Geometry of Hyperbolic Components in the Interior of the Mandelbrot Set

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Abstract

Let $\mathcal{M}$ denote the Mandelbrot set. This work was done in order to better understand the geometry of $\mathcal{M}$. We calculate the "width" of some hyperbolic components of $\mathrm{Int}(\mathcal{M})$, that accumulate on -2, called Tchebychev's components. It is done by means of the conformal radius of a component. We showed that it behaves asymptotically as 16^{-p} where p is the period of the component. This in turn proves that it has the smallest inner radius among all hyperbolic components of its period, up to a multiplicative constant not depending p. It also completes the picture that one gets by using the self similarity of the Mandelbrot set around -2, proved in [6].

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1 Personal Page

The motivation for a work in dynamical systems came from reading the popular science book *Chaos* by James Gleick [10]. The problem was to say something that will be equally innovative and exciting, but by using mathematics. This remains an intriguing but not yet well posed task.
I was directed to try and understand a very concrete family of dynamical systems, namely, the quadratic family. A romantic aim became careful analysis - but I have no regrets. I think that a good way for searching a role might be to try and understand the most simple case. I thank my advisor Prof. Genadi Levin for sharing this insight with me.
My opinion is that the geometry of $\mathcal{M}$ worth attention for its own, and even if the dynamics of the quadratic family will be "understood", $\mathcal{M}$ will stay a mystery.

2 Introduction

The dynamics of the *family of quadratic maps*

$$\{f_c(z) = z^2 + c\}_{c\in\mathbb{C}} \tag{2.1}$$

in the plane $\mathbb{C}$ is complicated, as best demonstrated by the fractal (cf. [7]) set, now called the Mandelbrot set

$$\mathcal{M} = \{c \in \mathbb{C} \quad | \quad f_c^n(0) \quad is \quad bounded\}.$$

$\mathcal{M}$ is interesting because of the following reasons:

1. "Dynamical": $\partial\mathcal{M}$ is the bifurcation set of 2.1.

2. "Geometrical": It is a very complicated set, and much isn't known (cf. 2.2). In fact, it is proved that $dim_H \partial\mathcal{M} = 2$.

3. "Physical". (cf. sec. 1).

The starting point of this dissertation is the correspondence between 1 and 2, of which a good example is the well known implication

$$MLC \Longrightarrow DHP, \tag{2.2}$$

where MLC stands for "Mandelbrot set is locally connected" conjecture and DHP stands for "Density of hyperbolic parameters" conjecture (cf. [12]).

In order to investigate $\partial\mathcal{M}$ it is natural to look at the hyperbolic components of Int($\mathcal{M}$) (cf. sec. 3). The MLC itself is tightly connected to the shape of the hyperbolic components of $\mathcal{M}$. For example, if the MLC is correct, then the diameter of the hyperbolic components of order p goes to 0 when $p \to \infty$. This fact for itself is not known yet. The following is a remarkable theorem by Douady and Hubbard:

Theorem 2.1. *Let W be a hyperbolic component of order p, and define*

$$\rho : W \to \mathbb{D}$$

by taking $\rho(\gamma)$ to be the multiplier of the attracting period p cycle of f_γ. Then ρ is one-to-one and onto $\mathbb{D}$.

Formally, if $a(\gamma)$ denotes a point on the attracting cycle of f_γ, then by (3.1)

$$\rho(\gamma) = 2^p a(\gamma)_1 \cdots a(\gamma)_p. \tag{2.3}$$

For every component W, $c_W = \rho^{-1}(0)$ is the only super-attracting parameter in W, and is called the center of W. The outcome of a careful differentiation at $\gamma = c_W$ of (2.3), (cf. [1]) is,

Lemma 2.2. $\rho'_\gamma \doteq \rho'_\gamma(\gamma) = 2(f_\gamma^{p-1})'(\gamma)^2 s_p(\gamma)$, *where*

$$s_p(\gamma) \doteq 1 + \frac{1}{2\gamma_1} + \cdots + \frac{1}{2^{p-1}\gamma_1\gamma_2\ldots\gamma_{p-1}} \tag{2.4}$$

It appears that ρ' is 1 over the conformal radius of W_γ at the center c_W (cf. sec. 4). We have used a computer in order to get a rough idea of what can be learned from Theorem 2.1 and Lemma 2.2. That required solving high degree polynomial equations (cf. sec. A.3). Then by using some general estimates on univalent functions (cf. sec. 4), we were able to make estimations on this formula within a limit case, namely, for Tchebychev's parameters. The corresponding components intersect the real line, and lie nearest to -2, the left edge of $\mathcal{M}$ (cf. sec. A.1). These evaluations, not only explain some of the computer results, but lead to the following theorems:

Theorem 2.3. *There exits a constant K such that the following: Let $W_{c^{(p)}}$ be the Tchebychev's component of order p. Then for every $p \in \mathbb{N}$ and every component W of order p,*

$$d(c_W, \partial W) \geq K d(c^{(p)}, \partial W_{c^{(p)}}),$$

Theorem 2.4. *There are positive constants $0 < K_1 < K_2$ such that*

$$K_1 16^{-p} < d(c^{(p)}, \partial W_{c^{(p)}}) < K_2 16^{-p},$$

for every $p \geq 3$.

In [6] it is proved that the Mandelbrot set is asymptotically self-similar around any Misiurewicz point (cf. sec. 6), in particular around -2. We show that Theorem 2.4 reflects better some aspects of $\mathcal{M}$ near -2.

3 Introduction to Complex Dynamics

A (discrete) dynamical system is a pair (X, f) where X is a set and $f : X \longrightarrow X$ is a function from X to itself. The objects of interest are the *orbits of* (X, f):

Definition 3.1. *Let (X, f) be a dynamical system. For every $x \in X$, the orbit of x is the sequence*

$$x, f(x), f(f(x)), ..., f^n(x), ...$$

There is no ambiguity in thinking of an orbit as a set and we will do this occasionally. To understand the dynamics will mean to classify the orbits. The most easy orbits to classify are the finite ones. Those are the preperiodic orbits.

