THE CHOICE OF NEW AXIOMS IN SET THEORY
(DRAFT)

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1. Introduction

The development of axiomatic set theory originated from the need for a rigorous investigation of the basic principles at the foundations of mathematics. The classical theory of sets ZFC offers a rich framework, nevertheless many crucial problems (such as the famous continuum hypothesis) cannot be solved within this theory. For this reason, set theorists have been exploring new axioms that would allow one to answer some fundamental questions that are independent from ZFC. Research in this area has led to consider several candidates for a new axiomatisation such as Large Cardinal Axioms, Forcing Axioms, Projective Determinacy and others. The legitimacy of such new axioms is, however, heavily debated and gave rise to extensive discussions around an intriguing philosophical problem: what is an axiom in mathematics? What aspects would distinguish an axiom from a hypothesis, a conjecture and other mathematical statements? The future of set theory very much depends on how we answer such questions. Self-evidence, intuitive appeal, fruitfulness are some of the many criteria that have been proposed. In the first part of this paper, we illustrate some classical views and the main challenges associated with these positions. In the second part, we outline a survey of the most promising candidates for a new axiomatization for set theory and we discuss to what extent those criteria are met. We assume basic knowledge of the theory ZFC.

2. Ordinary mathematics

Before we start our analysis of the axioms of set theory and the discussion about what criteria can legitimate those axioms, we should address a bold claim that seems to be quite popular among mathematicians, the claim that ‘ordinary mathematics needs much less than ZFC or ZF’. This view suggests that axioms such as the Axiom of Choice, or Infinity are not really needed for standard mathematical results, and certainly ordinary mathematics does not need new strong axioms such as Large cardinals axioms, Forcing Axioms etc. If so, then our goal of securing the axioms of ZFC and the new axioms would simply be irrelevant or a mere set theoretical concern (where set theory is not considered as a branch of standard mathematics).
The issue with this view is to clarify what counts as ‘ordinary mathematics’. For instance, the Axiom of Choice is heavily used in many fields such as Algebra, General Topology, Measure Theory and Functional Analysis; in particular, it is indispensable for proving the following theorems (among many others):

- Every field has an algebraic closure;
- Every subgroup of a free group is free;
- Vitali’s theorem on the existence of a set of reals that is not Lebesgue measurable;
- Hahn-Banach theorem;
- Every Hilbert space has an orthonormal basis;
- Baire Category theorem;
- Every Tychonoff space has a Stone-Čech compactification.

The analogous claim that ordinary mathematics does not need more than ZFC runs into a similar problem, as it is often the case that natural questions that were raised in what one might consider a standard mathematical framework turned out to be independent from ZFC, thus requiring stronger additional axioms to be settled. It is the case, for instance, for Whitehead’s problem in group theory, or in the study of regularity properties for sets of reals\(^1\), or for the Normal Moore Space Conjecture\(^2\), and many other problems.

Furthermore, the reader is certainly familiar with the famous Fermat’s conjecture recently demonstrated by Wiles who won the Abel prize for his outstanding result; what the reader might not be aware of is that the proof assumes Grothendieck’s universes whose existence require large cardinals (strongly inaccessible cardinals). McLarty undertook a rigorous investigation of the assumptions needed in Wiles’s proof and argued that Grothendieck’s universes are only used as a ‘scheme’; yet, there is still no proof that Fermat’s theorem can be demonstrated in a weaker system than Bounded Zermelo\(^3\) plus the existence of strongly inaccessible cardinals. Even if any reference to large cardinals will eventually be eliminated from Fermat’s theorem, it is quite significant that Wiles’s proof as it is was accepted by the community of number theorists.

After a mathematical problem is proven to be independent from ZFC, it is typically labeled as ‘just set theoretical’ or ‘vague’ and no longer mathematical in the traditional sense. Thus, the supporter of the claim that ordinary mathematics can be run

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\(^1\)Namely, which sets of reals are Lebesgue measurable, or have the perfect set property, or the Baire property.

\(^2\)The Normal Moore Space Conjecture is a topological problem whose solution was eagerly sought for many years until strong large cardinal assumptions turned out to be indispensable for it to be settled

\(^3\)Bounded Zermelo is Zermelo set theory with only bounded separation
in a much weaker theory than ZFC, or that it does not need more than ZFC, needs
to define the expression ‘ordinary mathematics’ in a way that justifies this attitude
towards such problems which, at first, seem to emerge naturally as intrinsically rele-
vant questions for mathematical research, then are abandoned after proven to require
stronger assumptions.

A simple move would be to regard independent problems as legitimate mathematical
questions that yet are ‘unsolvable’. In this perspective, then, any attempt to answer
such questions with stronger assumptions can only be seen as speculative. Surely,
many mathematicians navigate these lines of thoughts. For instance, when Nykos [17]
proved in 1980 the consistency of NMSC from a strongly compact cardinal, he titled
his paper ‘A provisional solution to the Normal Moore Space Conjecture’ which sug-
gests that he considered the large cardinal assumption expended in his proof as merely
hypothetical. Yet, as skeptical as one might be on the legitimacy of large cardinals
and other new axioms for set theory, the mathematical community does not seem to
consider those principles in the same way as hypotheses. Imagine, for instance, that
Wiles’s proof of Fermat’s conjecture were assuming Riemann hypothesis, would this
be considered to be a result worth publishing?

