From Large Cardinals to Large Combinatorial Properties

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Inaccessible cardinals

Definition

A cardinal $\kappa$ is **weakly inaccessible** if it satisfies

- for every cardinal $\gamma < \kappa$, $\gamma^+ < \kappa$;
- for every sequence $\langle \kappa_i \rangle_{i < \gamma}$ of ordinals less than $\kappa$ with $\gamma < \kappa$, the supremum $\sup_{i < \gamma} \kappa_i < \kappa$.

Definition

A cardinal $\kappa$ is **(strongly) inaccessible** if it satisfies

- for every cardinal $\gamma < \kappa$, $2^\gamma < \kappa$;
- for every sequence $\langle \kappa_i \rangle_{i < \gamma}$ of ordinals less than $\kappa$ with $\gamma < \kappa$, the supremum $\sup_{i < \gamma} \kappa_i < \kappa$. 
Inaccessible cardinals imply $ZFC$ is consistent

Von Neuman Hierarchy

\[
V_0 := \emptyset; \\
V_{\alpha + 1} := P(V_{\alpha}); \\
V_\alpha := \bigcup_{\beta < \alpha} V_{\beta}. \\
V := \bigcup_{\alpha \in \text{Ord}} V_{\alpha} \text{ is the set-theoretic universe.}
\]
The hierarchy of large cardinals

Vopenka’s Principle

→

Extendible

→

Supercompact

Superstrong

→

Strongly compact

→

Woodin

→

Strong

→

\(0^+\) exists

→

Measurable

→

Ramsey

→

Weakly Compact

→

Mahlo

→

Inaccessible
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Weakly compact, Strongly compact and Supercompact cardinals

Definition
Given a cardinal $\kappa$,
- $\kappa$ is weakly compact if any collection of sentences of the the infinitary language $L_{\kappa,\kappa}$ using at most $\kappa$ non-logical symbols, if $\kappa$-satisfiable, is satisfiable;
- $\kappa$ is strongly compact if any collection of sentences of the the infinitary language $L_{\kappa,\kappa}$, if $\kappa$-satisfiable, is satisfiable;

Proposition
A cardinal $\kappa$ is strongly compact if and only if, for every $\theta$, there exists an elementary embedding $j : V \rightarrow M$ of $V$ into an inner model $M$ with critical point $\kappa$ such that for every $X \subseteq M$ of size $\leq \theta$, there exists $Y \in M$ such that $X \subseteq Y$ and $M \models |Y| < j(\kappa)$.

Definition
A cardinal $\kappa$ is supercompact if and only if, for every $\theta$, there exists an elementary embedding $j : V \rightarrow M$ of $V$ into an inner model $M$ with critical point $\kappa$ such that $j(\kappa) > \theta$ and $M$ is closed by $\theta$-sequences.
Weakly compact, Strongly compact and Supercompact cardinals

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A cardinal $\kappa$ is supercompact if and only if, for every $\theta$, there exists an elementary embedding $j : V \rightarrow M$ of $V$ into an inner model $M$ with critical point $\kappa$ such that $j(\kappa) > \theta$ and $M$ is closed by $\theta$-sequences.
The Tree Property

A $\kappa$-tree, for a regular $\kappa$, is a tree of height $\kappa$ and levels of size $< \kappa$. 
A $\kappa$-tree, for a regular $\kappa$, is a tree of height $\kappa$ and levels of size $<\kappa$.

A regular cardinal $\kappa$ satisfies the tree property if, and only if, every $\kappa$-tree has a cofinal branch.
The Tree Property

A $\kappa$-tree, for a regular $\kappa$, is a tree of height $\kappa$ and levels of size $<\kappa$.

A regular cardinal $\kappa$ satisfies the tree property if, and only if, every $\kappa$-tree has a cofinal branch.
The Tree Property

Let $\kappa$ be a regular cardinal.