Definition 3.2. *Let (X, f) be a dynamical system and $x \in X$ some point.*
1. The orbit $(f^n(x))_{n=0}^{\infty}$ is called preperiodic if there are $n < m$ such that $f^n(x) = f^m(x)$.
2. For the minimal such n, m we say that the order is $m - n$.
3. If the minimal n is 0 then the orbit is called periodic and we call it cycle.
4. If the orbit of x is a cycle of order 1 then we say that x is a fixed point.

To further investigate the dynamics one usually assumes further structure, such as a topology or a measure on the space X, aside suitable compatibility of f. Then one can use terms as local or typical behavior. In the case of complex dynamics one assumes a complex structure, that is, assumes X is a Riemannian manifold and f is a holomorphic function. From now on, unless otherwise stated, we'll always assume a complex dynamical system $(\mathbb{C}, f)$ on background, where f is a nonlinear polynomial. We'll see that this makes the local behavior near a periodic orbit in some cases determined only by the derivatives on it, and the typical behavior strongly connected with the critical points of f.

For every $z \in \mathbb{C}$ denote $z_0 \doteq f^0(z) \doteq z$ and for every $n \in \mathbb{N}$

$$z_n \doteq f^n(z).$$

The following proposition is a most fundamental in complex dynamics, and it'll serve us as a definition for attracting/repelling cycle:

Proposition 3.3. *Assume $z \in \mathbb{C}$ is a fixed point. then:*
1. $|f'(z)| < 1$ iff z is attracting.
2. $|f'(z)| > 1$ iff z is repelling.

Proof. cf. [3] for the topological definition of reppeling/attracting fixed point and a proof of this theorem. □

This proposition motivates the definition of a multiplier:

Definition 3.4. *The multiplier of a cycle of order p is $\lambda = (f^p)'(z)$, where z is one of the points in the cycle.*

Evidently, by applying the chain rule we have

$$(f^p)'(z) = f'(z_1) f'(z_2) \cdots f'(z_p), \tag{3.1}$$

so the notion of a multiplier is well defined. A cycle is called *repelling, attracting, super-attracting, indifferent* according to whether its multiplier absolute value is $> 1, < 1, = 0, = 1$, respectively. It is said to be *hyperbolic* if it is repelling or attracting.

As much as the Proposition 3.3 has far reaching implications, we have a much stronger result.

Köenig's linearization theorem. *Assume z is a hyperbolic fixed point with multiplier $\lambda = f'(z)$. Then the equation*

$$\varphi \circ f = \lambda \cdot \varphi \tag{3.2}$$

$$\varphi(z) = 0, \quad \varphi'(z) = 1 \tag{3.3}$$

for univalent φ is locally solvable.
More explicitly, there exits a neighborhood U of z, and an univalent function

$$\varphi : U \cup f(U) \to \mathbb{C},$$

such that 3.3 holds and 3.2 holds inside U.

The proof may be found in [3] or [4]. As a matter of fact, this φ is a particular case of what one calls conjugation.

Definition 3.5. *Let (X, f), (Y, g) be complex dynamical systems. A conjugation between (X, f) and (Y, g) is a homeomorphism*

$$j : X \longrightarrow Y,$$

such that $j \circ f(x) = g \circ j(x)$, for every $x \in X$. If in addition j is holomorphic, it is called holomorphical conjugacy.

It is common to think of two conjugated systems as the same, because it induces a one-to-one onto correspondence between the orbits, and inside each orbit. So Theorem 3 actually states that a complex dynamical system behavior near a hyperbolic fixed point is holomorphically conjugated to its linear part, hence determined by its multiplier.

The next definition is followed by an example of a proposition with global meaning about the dynamics of f. The proof appears at **??**.

Definition 3.6. *Assume z_0 is an attracting fixed point. The basin of attraction of z_0 is the set*

$$B(z_0) = \{z \in \mathbb{C} \quad | \quad f^n(z) \longrightarrow z_0\}.$$

The immediate basin $B_0(z_0)$ is the component of B that contains z_0.

Proposition 3.7. *Assume z is an attracting fixed point. Then $B_0(z)$ contains a critical point of f.*

3.1 The Quadratic Case

The simplest analytic functions on $\mathbb{C}$ are polynomials. For polynomials of degree 1, dynamics is trivial: assume

$$f(z) = az + b$$

for some $a, b \in \mathbb{C}, a \neq 0$. If $a = 1$ this is just a translation by b. there are no periodic orbits, and every orbit is of the form $f^n(z) = z + nb$. Otherwise, if $a \neq 1$ then f is conjugated to $g(z) = az$ by the conjugation

$$j(z) = z + \frac{b}{1-a}.$$

g has a unique fixed point $z = 0$. If $|a| > 1$ then every other orbit diverges to ∞ and 0 is a repelling fixed point. If $|a| < 1$ then every orbit converges to 0 and 0 is an attractive fixed point. For $|a| = 1$, orbits lie on circles around 0. The dynamics on each circle is a translation by $arg(a)$ which is well understood.

The next case is degree 2. As in the linear case, by applying a linear conjugation if needed, every polynomial has the form

$$f(z) = z^2 + c,$$

where c is a complex parameter. We call this one-parameter family of polynomials the *quadratic family*. The dynamics of each f_c is not at all trivial (cf. [3] for pictures and theory), and the relation between dynamics of two close elements in 2.1 is the reason for introducing $\mathcal{M}$. Notice that 0 is the only critical point of f_c and hence have a crucial part in characterizing f_c dynamics. Parameters c for which f_c has an attracting periodic cycle of order p will be called attracting parameters of order p. Note that by Proposition 3.7, every f_c has at most 1 attracting cycle.

Definition 3.8. *For $p \in \mathbb{N}$, a hyperbolic component of order p is a connected component of Int($\mathcal{M}$) that contains only attracting parameters of order p.*

By the maximum principle, every component of Int($\mathcal{M}$) is simply connected.

4 Tools from Geometric Function Theory

In this work we have used the notion of conformal radius. Its usability is an outcome of some of the basic theorems of geometric function theory, which will be summarized in this section. (cf. [9]).
Let d denote the Euclidean distance.