In conclusion, if one wants to support the view that ordinary mathematics can all be
done in ZFC, or in a much weaker system than ZF, one needs to clarify the nature of
independent problems and of the additional assumptions needed for answering them.

3. INTRINSIC MOTIVATIONS

The word ‘axiom’ comes from the Greek αξιωµα ‘that which commends itself as
evident’. This is still the meaning of the word ‘axiom’ today in English language, in
fact as Feferman pointed out (see [2]), the English Oxford dictionary defines the word
‘axiom’ as

“a self-evident proposition requiring no formal demonstration to prove its truth,
but received and assented as soon as mentioned”.

At first, self-evidence seems a rather natural criterion for axioms, yet, it turns out
to be quite problematic from a philosophical point of view. In general, what counts
as obvious, self-evident, intuitively clear, inherently true is highly subjective. Thus,
claiming that the truth of axioms rests on self-evidence would inevitably lead to a
conception of mathematics as a social activity (that could be a perfectly legitimate
position, yet it would encounter strong oppositions).

“I can in no way agree to taking ‘intuitively clear’ as a criterion of truth in math-
ematics, for this criterion would mean the complete triumph of subjectivism and would
lead to a break with the understanding of science as a form of social activity.” (Markov 1962).

The self evidence criterion is also quite restrictive. Not only the new axioms considered in contemporary set theory can hardly be claimed to be self-evident (not even their strongest supporters claim they are self-evident), but even the axioms of ZFC are not strictly speaking obvious. The most outstanding counterexamples are provided by the Axiom of Choice and the Axiom of Infinity that, far from being immediately ‘received and assented as soon as mentioned’, were extensively debated and a mild skepticism still survives.

“The set theoretical axioms that sustain modern mathematics are self-evident in differing degrees. One of them – indeed, the most important of them, namely Cantor’s axiom, the so-called axiom of infinity – has scarcely any claim to self-evidence at all.” (Mayberry [15, p. 10])

Maddy’s analysis in [10] shows that even the less controversial axioms of ZF\Infinity were motivated by rather practical reasons. Consider for instance the Axiom of Foundation. First introduced in the form $A \notin A$ to block Russell’s paradox, it is nowadays adopted in its stronger version “every set is well-founded”. Reasons for reformulating the axiom in this way were not based on self-evidence, but originated in a practical reason, namely in the idea that “no field of set theory or mathematics is in any general need of sets which are not well-founded” (Fraenkel, Bar-Hillel and Levy [[4], p. 88]).

Non-well founded sets can actually be useful in mathematics, however today the Axiom of Foundation is better supported by the so-called ‘Iterative conception’. This consists roughly in the idea that sets must be obtained by an iterative process where at a first stage certain sets are secured ‘immediately’, then new sets can be obtained starting from the sets at the first level so to form a second level, and at each stage new sets can be defined from the ones obtained at the previous levels. Under the Axiom of Foundation, all sets can be obtained in this way, in fact the class of all sets $V$ coincide with the Von Neumann Universe which is defined as follows. The level zero $V_0$ is the empty set, then the first level $V_1$ contains just the empty set, at each successor stage $\alpha + 1$, the level $V_{\alpha+1}$ is defined as the set of all subsets of $V_\alpha$ (in fact $V_1$ coincides with $\mathcal{P}(V_0)$), at limit stages $\lambda$, we let $V_\lambda$ be the union of all $V_\alpha$ for $\alpha < \lambda$. The Von Neumann Universe is the class obtained from the union of all $V_\alpha$’s. The Axiom of Foundation is equivalent to $V$ being equal to the Von Neumann Universe which is the main expression of the iterative conception just described. This is often considered to be an intrinsic justification for the Axiom of Foundation, yet it is not strictly speaking obvious that such an iterative process would exhaust all possible sets (for instance, non-well founded sets would not fall into this hierarchy). On the other hand, this is a very useful and elegant description of the class of all sets, thus the
Axiom of Foundation has undoubtedly strong practical merits.

Let us reformulate the self-evidence requirement and consider the following criterion that we may call ‘\textit{intrinsic necessity}’.

\textit{An axioms must have some intuitive appeal, however the axiom may not be immediately obvious, but it should ultimately occur to us that what the axiom states is true and it could not be otherwise.}

Consider, for instance, the Axiom of Choice. While the well-ordering principle mainly encountered reluctance, the equivalent statement ‘the cartesian product of a collection of non-empty sets is non-empty’ seems to be better accepted by the mathematical community as a fundamental truth. Nevertheless, it is not obvious that what the Axiom of Choice states could not be otherwise. Indeed, as proven by Banach and Tarski, the Axiom of Choice is equivalent to a quite counterintuitive statement (Banach-Tarski paradox): given a solid ball in a 3-dimensional space, there exists a decomposition of the ball into a finite number of disjoint subsets, which can then be put back together in a different way to yield two identical copies of the original ball. Thus the Axiom of Choice challenges basic geometric intuition, leaving a shadow on its alleged intrinsic necessity. Even the Axiom of Foundation can hardly be justified by this criterion, as it is not clear why the cumulative hierarchy would be a necessary feature or the universe of sets.