**Theorem**

- (König’s Lemma 1936) $\aleph_0$ satisfies the tree property;
- (Aronszajn 1934) $\aleph_1$ does not satisfy the tree property;
- (Specker 1949) If $\tau^{<\tau} = \tau$, then the tree property fails at $\tau^+$;
- (Mitchell 1972) If $\text{Cons}(\text{ZFC} + \exists \kappa \text{ weakly compact})$, then for every regular $\tau$ such that $\tau^{<\tau} = \tau$, we have $\text{Cons}(\text{ZFC} + \tau^{++} \text{ has the tree property})$. 
Erdös & Tarski 1961

\( \kappa \) is weakly compact iff it is inaccessible and it satisfies the tree property.


\( \kappa \) is strongly compact iff it is inaccessible and it satisfies the strong tree property.


\( \kappa \) is supercompact iff it is inaccessible and it satisfies the super tree property.
Erdős & Tarski 1961

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The Strong Tree Property

Definition

Let $\lambda \geq \kappa$, a $(\kappa, \lambda)$-tree is a subset $F \subseteq \{f : X \to 2; \ X \in [\lambda]^{< \kappa}\}$ such that:

1. for all $f \in F$, if $X \subseteq \text{dom}(f)$, then $f \upharpoonright X \in F$;
2. for all $X \in [\lambda]^{< \kappa}$, $\text{Lev}_X(F) := \{f \in F; \ \text{dom}(f) = X\} \neq \emptyset$ and has size $< \kappa$. 
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$$f : X \to 2$$

$$X \quad < \kappa$$
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\[
\begin{array}{c}
\text{Y} \\
\text{UI} \\
\text{X}
\end{array}
\]

\[
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\]

\[
< \kappa
\]

\[
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\[ \begin{array}{c}
\xymatrix{
X 
& f : X \rightarrow 2 \\
Y \ar[ur] & \text{Lev}_X(F) \\
& Z \ar[ul]
} \\
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Definition

A cofinal branch for a $(\kappa, \lambda)$-tree $F$ is a function $b : \lambda \rightarrow 2$ such that $b \upharpoonright X \in \text{Lev}_X(F)$, for all $X \in [\lambda]^{<\kappa}$.

Definition

$\kappa$ (regular) satisfies the Strong Tree Property if for all $\lambda \geq \kappa$, every $(\kappa, \lambda)$-tree has a cofinal branch.
The Strong Tree Property

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The Super Tree Property

**Definition**

Let $F$ be a $(\kappa, \lambda)$-tree. A sequence $D := \langle d_X; X \in [\lambda]^{<\kappa} \rangle$ is an $F$-level sequence if $d_X \in \text{Lev}_X(F)$, for all $X \in [\lambda]^{<\kappa}$.
Definition

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The Super Tree Property

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Definition

Let $F$ be a $(\kappa, \lambda)$-tree and $D := \langle d_X; X \in [\lambda]^{<\kappa} \rangle$ an $F$-level sequence. An ineffable branch for $D$ is a cofinal branch $b : \lambda \to 2$ such that

$$\{ X \in [\lambda]^{<\kappa}; b \upharpoonright X = d_X \}$$

is stationary.

Definition

$\kappa$ satisfies the Super Tree Property if, for all $\lambda \geq \kappa$ and for all $(\kappa, \lambda)$-tree $F$, every $F$-level sequence has an ineffable branch.
The Super Tree Property

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Let $F$ be a $(\kappa, \lambda)$-tree. A sequence $D := \langle d_X; \ X \in [\lambda]^{<\kappa} \rangle$ is an $F$-level sequence if $d_X \in \text{Lev}_X(F)$, for all $X \in [\lambda]^{<\kappa}$.

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Definition

$\kappa$ satisfies the Super Tree Property if, for all $\lambda \geq \kappa$ and for all $(\kappa, \lambda)$-tree $F$, every $F$-level sequence has an ineffable branch.
If $\text{Cons}(ZFC + \exists \langle \kappa_n \rangle_{n < \omega}$ supercompact cardinals), then
$\text{Cons}(ZFC + \forall n \geq 2, \aleph_n \text{ has the Super Tree Property})$.