Theorem 4.1 (Köebe $\frac{1}{4}$-Theorem)**.** *Assume* $f : \mathbb{D} \to \mathbb{C}$ *is univalent,* $f(0) = 0$, $f'(0) = 1$. *Then*

$$d(0, \partial f(\mathbb{D})) \geq \frac{1}{4}.$$

Corollary 4.2. *Assume* f *is univalent on a domain* W, $z_0 \in W$, $f'(z_0) \neq 0$. *Then*

$$\frac{1}{4}|f'(z_0)|d(z_0, \partial W) \leq d(f(z_0), \partial(f(W))) \leq 4|f'(z_0)|d(z_0, \partial W)$$

Theorem 4.3 (Distortion Theorem)**.** *Assume* $f : \mathbb{D} \to \mathbb{C}$ *is univalent,* $f(0) = 0, f'(0) = 1$. *Then for every* $z \in \mathbb{D}$,

$$\frac{1 - |z|}{(1 + |z|)^3} \leq |f'(z)| \leq \frac{1 + |z|}{(1 - |z|)^3}$$

We will use an easy corollary of this theorem

Corollary 4.4. *Assume* $f : \mathbb{D} \to \mathbb{C}$ *is univalent,* $f(0) = 0$, $f'(0) \neq 0$. *Then for every* $z, w \in \mathbb{D}$,

$$\frac{\frac{1-|z|}{(1+|z|)^3}}{\frac{1+|w|}{(1-|w|)^3}} \leq \left|\frac{f'(z)}{f'(w)}\right| \leq \frac{\frac{1+|z|}{(1-|z|)^3}}{\frac{1-|w|}{(1+|w|)^3}}$$

Let $\mathbb{D}$ be the unit disk, W a bounded simply connected region in $\mathbb{C}$, and γ a point in W. By the Riemann mapping theorem there exist a *unique* univalent function

$$f : \mathbb{D} \longrightarrow W$$

such that $f(0) = \gamma$ and $f'(0) > 0$. So $f'(0)$ depends only on W and the choice of $\gamma \in W$.

Definition 4.5. $|f'(0)|$ *is called the conformal radius of* W *at* γ, *and is denoted by* $r(\gamma, W)$.

The importance of this notion is in the connection it has to geometry: by calculating the conformal radius at a point, one bounds the distance from a point to the boundary by factor 4. Actually, by Corollary 4.2 we have

$$\frac{1}{4}r(\gamma, W) \leq d(\gamma, \partial W) \leq 4r(\gamma, W).$$

In the next section, where we will be dealing with asymptotics, it'll be as good as knowing the distance.

5 Main theorem

5.1 Proof

Theorem 5.1. *There exist a constant M such that for every p and a super-attracting real parameter γ of order p,*

$$|\rho'_{c^{(p)}}(c^{(p)})| \geq M|\rho'_\gamma(\gamma)| \tag{5.1}$$

Proof. Let p be an integer and assume $p > 2$. Let $c^{(p)}$ denote the Tchebychev's parameter of order p and $\gamma \in \mathcal{M}$ be any real super-attracting parameter. The following is a basic lemma we need for the proof

Lemma 5.2. *For any $p \geq 3$:*

1. *For any $\gamma \in \mathcal{M}$, $|(f_\gamma^{p-1})'(\gamma)s_p(\gamma)| \leq \frac{1}{3}4^p - \frac{1}{3}$.*
2. *For any real $\gamma \in \mathcal{M}$, $|(f_{c^{(p)}}^{p-1})'(c^{(p)})| \geq |(f_\gamma^{p-1})'(\gamma)|$.*
3. *$\lim_{p\to\infty} s_p(c^{(p)}) = \frac{2}{3}$.*
4. *$\frac{3}{8} < s_p(c^{(p)}) < \frac{2}{3}$.*

Proof. 1. Notice that $\gamma \in \mathcal{M}$ so its orbit $\{\gamma_i\}_{i=1}^p$ satisfies $|\gamma_i| < 2$. By applying the triangle inequality,

$$|(f_\gamma^{p-1})'(\gamma)s_p(\gamma)| \leq \sum_{i=0}^{p-1} \left|2^{p-1-i}c_{p-1}\cdots c_{i+1}\right| \leq \sum_{i=0}^{p-1} 4^i = \frac{1}{3}(4^p - 1).$$

2. It is enough to prove that $|c_i^{(p)}| \geq |\gamma_i|$ for every $i \leq p-1$, because

$$|(f_\gamma^{p-1})'(\gamma)| = 2^{p-1}\gamma_1\cdots\gamma_{p-1}.$$

$|c^{(p)}| \geq |\gamma|$, because $c^{(p)}$ is Tchebychev's parameter. Now assume that there is $2 \leq i \leq p-1$ for which $|c_i^{(p)}| < |\gamma_i|$. $c^{(p)}$ and γ are real so taking this inequality squared and adding $c^{(p)} \leq \gamma$ yields

$$c_{i+1}^{(p)} = {c_i^{(p)}}^2 + c^{(p)} < {\gamma_i}^2 + \gamma = \gamma_{i+1}.$$

Since $c_{i+1}^{(p)} \geq 0$ ($i \geq 2$), we actually have

$$|c_{i+1}^{(p)}| < |\gamma_{i+1}|.$$

By repeating at most $p-2$ times we get

$$0 = c_p^{(p)} < \gamma_p = 0,$$

which is absurd. Equality holds iff $\gamma = c^{(p)}$.

3. Recall that by 2.4,

$$s_p(c^{(p)}) = 1 + \frac{1}{2c_1^{(p)}} + \cdots + \frac{1}{2^{p-1}c_1^{(p)}c_2^{(p)}\ldots c_{p-1}^{(p)}} = \sum_{i=0}^{m} \frac{1}{2^i c_1^{(p)}\cdots c_i^{(p)}}.$$

Let $\varepsilon < 1$ be positive and m be such that $\frac{1}{2^m} < \frac{\varepsilon}{3}$. For $p > m$

$$c^{(p)}\overrightarrow{p\to\infty} - 2,$$

$$\forall 2 \le i \le m \quad c_i^{(p)}\overrightarrow{p\to\infty}2,$$

and therefore by using limits arithmetic we get

$$\sum_{i=0}^{m} \frac{1}{2^i c_1^{(p)}\cdots c_i^{(p)}} \quad \overrightarrow{p\to\infty} \quad 1 - \sum_{i=1}^{m} 4^{-i}. \tag{5.2}$$