4. \textbf{Extrinsic motivations}

We argued that intrinsic motivations such as self-evidence, intrinsic necessity etc. are subjective and restrictive. Those considerations led Maddy to claim that axioms are mainly supported by \textit{extrinsic motivations}, namely by their success, or fruitfulness. The roots of this idea already appeared in Gödel [6]:

“Furthermore, however, even disregarding the intrinsic necessity of some new axiom, and even in case it had no intrinsic necessity at all, a decision about its truth is possible also in another way, namely, inductively by studying its “success”, that is, its fruitfulness in consequences and in particular in ‘verifiable’ consequences, i.e., consequences demonstrable without the new axiom, whose proofs by means of the new axiom, however, are considerably simpler and easier to discover, and make it possible to condense into one proof many different proofs.” (Gödel [6, p. 521] 1947)

However, Gödel and Maddy mean different things with the term ‘fruitful’. Let us discuss first Gödel’s view. For Gödel, the fruitfulness of an axiom is measured in terms of ‘\textit{verifiable consequences}’. This is a delicate notion that deserves several comments.
How can we verify a mathematical statement? Is this verification the result of an empirical process? Gödel was known to be a mathematical platonist and he believed in some sort of perception of mathematical entities analogous to our perception of physical objects. Thus, in Gödel’s view, a mathematical statements can be verified to the extent that our intuition provides us with an immediate perception of the mathematical objects involved; other statements cannot be verified directly, but they can be supported by strong enough extrinsic evidence as long as their consequences can be verified.

In more recent work, Magidor considers this mathematical verification to be directly connected to our empirical knowledge of the physical world:

“As far as verifiable consequences I consider the fact that these axioms [large cardinals] provide new $\Pi^1_0$ sentences which so far were not refuted. In some sense we can consider these $\Pi^1_0$ sentences as physical facts about the world that so far are confirmed by the experience.” (Magidor [14, p. 13])

Whatever meaning we accord to the expressions ‘mathematical verification’ and ‘mathematical evidence’, we should note that, as for natural sciences, a plurality of verifiable consequences cannot secure the theory with certainty, since even inconsistent mathematical theories can, at first, appear to have many verifiable consequences. Thus, we can only say of a given axiom or theory that it was not refuted so far. In other words, paraphrasing Popper, mathematical theories are not strictly speaking verifiable, they are only falsifiable; but this is not different from physics, chemistry or other sciences.

A more challenging fact is that mutually incompatible theories can all be ‘successful’ in the sense of both Gödel’s and Magidor’s quote. Consider, for instance, ZFC plus the axiom of measurable cardinals, and ZF+ V=L (more details about these theories will be provided in the second part of this paper). These two theories are incompatible, yet none of their consequences was ‘refuted’ so far.

In Maddy, the concepts of extrinsic justification and fruitfulness is different and it also changes over her various writings. In [11] she argues that mathematical entities such as sets are not abstract but concrete objects; it follows that an immediate empirical verification of mathematical statements is possible. In [12], she describes extrinsic justifications as an inductive process, where a first level mathematics is secured intrinsically, then new axioms can be justified via their consequences demonstrable in lower-level theories. Thus, in this view, the verification can be regarded as a proof that the mathematical statement in question can be derived from a more basic theory for which we have some sort of intrinsic evidence.
Over the time, Maddy seemed to developed a wider conception of extrinsic justifications that she describes as based on practical, inter-theoretic motivations. In [13] she explains the ‘success’ of an axiom or a theory on the basis of its effectiveness to achieve specific mathematical goals, Maddy refers to these sorts of motivations as based on the ‘proper methods’. So, for instance, the Axiom of Choice allows one to solve natural outstanding problems in various areas of mathematics, Projective Determinacy came into serious considerations as the result of a broader research for new axioms that might settle certain problems in analysis and set theory that could not be solved within ZFC, accordingly certain Large Cardinals axioms are justified as they imply Projective Determinacy and settle other problems that are independent from ZFC, and so on.

Once again, distinct incompatible theories can be successful, even in this sense. For instance, if many strong Large cardinals axioms such as the axiom of measurable cardinals can be justified in this way, even the axiom of constructibility $V=\text{L}$ can be viewed as an effective mean to achieve specific mathematical goals: $V=\text{L}$ settles the continuum problem as it implies the Continuum Hypothesis (and even GCH), it also implies the Axiom of Choice (which can be used itself for proving classical fundamental theorems in mathematics), it settles many other questions that are independent from ZF, for instance it implies the negation of the Suslin’s hypothesis.