If $\text{Cons}(ZFC + \exists \langle \kappa_n \rangle_{n < \omega}$ supercompact cardinals), then
$\text{Cons}(ZFC + \aleph_{\omega + 1} \text{ has the Strong Tree Property})$. 
Weiss

Let $n \geq 2$, if $\text{Cons}(\text{ZFC} + \exists \kappa \text{ supercompact})$, then $\text{Cons}(\text{ZFC} + \aleph_n \text{ has the Super Tree Property})$.

\[ \kappa \quad \text{inaccessible} \quad + \text{super tree property} \]

use $\mathcal{M}(\aleph_n, \kappa)$. 
The Super Tree Property at Small Cardinals

Weiss

Let $n \geq 2$, if $\text{Cons}(\text{ZFC} + \exists \kappa \text{ supercompact})$, then $\text{Cons}(\text{ZFC} + \aleph_n \text{ has the Super Tree Property})$.

\[ \kappa = \aleph_{n+2} \]

+ super tree property

use $M(\aleph_n, \kappa)$. 
The strong and super tree properties at small cardinals

The Super Tree Property at Small Cardinals

Fontanella - Main Theorem 1

If $\text{Cons}(\text{ZFC} + \exists \langle \kappa_n \rangle_{n<\omega}$ supercompact cardinals), then $\text{Cons}(\text{ZFC} + \forall n \geq 2, \mathbb{N}_n$ has the Super Tree Property $)$. 

\[ \begin{align*} \lambda & \quad \text{inaccessible} \\ & \quad + \text{super tree property} \end{align*} \]

\[ \begin{align*} \kappa & \quad \text{inaccessible} \\ & \quad + \text{super tree property} \end{align*} \]
The Super Tree Property at Small Cardinals

Fontanella - Main Theorem 1

If \( \text{Cons}(\text{ZFC} + \exists \langle \kappa_n \rangle_{n<\omega} \text{ supercompact cardinals}) \), then \( \text{Cons}(\text{ZFC} + \forall n \geq 2, \aleph_n \text{ has the Super Tree Property}) \).

\[
\begin{align*}
\lambda & \quad \text{inaccessible} \\
& \quad + \text{super tree property} \\
\kappa & \quad = \aleph_2 \\
& \quad + \text{super tree property}
\end{align*}
\]

\( M(\aleph_0, \kappa) \)
Fontanella - Main Theorem 1

If $\text{Cons}(\text{ZFC} + \exists \langle \kappa_n \rangle_{n < \omega}$ supercompact cardinals), then $\text{Cons}(\text{ZFC} + \forall n \geq 2, \aleph_n$ has the Super Tree Property $)$. 

\[ \lambda = \aleph_3 \]
\[ \lambda \text{ – super tree property} \]

\[ \kappa = \aleph_2 \]
\[ \kappa \text{ – super tree property} \]

\[ M(\aleph_0, \kappa) \ast M(\aleph_1, \lambda) \]
Cummings and Foreman’s Iteration

\[ \langle \kappa_n \rangle_{n<\omega} \] supercompact cardinals

At stage \( n + 1 \), we force with \( Q_n \).

1. \( Q_n \) makes \( \kappa_n = \kappa_{n+2} \) while preserving the super tree property at \( \kappa_n \);
2. \( Q_n \) anticipates the tail of the iteration \( Tail_{n+1} \) (using the Laver function \( L_n \)).
The strong and super tree properties at small cardinals

\( \kappa_0 \)

- inaccessible
- + super tree property

\( \kappa_1 \)

- inaccessible
- + super tree property

\( \kappa_n \)

- inaccessible
- + super tree property

\( \kappa_0 \)

- inaccessible
- + super tree property
The strong and super tree properties at small cardinals

\[ \kappa_0 \text{ super tree property} \]

\[ \kappa_1 \text{ inaccessible + super tree property} \]

\[ \kappa_n \text{ inaccessible + super tree property} \]

\[ = \kappa_2 \]