For the m we chose certainly there is

$$\sum_{i=m+1}^{\infty} 4^{-i} = 4^{-m}\sum_{i=1}^{\infty} 4^{-i} < \left(\frac{\varepsilon}{3}\right)^2 \frac{1}{3} < \frac{\varepsilon}{3}.$$

Also, by 5.2 there exits p_0 such that for every $p > p_0$

$$\left|\sum_{i=0}^{m} \frac{1}{2^i c_1^{(p)}\cdots c_i^{(p)}} - (1 - \sum_{i=1}^{m} 4^{-i})\right| < \frac{\varepsilon}{3}$$

And therefore, finally, for every $p > p_0$,

$$\left|\sum_{i=0}^{p-1} \frac{1}{2^i c_1^{(p)}\cdots c_i^{(p)}} - \frac{2}{3}\right| <$$

$$< \left|\sum_{i=m+1}^{p-1} \frac{1}{2^i c_1^{(p)}\cdots c_i^{(p)}}\right| + \left|\sum_{i=0}^{m} \frac{1}{2^i c_1^{(p)}\cdots c_i^{(p)}} - (1 - \sum_{i=1}^{m} 4^{-i})\right| + \left|1 - \sum_{i=1}^{m} 4^{-i} - \frac{2}{3}\right| <$$

$$< \frac{1}{2^m} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

4. For every $p \ge 3$ we have $1.7 < |c^{(p)}| < 2$ (cf. Example A.2 on the appendix), so for every $1 < i < p$

$$1.3 < c_{p-1}^{(p)} < c_i^{(p)} < |c^{(p)}| < 2.$$

So

$$\left|\sum_{i=0}^{p-1} \frac{1}{2^i c_1^{(p)} \cdots c_i^{(p)}}\right| = 1 - \sum_{i=1}^{p-1} \frac{1}{\left|2^i c_1^{(p)} \cdots c_i^{(p)}\right|} > 1 - \sum_{i=1}^{\infty} 2.6^{-i} = 1 - \frac{10}{26} \frac{1}{1 - \frac{10}{26}} = \frac{3}{8},$$

and

$$\left|\sum_{i=0}^{p-1} \frac{1}{2^i c_1^{(p)} \cdots c_i^{(p)}}\right| = 1 - \sum_{i=1}^{p-1} \frac{1}{\left|2^i c_1^{(p)} \cdots c_i^{(p)}\right|} < 1 - \sum_{i=1}^{\infty} 4^{-i} = 1 - \frac{1}{3} = \frac{2}{3}.$$

$\square$

To prove the theorem, recall that by Lemma 2.2

$$\rho'_\gamma \doteq \rho'_\gamma(\gamma) = 2(f_\gamma^{p-1})'(\gamma)^2 s_p(\gamma),$$

so we must show that there exists M such that for every $p > 2$,

$$|(f_{c^{(p)}}^{p-1})'(c^{(p)})^2 s_p(c^{(p)})| \geq M|(f_\gamma^{p-1})'(\gamma)^2 s_p(\gamma)|.$$

By Lemma 5.2.2 it is enough to prove that

$$|(f_{c^{(p)}}^{p-1})'(c^{(p)})||s_p(c^{(p)})| \geq M|(f_\gamma^{p-1})'(\gamma) s_p(\gamma)|.$$

By Lemma 5.2.1 and Lemma 5.2.4 it is enough to prove that

$$|(f_{c^{(p)}}^{p-1})'(c^{(p)})|\frac{3}{8} \geq M\frac{1}{3}4^p,$$

or equivalently

$$|(f_{c^{(p)}}^{p-1})'(c^{(p)})| \geq M\frac{32}{9}4^{p-1}. \tag{5.3}$$

Denote

$$c = c^{(p)}, \quad f = f_c$$

and let

$$b = b_c \doteq \frac{1 + \sqrt{1 - 4c}}{2} \tag{5.4}$$

be the repelling fixed point of f, with multiplier

$$\beta \doteq \beta_c \doteq 2b.$$

We first show that there exists a K such that for every $p > 2$

$$\left|(f_{c^{(p)}}^{p-1})'(c^{(p)})\right| \geq K\beta^{p-1}, \tag{5.5}$$

and then proving $\exists K' \quad \forall p > 2$

$$\beta^{p-1} \geq K' 4^{p-1} \tag{5.6}$$

will end the proof with

$$M = \frac{9}{32} K K'. \tag{5.7}$$

By Theorem 3, there exists an univalent conjugation $\varphi = \varphi_c$ from a neighborhood $U = U_c$ of b to $\mathbb{C}$ for which $\forall z \in U$,

$$f(z) = \varphi^{-1}(\beta \varphi(z)) \tag{5.8}$$

$$\varphi(b) = 0, \quad \varphi'(b) = 1 \tag{5.9}$$

By applying the last equation m times and composing with φ from the left we get

$$\varphi(f^m(z)) = \beta^m \varphi(z), \tag{5.10}$$

$\forall z \in f^{-m}(U)$, where f^{-m} is $(f^{-1})^m$ and f^{-1} is the appropriate branch of

$$f^{-1}(z) = \sqrt{z - c}$$

in

$$\mathbb{C} \setminus \{z \in \mathbb{R} | z < c\}$$

for which $f^{-1}(b) = b$. Note that for this branch

$$f^{-1}(U) \subseteq U.$$

Now, by 5.10 we can extend φ for all z but for the ray $\{z \in \mathbb{R} | z < c\}$. Indeed, If

$$z \notin \{z \in \mathbb{R} | z < c\}$$

then there is $m \in \mathbb{N}$ such that $f^{-m}(z) \in U$, so define

$$\varphi(z) = \beta^m \varphi(f^{-m}(z)).$$

For any $m' \in \mathbb{N}$ such that $f^{-m'}(z) \in U$, we have by 5.10

$$\beta^{m'} \varphi(f^{-m'}(z)) = \beta^{m'} \varphi(f^{-m'+m}(f^{-m}(z))) = \beta^m \varphi(f^{-m}(z)),$$

proving that φ is well defined in

$$\mathbb{C} \setminus \{z \in \mathbb{R} | z < c\}.$$

Note also that 5.10 stays correct. Moreover, φ stays univalent. Indeed, if $\varphi(z_1) = \varphi(z_2)$ then for some m, $f^{-m}(z_1), f^{-m}(z_2) \in U$ and again by 5.10

$$\varphi(f^{-m}(z_1)) = \beta^{-m} \varphi(z_1) = \beta^{-m} \varphi(z_2) = \varphi(f^{-m}(z_2)).$$