Thus, even Maddy’s approach is not immune to set theoretical pluralism; hence if one wants to defend the view that only one theory of sets is legitimate, as Maddy seems to do, then additional motivations are needed for choosing a specific theory over the others. Maddy’s suggestion is to appeal on the maximality principle. Roughly, this consists in the idea that we should prefer the theory that maximizes the concept of set. For instance, the concept of set underlying large cardinals seems to be wider than the one associated with the axioms of constructibility which is often ruled out as ‘too restrictive’. Reference to this ‘maximize rule’ can be found for instance in Drake [1], Moschovakis [16] and Scott [20]. Nevertheless, the alleged restrictiveness of the axiom of constructibility was recently refuted by Hamkins [7] who proved, roughly, that the axiom of constructibility is reach enough to allow one to talk about the concept of sets in the sense of large cardinals within a model of $V=\text{L}$. We will discuss this further in Section 6.

Finally, we can remark that the ‘extrinsic approach’ makes axioms depend on their consequences, not the converse. This conflicts with the traditional view that considers axioms to be the starting point for demonstration from which, ideally, the truth of the other mathematical statements can be derived. Here, the situation is reversed: the consequences of an axiom legitimate the axiom, or they lead us to reject it when we have some ‘counter-evidence’ for such mathematical consequences. In this picture,
then, mathematics resembles Quine’s web of belief, namely any part of mathematics, including axioms, could be altered in the light of ‘evidence’.

5. AXIOMS AS DEFINITIONS

To conclude our analysis of the possible conceptions of axioms, we should also mention an entirely different approach: we could view the axioms of a theory, not as assertions, but rather as definitions of some basic concepts. The roots of this conception can be found in Hilbert:

“In my opinion, a concept can be fixed logically only by its relations to other concepts. These relations formulated in certain statements, I call axioms, thus arriving at the view that axioms (perhaps together with propositions assigning names to concepts) are the definitions of the concepts”. (Hilbert [5])

In order to understand this quote, imagine we were asked to define every geometrical notion. Then, for instance, we would define the notion of ‘triangle’ as a ‘polygon’ with three ‘edges’ and three ‘vertices’. The notion of ‘polygon’ is defined from the notion of ‘plane’, the notion of ‘edges’ can be defined from the notion of ‘line’, and ‘vertex’ is defined from ‘point’. At this point we are supposed to define ‘plane’, ‘edges’ and ‘line’, but if we find a concept χ (or more concepts) that defines these notions, we will have to find a definition for χ and so on. Thus, as Hilbert says, we can define a concept only if we put it in relation to other concepts. We can stop this process, if we define the notions of ‘point’, ‘line’ and ‘plane’ by describing their relations to one another through certain axioms. In [3], I revisit this view and argue that set theory can be regarded as definition: we can define every mathematical notion, including the notion of ‘function’ and ‘number’ from the notion of ‘set’ which can then be ‘defined’ through a series of axioms.

We should immediately clarify a possible source of misunderstanding. In Logic, the expression axiomatic definition may refer to the definition of a symbol that is not included in a given language through a series of sentences in the language: suppose we have a theory T in a language \( \mathcal{L} \) and we want to define a new symbol R (a singular term, a predicate or a function) that is not in the language \( \mathcal{L} \), we may be able to give a definition of R through sentences in the expanded language \( \mathcal{L} + \{R\} \). This sort of definitions are called implicit definitions. Beth definability theorem establishes that, in first order logic, every implicit definition is equivalent to an explicit definition, namely one that depends on a formula of the original language \( \mathcal{L} \), but the kind of axiomatic definitions that we are referring to has nothing to do with such syntactical definitions, what is defined is a ‘schema of concept’:
"[...] it is surely obvious that every theory is only a scaffolding or schema of concepts together with their necessary relations to one another, and that the basic elements can be thought in any way one likes. If in speaking of my points I think of some system of things, e.g. the system: love, law, chimney-sweep... and then assume all my axioms as relations between these things, then my propositions, e.g. Pythagoras’ theorem, are also valid for these things. [...] At the same time, the further a theory has been developed and the more finely articulated its structure, the more obvious the kind of application it has in the world of appearances and it takes a very large amount of ill will to want to apply the more subtle propositions of plane geometry or of Maxwell’s theory of electricity to other appearances than the ones for which they were meant...” (Hilbert [5])

As definitions, axioms are neither true nor false per se. Nevertheless, this view does not entail necessarily a rejection of the idea that mathematics is a body of truths. For instance, Hilbert wrote:

"[...] if the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist. This is for me the criterion of truth and existence.” (Hilbert [5])

On the other hand, today we know from Gödel’s incompleteness results that Hilbert’s project cannot be accomplished. Therefore, a confirmation of the truth of mathematical theories such as set theory would require a different source. In [3], I argue that this definitional view leads naturally to pluralism in set theory, but we will not discuss this further.

6. The Axiom of Constructibility

We now illustrate the main candidates for new axioms considered in contemporary set theory. The oldest one is certainly the Axiom of Constructibility V=L, that asserts that every set is constructible, namely every set belongs to Gödel’s constructible universe $L$. $L$ is inductively defined as follows:

- $L_0 = \emptyset$;
- $L_{\alpha+1}$ is the set of all subsets $a$ of $L_\alpha$ that are definable with parameters in $L_\alpha$ (i.e. there is a formula $\varphi(x,a_1,\ldots,a_n)$ with parameters $a_i \in L_\alpha$ such that $a = \{x \in L_\alpha; \ L_\alpha \models \varphi(x,a_1,\ldots,a_n)\}$);
- when $\lambda$ is a limit ordinal, $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$.