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The strong and super tree properties at small cardinals

\[ \kappa_0 \quad \Rightarrow \quad \kappa_1 \Rightarrow \kappa_n \quad \text{inaccessible} \quad + \quad \text{super tree property} \]

\[ \kappa_1 = \aleph_3 \quad + \quad \text{super tree property} \]

\[ \kappa_0 = \aleph_2 \quad + \quad \text{super tree property} \]
The strong and super tree properties at small cardinals

\[ \kappa_n = \aleph_{n+2} \]

+ super tree property

\[ \kappa_n = \aleph_{n+2} \]

\[ \kappa_1 = \aleph_3 \]

+ super tree property

\[ \kappa_0 = \aleph_2 \]

+ super tree property

\[ \mathcal{Q}_0 \ast \dot{\mathcal{Q}}_1 \ast \cdots \ast \dot{\mathcal{Q}}_n \ast \cdots \]
Fontanella - Main Theorem 2

If $\text{Cons}(\text{ZFC} + \exists \langle \kappa_n \rangle_{n<\omega} \text{ supercompact cardinals})$, then $\text{Cons}(\text{ZFC} + \mathcal{R}_{\omega+1} \text{ has the Strong Tree Property})$.

Magidor & Shelah 1996

If $\nu$ is a singular limit of strongly compact cardinals, then $\nu^+$ satisfies the Tree Property.

Fontanella 2012 - Key Lemma

If $\nu$ is a singular limit of strongly compact cardinals, then $\nu^+$ satisfies the Strong Tree Property.
The Strong Tree Property at $\aleph_{\omega+1}$

Fontanella - Main Theorem 2

If $\text{Cons}(\text{ZFC} + \exists \langle \kappa_n \rangle_{n<\omega}$ supercompact cardinals), then $\text{Cons}(\text{ZFC} + \aleph_{\omega+1}$ has the Strong Tree Property).

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If $\nu$ is a singular limit of strongly compact cardinals, then $\nu^+$ satisfies the Tree Property.

Fontanella 2012 - Key Lemma

If $\nu$ is a singular limit of strongly compact cardinals, then $\nu^+$ satisfies the Strong Tree Property.
The Strong Tree Property at Successors of Singular Cardinals

$\nu^+$ strongly tree property

$\nu$ strongly compact

$\kappa_n$ strongly compact

$\kappa_1$ strongly compact

$\kappa_0$ strongly compact
\( \nu^+ \) strong tree property

\( \nu \) strongly compact

\( \kappa_n \) strongly compact

\( \kappa_1 \) strongly compact

\( \kappa_0 \) strongly compact

\( \mu \) \( \text{cof}(\mu) = \omega \)

\( \mathbb{I}_\mu := \text{Coll}(\omega, \mu) \times \text{Coll}(\mu^+, < \kappa_0) \times \prod_{n<\omega} \text{Coll}(\kappa_n, < \kappa_{n+1}) \)
\( \nu^+ = \aleph_{\omega+1} \) strong tree property

\( \nu = \aleph_{\omega} \)

\( \kappa_n = \aleph_{n+2} \)

\( \kappa_1 = \aleph_3 \)

\( \kappa_0 = \aleph_2 \)

\( \mu \quad \text{cof}(\mu) = \omega \)

\[ \mathbb{L}_\mu := \text{Coll}(\omega, \mu) \times \text{Coll}(\mu^+, < \kappa_0) \times \prod_{n<\omega} \text{Coll}(\kappa_n, < \kappa_{n+1}) \]
Conclusions

From large cardinals to large combinatorial properties

- Can every regular cardinal satisfy the tree property, strong tree property or the super tree property?
- What are the consequences of the "strong compactness" or "supercompactness" of these small cardinals?
- Can we characterize all large cardinals in terms of combinatorial properties?
Thank you.