Now by applying the fact φ is one-to-one in U we get

$$f^{-m}(z_1) = f^{-m}(z_2).$$

Applying f^m we get $z_1 = z_2$, which proves the claim. At last, by differentiating 5.10 for $m = p - 1$ we get,

$$\varphi'(f^{p-1}(z))(f^{p-1})'(z) = \beta^{p-1}\varphi'(z)$$

Substitute $z = -c$, $f^{p-1}(-c) = 0$,

$$(f^{p-1})'(-c) = -(f^{p-1})'(c),$$

so, dividing by $\varphi'(0)$ we get,

$$|(f^{p-1})'(c)| = \beta^{p-1}\left|\frac{\varphi'(-c)}{\varphi'(0)}\right|.$$

By using Theorem 4.4 one can get a uniformed bound on

$$|\frac{\varphi'(-c)}{\varphi'(0)}|.$$

Actually, define

$$A \doteq \mathbb{C} \setminus \{z \in \mathbb{R} | z \le c\},$$
$$B \doteq \{z \in \mathbb{C} | Re(z) > 0\},$$
$$\varphi : A \to \mathbb{C},$$
$$f : B \to A, \quad f(z) = z^2 + c,$$
$$h : \mathbb{D} \to B, \quad h(z) = -\sqrt{-2c}\frac{z+1}{z-1},$$
$$l \doteq f \circ h : \mathbb{D} \to A,$$
$$l(z) = (-\sqrt{-2c}\frac{z+1}{z-1})^2 + c = -2c(\frac{z+1}{z-1})^2 + c.$$

Then

$$f(\sqrt{-2c}) = -c,$$
$$f(\sqrt{-c}) = 0,$$
$$h(0) = \sqrt{-2c},$$
$$h(-3+2\sqrt{2}) = -\sqrt{-2c}\frac{-3+2\sqrt{2}+1}{-3+2\sqrt{2}-1} = -\sqrt{-2c}\frac{2(\sqrt{2}-1)}{-2\sqrt{2}(\sqrt{2}-1)} = \sqrt{-c},$$

so

$$l(0) = -c,$$
$$l(-3+2\sqrt{2}) = 0$$

Now, $\varphi \circ l$ is a composition of univalent functions, thus is univalent, so by Theorem 4.4

$$\left|\frac{(\varphi\circ l)'(0)}{(\varphi\circ l)'(-3+2\sqrt{2})}\right| \geq \frac{\frac{1-0}{(1+0)^3}}{\frac{1+|-3+2\sqrt{2}|}{(1-|-3+2\sqrt{2}|)^3}} = \frac{(2\sqrt{2}-2)^3}{4-2\sqrt{2}} = 2\sqrt{2}(\sqrt{2}-1)^2.$$

On the other hand, the left hand side equals

$$\left|\frac{\varphi'(-c)}{\varphi'(0)}\right|\left|\frac{l'(0)}{l'(-3+2\sqrt{2})}\right|.$$

But

$$l'(z) = 8c\frac{z+1}{(z-1)^3}.$$

Hence

$$\left|\frac{l'(0)}{l'(-3+2\sqrt{2})}\right| = \left|\frac{-8c}{8c\frac{-2+2\sqrt{2}}{(-4+2\sqrt{2})^3}}\right| = \frac{(4-2\sqrt{2})^3}{2\sqrt{2}-2} = 8\sqrt{2}(\sqrt{2}-1)^2.$$

Finally

$$\left|\frac{\varphi'(-c)}{\varphi'(0)}\right| 8\sqrt{2}(\sqrt{2} \geq 2\sqrt{2}(\sqrt{2}-1)^2.$$

so,

$$\left|\frac{\varphi'(-c)}{\varphi'(0)}\right| \geq \frac{1}{4}. \tag{5.11}$$

So,

$$\frac{1}{4}\beta^{p-1} \leq \left|(f^{p-1})'(c)\right|,$$

for every p. This proves 5.5, with

$$K = \frac{1}{4}. \tag{5.12}$$

To prove 5.6, assume $c = -2+\varepsilon$ for some $\varepsilon > 0$. Note that by 5.10 we already have

$$\sqrt{-c} = f^{p-1}(0) = f^{p-2}(c) = f^{p-2}(-c) = \varphi^{-1}(\beta^{p-2}\varphi(-c)),$$

so

$$\varphi(-c) = \frac{\varphi(\sqrt{-c})}{\beta^{p-2}}. \tag{5.13}$$

But
$$|\varphi(-c)| = |\varphi(-c) - \varphi(b)| = (b+c)|\varphi'(\vartheta)|$$
where ϑ is in $[-c, b]$.

By substituting c in 5.4 we get

$$b = \frac{1}{2} + \sqrt{\frac{9}{4} - \varepsilon} = \frac{1}{2} + \frac{3}{2}\sqrt{1 - \frac{4}{9}\varepsilon},$$

and so by Taylor's expansion for $\sqrt{1 - \frac{4}{9}\varepsilon}$ we achieve (calculations are given in the appendix)
$$b > 2 - \frac{\varepsilon}{2} \tag{5.14}$$
so
$$b + c > \frac{\varepsilon}{2}.$$
Putting this in 5.13 gives

$$\varepsilon < 2(b+c) = 2\left|\frac{\varphi(-c)}{\varphi'(\theta)}\right| = \frac{2}{\beta^{p-2}}\left|\frac{\varphi(\sqrt{-c})}{\varphi'(\theta)}\right| = \frac{2}{\beta^{p-2}}\left|\frac{\varphi'(\eta)(\sqrt{-c} - b)}{\varphi'(\theta)}\right|.$$

where η is in $[\sqrt{-c}, b]$. $-c > 1$ and $b < 2$, hence

$$\left|(\sqrt{-c} - b)\right| < 1$$

so finally,
$$\varepsilon < \frac{2}{\beta^{p-2}}\left|\frac{\varphi'(\eta)}{\varphi'(\theta)}\right|$$
Here again, by the same methods applied above (cf. sec. A.2), one can get