Finally, $L := \bigcup_{\alpha\in\text{Ord}} L_\alpha$ ($L$ is a class). The constructible universe was introduced by Gödel in 1938 to prove the consistency of the continuum hypothesis. In fact, the axiom of constructibility implies the generalised continuum hypothesis. Moreover, it implies the Axiom of Choice, and it settles many other questions that are independent...
from ZF, for instance it implies the negation of the Suslin’s hypothesis.

Sentiment in favour of the Axiom of Constructibility can be found for example in Fraenkel, Bar-Hillel and Levy [4]. Nevertheless, the axiom of constructibility counts few supporters among contemporary set theorists. Gödel himself did not consider \( V = L \) as a valid candidate axiom for set theory as he believed CH was actually false. On the other hand, there is no clear evidence for the Continuum Hypothesis or its negation that would count as a corroboration or a falsification for the Axiom of Constructibility. Here we have a lucid example of how the extrinsic approach prioritizes the consequences over the axiom. Similarly, Maddy suggests that \( V=L \) should be rejected on the basis of the fact that it implies the existence of a \( \Delta^1_2 \) non-measurable set of reals, but there is no evidence for the claim that every set of reals is Lebesgue measurable (we will discuss Lebesgue measurability in Section 9).

"There are also extrinsic reasons for rejecting \( V = L \), most prominently that it implies the existence of a \( \Delta^1_2 \) well-ordering of the reals, and hence that there is a \( \Delta^1_2 \) set which is not Lebesgue measurable." (Maddy [10])

The strongest objections to the Axiom of Constructibility are related to its apparent restrictiveness. In fact, \( L \) is provably the smallest inner model of ZFC (i.e. the smallest class satisfying the axioms of ZFC and containing all the ordinals). It follows that, if we consider other axioms such as the Axiom of Measurable cardinals (a large cardinal axiom that establishes the existence of a measurable cardinal), despite the incompatibility of this axiom with \( V=L \), it is always possible to talk about the concept of set in the sense of \( V=L \) within the theory \( ZFC + \exists \kappa \) measurable, while the converse seems not possible.

"The language of set theory as used by the believer in \( V=L \) can certainly be translated into the language of set theory as used by the believer in measurable cardinals, via the translation \( \varphi \mapsto \varphi^L \). There is no translation in the other direction." (Steel [2])

Actually, as Hamkins showed in [7], there is a sense in which the converse is also possible.

"even if we have very strong large cardinal axioms in our current set-theoretic universe \( V \), there is a much larger universe \( V^+ \) in which the former universe \( V \) is a countable transitive set and the axiom of constructibility holds."

This means that, even the Axiom of Constructibility is reach enough to allow us to talk about the concept of sets in the sense of large cardinals within a model of \( V = L \). Thus, the Axiom of Constructibility is less restrictive as it looks.
7. Large cardinals axioms

Let us discuss now Large Cardinals Axioms. There is a whole hierarchy of large cardinals axioms, we will discuss just some notions. A large cardinal is any uncountable cardinal $\kappa$ which is at least weakly inaccessible, namely which satisfies the following two properties:

1. For every cardinal $\gamma < \kappa$, we have $\gamma^+ < \kappa$;
2. For every subset $X \subseteq \kappa$ of size $< \kappa$, we have $\sup X < \kappa$.

If we replace the condition 1 with the stronger 'for every cardinal $\gamma < \kappa$, we have $2^\gamma < \kappa$, then we have the notion of strong inaccessible cardinal; we will simply call inaccessible the strong inaccessible cardinals. We can observe that the properties above are all satisfied by $\aleph_0$, thus a large cardinal axiom establishes that there are other cardinals than $\aleph_0$ with those properties. There is no precise definition of what a large cardinal axiom is, but we can say that all large cardinals axioms establish or imply the existence of large cardinals with stronger and stronger properties.

The existence of a weakly inaccessible cardinal yields the consistency of ZFC, as if $\kappa$ is a weakly compact cardinal, then $L_\kappa$ is a model of ZFC. Since, by Gödel’s incompleteness theorem, ZFC cannot prove its own consistency, it follows that the existence of large cardinals cannot be proven within ZFC. The consistency of large cardinals axioms cannot be proven either from the consistency of ZFC. This marks an important difference between large cardinals axioms and other kind of axioms such as $V = L$ whose consistency can be proven relative to the consistency of ZFC.

Let us discuss some intrinsic motivations for the existence of inaccessible cardinals.

- **Uniformity.** The universe of sets should be uniform, in the sense that “it doesn’t change its character substantially as one goes over from smaller to larger sets or cardinals, i.e., the same or analogous states of affairs reappear again and again (perhaps in more complicated versions)” (Wang [22, pp. 189-90], see also Kanamori and Magidor [9], Solovay, Reinhardt and Kanamori [21], and Reinhardt [19]); $\aleph_0$ is the first cardinal with the properties (1) and (2) above, hence a cardinal with the same property must reappear at higher levels.

