KR-theory with a sign choice

For some applications to physics, we need another variant KR_α of KR-theory for Real spaces (X, ι) with an added decoration: a choice α of a $\pm$ sign on each component of the fixed set X^ι. This is required to have the following properties (which almost but don't quite determine it). Let $Y^\pm$ be the union of the fixed set components with sign $\pm$, and let $Z = X \smallsetminus X^\iota$. Let $X^\pm = Z \cup Y^\pm$. Thus $X = X^+ \cup X^-$, $X^\iota = Y^+ \cup Y^-$, and $X^+ \cap X^- = Z$.

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1. $KR^*_\alpha(X^+) = KR^*(X^+)$, and $KR^*_\alpha(X^-) = KH^*(X^-) \cong KR^{*+4}(X^-)$. ($KH$ is the quaternionic analogue of KR. The dimension shift by 4 comes from the fact that $KSp^* \cong KO^{*+4}$.)

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2. KR^*_α satisfies Bott periodicity with period 8 and is functorial for maps of Real spaces preserving the sign decoration.

Constructing KR-theory with a sign choice

Given a Real space (X, ι), we attach the real C^*-algebra $C_0(X^+, \iota) \otimes \mathcal{K}_{\mathbb{R}}$ to X^+. This has topological K-theory $KR^{-*}(X^+)$. To X^- we attach the real C^*-algebra $C_0(X^-, \iota) \otimes \mathbb{H} \otimes \mathcal{K}_{\mathbb{R}}$, which has topological K-theory $KH^{-*}(X^-)$. These algebras "agree" over Z since $\mathbb{H}$ splits over $\mathbb{C}$ (in the sense of the theory of central simple algebras). So we clutch together and get an algebra whose topological K-theory we can call $KR_\alpha^{-*}(X)$. There is an exact sequence

$$\cdots \to KR^{-j}(Z) \to KR_\alpha^{-j}(X) \to KO^{-j}(Y^+) \oplus KSp^{-j}(Y^-) \xrightarrow{\delta} KR^{-j+1}(Z) \to \cdots .$$

General twisted KR-theory

All of these twistings, including the sign choice on the fixed sets, have been unified in work of Moutuou. He constructs and computes a graded Brauer group of graded real CT algebras over a Real space (X, ι). The equivalence relation is Morita equivalence over X and the group operation is graded tensor product (over X).

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$$BrR(X, \iota) \cong H^0(X^\iota, \mathbb{Z}/2) \oplus H^2_\iota(X, \mathcal{S}),$$

where the first summand is the group of sign choices and the second group is equivariant sheaf cohomology for the Real sheaf $\mathcal{S}$ of germs of S^1-valued continuous functions and we use the complex conjugation involution on S^1. The second summand encodes the (Real) Dixmier-Douady class. When $X = X^\iota$, this summand reduces to $H^2(X, \mathbb{Z}/2)$, since $\mathbb{Z}/2$ is the fixed-point subgroup of the circle.

KR theories for elliptic curve orientifolds

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IIB	T^2 with w_2	$S^{2,0} \times S^{2,0}$	$KO^{*-1} \oplus KO^{*-1} \oplus K^*$
IIB	$\{++++\}$	$S^{1,1} \times S^{1,1}$	$KO^{*+2}(T^2)$
IIB	$\{++--\}$	$S^{1,1} \times S^{1,1}$	$KSC^{*+2} \oplus KSC^{*+1}$
IIB	$\{+++-\}$	$S^{1,1} \times S^{1,1}$	$KO^{*+1} \oplus KO^{*+1} \oplus K^*$
IIB	$\emptyset$	$S^{2,0} \times S^{0,2}$	$KSC^* \oplus KSC^{*-1}$

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IIB	$\emptyset$	$S^{2,0} \times S^{0,2}$	$KSC^* \oplus KSC^{*-1}$
IIA	$S^1 \amalg S^1$	$S^{1,1} \times S^{2,0}$	$KO^{*+1}(T^2)$
IIA	$S^1_+ \amalg S^1_-$	$S^{1,1} \times S^{2,0}$	$KSC^{*+1} \oplus KSC^*$
IIA	S^1	not a product	$KO^* \oplus KO^* \oplus K^{*-1}$
IIA	$\emptyset$	$S^{1,1} \times S^{0,2}$	$KSC^{*+1} \oplus KSC^*$

Connections with physics

The chart on the last slide nicely matches what was predicted by physicists just using physical T-duality arguments. The orientifold theories on an elliptic curve fit into 3 families, where the theories in a family are related to one another by T-dualities. To these 3 families we could add a 4th family, the case of ordinary type IIA or IIB string theory with no orientifold structure (or equivalently, orientifolds on a disjoint union of two elliptic curves, where the involution switches the two factors), for which we get $K^*(T^2)$.

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Confirmation from Algebraic Geometry

It turns out that algebraic geometry provides independent evidence for the T-duality groupings. The arguments for this are a bit complicated, but basically the idea is to look at the Jacobi/Legendre normal forms and study the behavior of the Jacobi functions under the involutions. Complex conjugate zeros of the parameterizing Jacobi functions are associated to O^+-O^- pairs, whereas real zeros correspond to (unpaired) O^+-planes. Again everything matches with the physics predictions.

Connection with the Baum-Connes Conjecture

So far we just computed the twisted KR groups and observed that within the T-duality groupings, they are abstractly isomorphic. But from where does one get the actual isomorphisms between these groups?

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So far we just computed the twisted KR groups and observed that within the T-duality groupings, they are abstractly isomorphic. But from where does one get the actual isomorphisms between these groups? It turns out these come from the real Baum-Connes conjecture, which asserts that for a discrete group G, there is an isomorphism (given by a specific index map)

$$\mu\colon\ KO_*^G(\mathcal{E}G) \to KO_*(C^*_{\mathbb{R}}(G)).$$

Here $\mathcal{E}G$ is the universal proper G-space. If G is torsion-free, the left-hand side is just $KO_*(BG)$. The conjecture is a *theorem* for G amenable and in many other cases.

Example

Let's take G to be the following solvable group:

$$G = \langle a, b, c \mid ab = cba, c^2 = 1 \rangle.$$

For this group, $\mathcal{E}G = \mathbb{R}^2$ and each side of the Baum-Connes conjecture splits into two pieces. The index map μ canonically splits into a direct sum of two different Baum-Connes maps. First of these is the Baum-Connes map for $\mathbb{Z}^2$ (corresponding to setting $c = 1$), which gives an isomorphism $KO_j(T^2) \xrightarrow{\cong} KR^{-j}(S^{1,1} \times S^{1,1})$, or (after applying Poincaré duality) $KO^{2-j}(T^2) \xrightarrow{\cong} KR^{-j}(S^{1,1} \times S^{1,1})$.

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