$$\left|\frac{\varphi'(\eta)}{\varphi'(\theta)}\right| < 2. \tag{5.15}$$

Now we have a bound on ε, namely

$$\varepsilon < 4\beta^{-p+2}.$$

This in turn yields the wanted estimation on β^p. $\beta = 2b > 4 - 4\varepsilon$ so by the inequality
$$\forall\delta > 0, \delta p < 1 \qquad (1-\delta)^p > 1 - \delta p,$$
we get
$$\beta^p > (4 - 4\varepsilon)^p = 4^p(1-\varepsilon)^p > 4^p(1 - \varepsilon p) > K'4^p,$$

with $K' = \frac{1}{2}$ for every $p > 4$. For $p = 3$, $p = 4$ it also holds by check. This ends the proof, with (cf. 5.7)

$$M = \frac{9}{256}. \tag{5.16}$$

□

The same result is true for any super-attracting parameter $\gamma \in \mathcal{M}$, not necessarily real, although a bigger constant is needed.

Theorem 5.3. *There exist a constant M' such that for every p and a super-attracting parameter $\gamma \in \mathcal{M}$ of order p,*

$$|\rho'_{c^{(p)}}(c^{(p)})| \geq M'|\rho'_\gamma(\gamma)| \tag{5.17}$$

Proof. Lemma 5.2 still holds in this general case, with a slight reformulation of 5.2.2:

Lemma 5.4. *For any $\gamma \in \mathcal{M}$ $|(f^{p-1}_{c^{(p)}})'(c^{(p)})| \geq \frac{1}{8}|(f^{p-1}_\gamma)'(\gamma)|$.*

Proof. In the proof of Theorem 5.1 we had proved

$$|(f^{p-1}_{c^{(p)}})'(c^{(p)})| \geq \frac{1}{8}4^{p-1},$$

(cf. 5.3, 5.16). But

$$|(f^{p-1}_\gamma)'(\gamma)| = 2^{p-1}|\gamma_1| \cdots |\gamma_{p-1}|$$

and each $|\gamma_i|$ is less than 2, so

$$|(f^{p-1}_\gamma)'(\gamma)| \leq 4^{p-1}$$

which ends the proof of the lemma. □

Similar to what we have done in the proof Theorem 5.1, take

$$M' = \frac{1}{8}M$$

where M is as given in 5.16. We wish to prove,

$$|(f^{p-1}_{c^{(p)}})'(c^{(p)})^2 s_p(c^{(p)})| \geq M'|(f^{p-1}_\gamma)'(\gamma)^2 s_p(\gamma)|.$$

By the preceding lemma, it is enough to prove that

$$|(f^{p-1}_{c^{(p)}})'(c^{(p)})||s_p(c^{(p)})| \geq M|(f^{p-1}_\gamma)'(\gamma)s_p(\gamma)|.$$

Applying Lemma 5.2.1 and Lemma 5.2.4, it is enough to prove

$$|(f_{c^{(p)}}^{p-1})'(c^{(p)})|\frac{3}{8} \geq M\frac{1}{3}4^p,$$

substitute $M = \frac{9}{256}$ we get

$$|(f_{c^{(p)}}^{p-1})'(c^{(p)})| \geq \frac{1}{8}4^{p-1}$$

which we already know. $\square$

5.2 Corollaries

Theorem 2.3 is now a direct consequence of Theorem 5.3 and the geometric interpretation of the conformal radius (cf. Definition 4.5 and discussion there). To prove Theorem 2.4, notice that

$$\rho'_{c^{(p)}}(c^{(p)}) = 2(f_{c^{(p)}})'(c^{(p)})^2 s_p(c^{(p)}).$$

During the proof of Theorem 5.1 we have proved that

$$\frac{1}{8}4^{p-1} < \left|(f^{p-1})'(c^{(p)})\right|$$

and that $\frac{3}{8} < s_p < \frac{2}{3}$ (cf. Lemma 5.2.4). Also,

$$\left|(f^{p-1})'(c^{(p)})\right| < 4^{p-1}$$

because $|c_i^{(p)}| < 2$ for every $1 \leq i \leq p-1$. So the theorem is proved.

6 Self similarity of the Mandelbrot set

The "self-similarity" of the Mandelbrot set is one of the main characteristics of its geometry. In his book [7], Mandelbrot shares a certain feeling of his that $\mathcal{M}$ has a strong self-similarity property. It is far from being clear what he means. In 1990 a much more concrete form of self-similarity of $\mathcal{M}$ is proved by Tan Lei. The notion of self-similarity has an intuitive meaning hence details are important. Details are taken from [6].

Definition 6.1. *The Hausdorff distance between $A, B \subseteq \mathbb{C}$ is*

$$d(A,B) \doteq max\{sup_{x\in A}d(x,B), sup_{x\in B}d(x,A)\}.$$

For $r > 0$, denote by D_r the disc of radius r centered at 0, and define

$$A_r = (A \cap D_r) \cup \partial D_r$$

(we add the boundary of D_r to exclude empty intersection). For $a \in \mathbb{C}$, denote by

$$T_a : \mathbb{C} \to \mathbb{C}$$

the translation by a, $z \mapsto z + a$.
Let $A \subseteq \mathbb{C}$ be a closed set, $x \in A$, $\rho \in \mathbb{C}$, $|\rho| > 1$.

Definition 6.2. *A is ρ-self-similar around x if there is $r > 0$ for which*

$$(\rho T_{-x} A)_r = (T_{-x} A)_r$$

A is asymptotically ρ-self-similar around x if there is $r > 0$, and a set $B \subseteq \mathbb{C}$, called the model of A around x, for which

$$\lim_{n \to \infty} d((\rho^n T_{-x} A)_r, B_r) = 0$$

Definition 6.3. *$c \in \mathcal{M}$ is called Misiurewicz parameter if 0 is preperiodic but not periodic.*

Theorem 6.4. *Let c be a Misiurewicz parameter. Let l and p be minimal such that $a \doteq f_c^l(c)$ is a cycle of order p, and assume $\rho \doteq (f_c^p)'(a)$ satisfies*

$$|\rho| > 1.$$

then:
1. J_c is asymptotically ρ-self-similar around c. Denote its model by Z.
2. There exist $\lambda \in \mathbb{C}$ such that $\mathcal{M}$ is asymptotically ρ-self-similar around c and λZ is the model.