- **Inexhaustibility.** The universe of all sets is too rich to be exhausted by some basic operations such as power set or replacement, therefore there must be a cardinal which is not generated by those operations (see e.g. Gödel [6], Wang [22] or Drake [1]); such a cardinal can be proven to be inaccessible.

- **Reflection.** The universe of sets is too complex to be completely described by some property, hence anything that is true of the entire universe, must be true also at some initial segment of it, it must “reflect” at some $V_\kappa$. In particular
there must be a $V_\kappa$ which is also closed by the power set and replacement operations; then $\kappa$ can be proven to be inaccessible.

All these arguments seem to rest on mathematical platonism in an essential manner, as they appeal on some specific conception of “the universe of sets” as uniform, inexhaustible, indescribable and so on. But even assuming a platonic point of view, how do we know that the universe of sets has such features? Some issues arise, for instance, with the claim of uniformity. In fact, there are properties that do hold at $\aleph_0$ and do not occur at higher cardinals. For instance, Ramsey’s theorem establishes that for every $n, m < \aleph_0$ and for every coloring of the $n$-tuples of $\aleph_0$ into $m$ colors, we can find a set $H \subseteq \aleph_0$ of size $\aleph_0$ such that all the $n$-tuples of $H$ have the same color, this is called a homogeneous set; on the other hand, it can be proven that no uncountable cardinal can satisfy the same property: if we replace $\aleph_0$ with an uncountable $\kappa$, we get a statement that is provably false in ZFC.

Typically, large cardinals generalize properties of $\aleph_0$. For instance, the notions of Ramsey cardinal, Erdős cardinal, weakly compact cardinals and others can be defined as special generalizations of the theorem of Ramsey that we just mentioned; some limitations are necessary because as we said the direct generalization of Ramsey Theorem to an uncountable cardinal is provably false in ZFC. We consider, for example, the axiom of weakly compact cardinals which establishes the existence of an uncountable cardinal $\kappa$ such that for every for every coloring of the pairs of ordinals of $\kappa$ into less than $\kappa$ many colors there is a homogeneous set of size $\kappa$. Once again, we stress the fact that generalizations are dangerous as they may lead to inconsistencies as in the case above.

The axiom of weakly compact cardinals can also be defined as a generalization of Compactness theorem to the infinitary language $\mathcal{L}_{\kappa,\kappa}$. Given two infinite cardinals $\kappa, \lambda$, we denote by $\mathcal{L}_{\kappa,\lambda}$ the infinitary language that roughly allows conjunctions and disjunctions of less than $\kappa$ many formulas, and quantifications over less than $\lambda$ many variables. Thus, for instance $\mathcal{L}_{\omega,\omega}$ corresponds to first order logic. An uncountable cardinal $\kappa$ is weakly compact if, and only if, whenever we have a theory $T$ in $\mathcal{L}_{\kappa,\kappa}$ with at most $\kappa$ non logical symbols, if $T$ is $<\kappa$-satisfiable (i.e. every family of less than $\kappa$ many sentences of $T$ is satisfiable), then $T$ is satisfiable. If we remove the restriction to ‘theories that have at most $\kappa$ non-logical symbols, we have the notion of strongly compact cardinal. Other large cardinals axioms can be defined as generalizations of compactness theorem. Such generalizations imply interesting ‘compactness results’, namely principles where, given some structure, we assume that all its smaller sub-structures satisfy a certain property and we deduce that the whole structure satisfy the same property. For instance assuming a strongly compact cardinal $\kappa$ it is possible to prove that every abelian group of size at least $\kappa$ is free abelian whenever all its smaller subgroups are free abelian. The axiom of constructibility on the other hand
is the paradise of incompactness results (for instance compactness for the freeness of abelian groups is actually false in $V = L$). It is not possible to decide for one axiom over the other on the basis of such compactness results because ZFC proves both compactness and incompactness results. For instance, König’s lemma can be regarded as a compactness result$^4$, but on the other hand its generalization to $\aleph_1$ is provably false in ZFC (there are Aronszajn trees).

We can see that the notion of weakly compact cardinal can be defined both as a combinatorial and model-theoretic notion. The same occurs for other large cardinals, namely it is often the case that certain mathematical problems arising in completely different contexts and fields lead to the same large cardinal notions. This fact is sometimes considered to be an intrinsic motivation for large cardinals, but it is not clear that this can actually be considered an evidence for these axioms.

The most powerful large cardinals axioms are the ones that can be defined as elementary embeddings of $V$ into some inner model of ZFC. We discuss some of these notions in the next section.

8. MEASURABLE CARDINALS AND ELEMENTARY EMBEDDINGS

In the history of large cardinals axioms the introduction of measurable cardinals was probably the most crucial step as it lead to the theory of elementary embeddings that are extremely useful in solving set theoretical problems and answering other mathematical questions. Let us discuss, then, these notions.

In 1902, Lebesgue formulated the measure problem: he asked whether there is a function that associates to every bounded set of reals a real number between 0 and 1 and such that the function is not identically 0, it is translation invariant and countably additive. Motivated by this question, he introduced his famous Lebesgue measure (a function with these properties) and asked whether every bounded set of reals was Lebesgue measurable, namely whether his measure was defined over every bounded set of reals. Vitali soon found a counterexample under the Axiom of Choice, the problem was then reformulated by replacing the condition of translation invariance with ‘every singleton must have measure 0’, the minimal request for avoiding trivial solutions. The problem was still proven to be independent from ZF, in fact a counterexample can be built under CH. At this point Banach realized that the problem did not depend on the structure of $\mathbb{R}$, and it could be reformulated for a general set $S$: is there a function $\mu : \mathcal{P}(S) \rightarrow [0,1]$ which is not identically 0, assigns to every singleton the value 0 and is countably additive? The solution of this problem comes

$^4$Given a tree of height $\omega$ whose levels are finite, if every finite subtree has a branch of the same length as the height subtree, then the whole tree also has a branch of the same length as the height of the tree
down to the existence of certain large cardinal, the *real valued measurable cardinals*. A cardinal \( \kappa \) is real valued measurable if every set of size \( \kappa \) has a measure \( \mu \) with the properties above which moreover is \( \kappa \)-additive, namely for every family \( \{X_\alpha\}_\alpha \) of less than \( \kappa \) many sets, \( \mu(\bigcup_\alpha X_\alpha) = \sum_\alpha \mu(X_\alpha) \). This is an example of a notion that can be justified extrinsically by Maddy’s ‘proper methods’, namely it arose naturally as the solution to a specific mathematical problem.

Now, if we require that not only every set of size \( \kappa \) has a measure, but also the measure takes just two values 0 or 1 we have a fundamental notion in set theory, the notion of *measurable* cardinal. In fact this notion has an extremely powerful characterization: \( \kappa \) is measurable if and only if one can define a non-trivial elementary embedding \( j : V \to M \) where \( M \) is a transitive class, such that \( \kappa \) is the least cardinal that is moved by \( j \). By using this characterization, Scott was able to prove that if there is a measurable cardinal, then \( V \neq L \). Thus measurable cardinals and every stronger large cardinal are incompatible with the axiom of constructibility.

Many powerful large cardinal notions can be defined in terms of elementary embeddings where we require the transitive class \( M \) to be more and more ‘close’ to \( V \). These notions have weak intrinsic justifications, in fact the ultimate large cardinal notion expressible in terms of elementary embeddings is provably inconsistent with ZFC. This is the notion of *Reinhardt cardinal* which is a cardinal \( \kappa \) such that there is a non trivial embedding \( j \) of \( V \) into itself where \( \kappa \) is the least cardinal which is moved by \( j \).

Large cardinals axioms that establish the existence of elementary embeddings are more successfully justified by their fruitfulness, as they settle a number of questions that are independent from ZFC. The most remarkable application of such cardinals is the theory of projective sets that under these cardinals gets a very elegant and coherent description. In fact, the existence of infinitely many Woodin cardinals implies that every projective set of reals is Lebesgue measurable, has the perfect set property and the Baire property, these are called *regularity properties*. This brings us to discuss Determinacy hypotheses.

9. **Determinacy Hypotheses**

The study of regularity properties dates back to the earliest 20th century from the work of the French analysts Borel, Baire and Lebesgue. The subject grew up as an independent discipline, known as descriptive set theory, then about 40 years later it was shown that the open questions that descriptive set theorists were trying to solve

\[^5\text{A function } j : V \to M \text{ is an elementary embedding if for every formula } \varphi \text{ and parameters } a_1, \ldots, a_n \text{ one has } V \models \varphi(a_1, \ldots, a_n) \text{ if and only if } M \models \varphi(j(a_1), \ldots, j(a_n)).\]
(namely whether every set of real has the regularity properties above) could not be answered within ZFC (as we have seen for Lebesgue measurability). In 1962 Mycielski and Steinhaus introduced the Axiom of Determinacy AD which was proven to solve such problems. AD is the assertion that every set of reals is determined, that means that for every set of reals $A$, one of the two players has a winning strategy in the following game of length $\omega$. We regard $A$ as a subset of $\omega^\omega$ (in set theory a real is an omega sequence of natural numbers), the two players I and II alternatively choose natural numbers $n_0, n_1, n_2, \ldots$. At the end of the game a sequence $\langle n_i; i \in \mathbb{N} \rangle$ is generated, player I wins if and only if the sequence belongs to $A$.

The Axiom of Determinacy implies that all sets of reals are Lebesgue measurable, have the perfect set property and the Baire property. Moreover, the statement that every set of reals has the perfect set property implies a weak form of the continuum hypothesis: every uncountable set of reals has the same cardinality as the full set of reals. On the other hand AD implies the negation of of the generalised continuum hypothesis. Despite its fruitfulness, AD was never seriously considered as a valid candidate new axiom for set theory as it contradicts the Axiom of Choice. So, once again the priority goes on the consequences rather than the axiom, but the consequence (here the Axiom of Choice) is itself in need for a justification. Anyway, this led to investigate two distinct directions. The first approach was to assume AD in a quite natural subuniverse, namely $L(\mathbb{R})$, together with AC in the full universe $V$ ($L(\mathbb{R})$ is the smallest transitive inner model of ZF containing all the ordinals and the reals). The second approach was to consider a weakening of AD, called Projective Determinacy PD. PD asserts that every projective set of reals is determined. PD implies that every projective set of reals is Lebesgue measurable, has the perfect set property and the Baire property, and unlike AD, Projective Determinacy is not known to contradict the Axiom of Choice. Projective Determinacy follows from the existence of infinitely Woodin cardinals and this is the reason why this large cardinal assumption implies that every projective set of reals has the regularity properties above.