In [6] λ is given explicitly.

Example 6.5. $c = -2$

$c = -2$ is Misiurewicz parameter because $f_c^2(0) = f_c(-2) = 2$ is a fixed point, and the multiplier is $\rho = 4$. In this case, $J_c = [-2, 2]$. So by Theorem 6.4, $\mathcal{M}$ is asymptotically self-similar to a straight line. The theorem tells us nothing about the rate of convergence, however we can get

$$d(c^{(p)}, \partial W_{c^{(p)}}) = o(4^{-p}).$$

This follows from the fact that $d(c^{(p)}, -2) = O(4^{-p})$. By Theorem 2.4, we know that there are $0 < K_1 < K_2$ such that

$$K_1 16^{-p} < d(c^{(p)}, \partial W_{c^{(p)}}) < K_2 16^{-p},$$

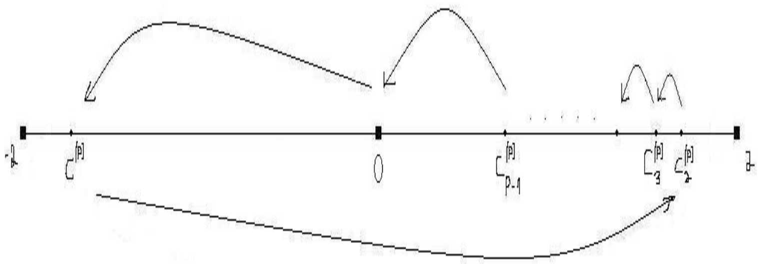
which is a more concrete result.

It is interesting to search for the shape of these components, for instance, what's the diameter of $W_{c^{(p)}}$? Again by Theorem 6.4, we know that the diameter of $W_c^{(p)}$ is $o(4^{-p})$, but is this all that one can say?

A Appendix

A.1 Existence of Tchebychev's Parameter

We are looking for a cycle of the type presented on the following picture:



For $p = 2$ this is just a real super-attracting cycle of order 2, which we know to have only for $c = -1$.

Proposition A.1. *For every $2 \leq p \in \mathbb{N}$ there is a unique non-positive parameter $c^{(p)} \in \mathcal{M}$ such that*

$$c_2^{(p)} \geq ... \geq c_p^{(p)} = 0.$$

Moreover, for every $2 \leq k \leq p, c < c^{(p)}$

$$c_k'(c) < -1 \tag{A.1}$$

and in particular, $c_k(c)$ is decreasing on $[-2, c^{(p)}]$.

Proof. For $p = 2$, $c^{(2)} = -1$, and for every $c < c^{(2)}$,

$$c_2'(c) = 2c + 1 < -1.$$

It is unique since such a parameter must satisfy $c^2 + c = 0$, and the only solutions are 0 and -1. Proceed by induction. Assume $c^{(p)}$ exits and unique, and that for every $2 \leq k \leq p$ and $c < c^{(p)}$ A.1 is satisfied. Observe the equation

$$c_p(c) - \sqrt{-c} = 0.$$

For $c = -2$

$$c_p(-2) - \sqrt{2} = 2 - \sqrt{2} > 0,$$

and for $c = c^{(p)}$,

$$c_p(c^{(p)}) - \sqrt{-c^{(p)}} = -\sqrt{-c^{(p)}} < 0.$$

Differentiation of $c_p(c) - \sqrt{-c}$ gives

$$c_p'(c) + \frac{1}{2\sqrt{-c}} \leq -\frac{1}{2} < 0, \quad c \in [-2, c^{(p)}].$$

The last inequality follows the fact that $c_p'(c) < -1$, which we know by induction, and $c < c^{(p)} \leq -1$. So $c_p(c) - \sqrt{-c}$ is decreasing on $[-2, c^{(p)}]$, thus, by continuity, it has a unique zero there. This zero is $c^{(p+1)}$. By the induction, for every $2 \leq k \leq p$

$$c_k(c^{(p)}) > 0,$$

and since for every $2 \leq k \leq p$

$$c_k^{(p+1)} > c^{(p+1)} > -2,$$

we get

$$c_k^{(p+1)} = (c_{k-1}^{(p+1)})^2 + c^{(p+1)} < c_{k-1}^{(p+1)} + c_(k-1)^{(p+1)} + c^{(p+1)} < c_k^{(p+1)}.$$

So,

$$c_2^{(p+1)} \geq ... \geq c_{p+1}^{(p+1)} = 0,$$

as needed. Further, we prove that A.1 is satisfied

$$c_{p+1}'(c) = 2c_p(c)c_p'(c) + 1 < -1, \quad c \in [-2, c^{(p+1)}]$$

For every $c \in [-2, c^{(p+1)}] \subseteq [-2, c^{(p)}$ we have by the induction,

$$c_p(c) \geq c_p(c^{(p+1)}) = \sqrt{-c^{(p+1)}} > 1$$

and

$$c_p'(c) < -1,$$

hence

$$c_{p+1}'(c) < 1$$

on $[-2, c^{(p+1)}]$. We now have to prove uniqueness. Certainly there isn't another such parameter in $[-2, c^{(p)}]$. This follows the fact that the equation

$$c_p(c) - \sqrt{-c} = 0$$

has a unique solution in this interval. Also, for every

$$c^{(k-1)} < c < c^{(k)}, \quad 2 < k \leq p$$

must be

$$c_k(c) < 0,$$

for otherwise we would have another zero of

$$c_k(c) - \sqrt{-c}$$

in $[-2, c^{(k-1)}]$ which is a contradiction. Lastly, in $[-1, 0]$ there are no super-attracting parameters but -1, and 0, which ends the proof of uniqueness. $\square$

Finally, Tchebychev's parameter is defined for every $p \in \mathbb{N}$, and is denoted by $c^{(p)}$ ($c^{(1)} \doteq 0, c^{(2)} \doteq -1$). Note that for $c = c^{(p)}$, f_c has a super-attracting period p cycle, and that

$$c^{(p)} \downarrow_{p\to\infty} -2.$$

Example A.2. $p = 3$

If, for instance $c = -1.7$, then

$$c_2 = (-1.7)^2 - 1.7 = 2.89 - 1.7 = 1.73$$

and

$$c_3 = 1.19^2 - 1.7 = -0.26 < 0.$$

For $c = -1.8$ we get

$$c_2 = (-1.8)^2 - 1.8 = 3.24 - 1.8 = 1.44$$

and

$$c_3 = 1.44^2 - 1.8 = 2.0736 - 1.8 = 0.2736 > 0.$$

So

$$-1.8 < c^{(3)} < -1.7$$

(we shall use this estimation in the proof of Lemma 5.2.4). See the Appendix A.3 for estimated values of the first six $c^{(p)}$s.