10. Ultimate L

As we said, the Axiom of Constructibility and the Axiom of Determinacy both decide the continuum problem (the former implies GCH, the latter implies a weak form of CH, but it also implies the negation of the generalised continuum hypothesis). Large cardinals axioms, on the other hand, do not decide the size of the continuum. So, a quite promising direction of research was considered which combine large cardinals with $L$ and it may shed light on the size of the continuum, this approach is known as $V=\text{Ultimate } L$. 
To understand this view, consider the intuition behind the Axiom of Constructibility: $\text{L}$ is build up from a cumulative process where each stage is obtained from the previous one by a canonical operation, namely by taking the definable subsets of the previous stage. The universe of sets resulting from this process is quite restrictive as only few ‘canonical’ sets are accepted at each stage. The idea behind $\text{V}=\text{Ultimate L}$ is that, while we want large cardinals to exist in the universe of sets, we only want to include sets that are canonical or necessary after a fashion. Ultimate $\text{L}$, proposed by Woodin, is the alleged inner model for supercompact cardinals. Roughly this is an $\text{L}$-like model where lives a supercompact cardinal. Such a model was not build yet and it is an open problem whether it can actually be found, but it can be proven that if the construction of the Ultimate $\text{L}$ is successful, then it would contain also all the stronger large cardinals (i.e. stronger than supercompact cardinals). More importantly, $\text{V}=\text{Ultimate L}$ would imply $\text{CH}$.

Magidor, however, expressed some doubts about this approach:

“It is very likely that the Ultimate $\text{L}$, like the old $\text{L}$, will satisfy many of the combinatorial principles like $\Diamond_{\omega_1}$. These principles are usually the reason that “$\text{L}$ is the paradise of counter examples”. They allow one to construct counter examples to many elegant conjectures. (The Suslin Hypothesis is a famous case).” (Magidor [14])

11. Forcing Axioms

Moreover, as for the Axiom of Constructibility, $\text{V}=\text{Ultimate L}$ rests on the idea of a limitation of the concept of set through a cumulative process, while other views rely on the opposite slogan that the concept of set should be as rich as possible. The most important example of such a liberal view is given by Forcing axioms. Forcing is the main tool for proving independence results in set theory. There are essentially two main approaches for building models of set theory and proving consistency results: one is through inner models which are obtained roughly by ‘restricting’ $\text{V}$ into a subclass; the other is by forcing where, conversely, $\text{V}$ is expanded to a larger universe. Forcing axioms roughly establish that anything that can be forces by some ‘nice’ forcing notions (a forcing is simply a partially ordered set) is a set in the universe. For instance, the two most fruitful forcing axioms, PFA and MM, are the following statements.

The Proper Forcing Axiom PFA states that if $\text{P}$ is a forcing notion that is proper and $\text{D}$ is a collection of $\aleph_1$ many dense subsets of $\text{P}$, then there is a generic filter that meets all the dense sets in $\text{D}$.

Roughly, this says that anything that can be forced by a proper forcing is a set in the universe.
Martin’s Maximum MM asserts that if P is a forcing notion that preserves stationary subsets of \( \omega_1 \) and D is a collection of \( \aleph_1 \) many dense subsets of P, then there is a generic filter that meets all the dense sets in D.

Roughly, this says that anything that can be forced by a forcing that preserves stationary subsets of \( \omega_1 \) is a set in the universe. We will not discuss these notions in the details, we should only point out that MM is the strongest possible version of a forcing axiom and it was proven to be consistent relative to the existence of a supercompact cardinal (this provides another motivation for large cardinals axioms). Forcing axioms settle many important questions that cannot be answered within ZFC, but more importantly they find remarkable applications in cardinal arithmetic. In fact, Foreman Magidor and Shelah proved in 1988 that Martin’s Maximum settles the size of the continuum, it implies that the \( 2^{\aleph_0} = \aleph_2 \). Later in 1992, Todorcević and Veličković showed that even the weaker axiom PFA implies that the size of the continuum is \( \aleph_2 \). Other remarkable applications of forcing axioms include the singular cardinals hypothesis (from PFA), the Axiom of Determinacy in L(\( \mathbb{R} \)), the statement that any two \( \aleph_1 \)-dense subsets of \( \mathbb{R} \) are isomorphic (from PFA), every automorphism of the Boolean algebra \( \mathcal{P}(\omega)/\text{fin} \) is trivial (from PFA), the \( \aleph_2 \)-saturation of the ideal of non stationary sets on \( \omega_1 \) (from MM), and the reflection of stationary subsets of \( \kappa \) for any regular cardinal \( \kappa \geq \omega_2 \) (from MM).

References