A.2 Calculations for Section 5

Proof of Calculation 5.14
For x, $\quad |x| < \frac{1}{2}$ we have the following inequality

$$\sqrt{1-x} = 1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3-... > 1-\frac{1}{2}x-\frac{1}{8}x^2(1+x+x^2+...) = 1-\frac{1}{2}x-\frac{1}{4}x^2.$$

In our case, $c = c^{(p)} = -2 + \varepsilon$ for $p > 2$, so $\varepsilon < \frac{1}{2}$ and we get

$$b = \frac{1}{2} + \frac{3}{2}\sqrt{1 - \frac{4}{9}\varepsilon} > \frac{1}{2} + \frac{3}{2}(1 - \frac{1}{2}(\frac{4}{9}\varepsilon) - \frac{1}{4}(\frac{4}{9}\varepsilon)^2) = 2 - \frac{1}{3}\varepsilon - \frac{2}{27}\varepsilon^2 > 2 - \frac{1}{2}\varepsilon.$$

Finally,

$$b > 2 - \frac{1}{2}\varepsilon,$$

as needed.
Proof of Calculation 5.15
We wish to bound from above

$$\left|\frac{\varphi'(\eta)}{\varphi'(\theta)}\right|,$$

where

$$\eta \in [\sqrt{-c}, b], \quad \theta \in [-c, b].$$

Recall that $A = \mathbb{C} \setminus \{z \in \mathbb{C} | \quad z < c\}$,

$$\varphi : A \longrightarrow \mathbb{C}$$

is univalent, and that

$$l : \mathbb{D} \longrightarrow A$$

defined by

$$l(z) = -2c(\frac{z+1}{z-1})^2 + c$$

is univalent and onto, with derivative

$$l'(z) = 8c\frac{z+1}{(z-1)^3}$$

Denote $\alpha_1 = l^{-1}(\eta)$ and $\alpha_2 = l^{-1}(\theta)$. So by Theorem 4.4 applied to $\varphi \circ l$, we get

$$\left|\frac{\varphi'(\eta)}{\varphi'(\theta)}\right| = \left|\frac{l'(\alpha_2)}{l'(\alpha_1)}\right| \left|\frac{\varphi \circ l'(\alpha_1)}{\varphi \circ l'(\alpha_2)}\right| \leq \left|\frac{\frac{\alpha_2+1}{(\alpha_2-1)^3}}{\frac{\alpha_1+1}{(\alpha_1-1)^3}}\right| \frac{\frac{1+|\alpha_1|}{(1-|\alpha_1|)^3}}{\frac{1-|\alpha_2|}{(1+|\alpha_2|)^3}} =$$

$$= \left|\frac{(1+|\alpha_2|)^3(\alpha_2+1)}{(\alpha_2-1)^3(1-|\alpha_2|)} \frac{(\alpha_1-1)^3(1+|\alpha_1|)}{(\alpha_1+1)(1-|\alpha_1|)^3}\right| \leq \left(\frac{1+|\alpha_1|}{1-|\alpha_1|}\right)^4 \left(\frac{1+|\alpha_2|}{1-|\alpha_2|}\right)^4,$$

Define a branch of $f^{-1}(z) = \sqrt{z-c}$ on A by $f^{-1}(b) = b$. It is easy to verify that

$$l^{-1}(z) = \frac{\sqrt{z-c} - \sqrt{-2c}}{\sqrt{z-c} + \sqrt{-2c}}$$

is a branch of $l^{-1}(z)$. On the real line

$$l^{-1}(z)$$

is an increasing function, and we will use it to bound

$$\left(\frac{1+|\alpha_1|}{1-|\alpha_1|}\right)^4$$

and

$$\left(\frac{1+|\alpha_2|}{1-|\alpha_2|}\right)^4$$

For α_2:

$$\left(\frac{1+|\alpha_2|}{1-|\alpha_2|}\right)^4 \leq \left(\frac{1+\frac{\sqrt{b-c}-\sqrt{-2c}}{\sqrt{b-c}+\sqrt{-2c}}}{1-\frac{\sqrt{b-c}-\sqrt{-2c}}{\sqrt{b-c}+\sqrt{-2c}}}\right)^4 = \sqrt{\frac{b-c}{-2c}}^4 = \left(\frac{b-c}{-2c}\right)^2 \leq$$

$-c > 1.7$ and $b < 2$ thus

$$\leq \left(\frac{1}{2}(\frac{2}{1.7}+1)\right)^2 = 1\frac{3}{34}^2 < 1.1^2 < \sqrt{2}$$

For α_1: It is bounded by the maximum between

$$\left(\frac{1+\frac{\sqrt{b-c}-\sqrt{-2c}}{\sqrt{b-c}+\sqrt{-2c}}}{1-\frac{\sqrt{b-c}-\sqrt{-2c}}{\sqrt{b-c}+\sqrt{-2c}}}\right)^4$$

and

$$\left(\frac{1+\frac{\sqrt{-2c}-\sqrt{\sqrt{-c}-c}}{\sqrt{-2c}+\sqrt{\sqrt{-c}-c}}}{1-\frac{\sqrt{-2c}-\sqrt{\sqrt{-c}-c}}{\sqrt{-2c}+\sqrt{\sqrt{-c}-c}}}\right)^4.$$

The first expression is already estimated. For the second,

$$\left(\frac{1+\frac{\sqrt{-2c}-\sqrt{\sqrt{-c}-c}}{\sqrt{-2c}+\sqrt{\sqrt{-c}-c}}}{1-\frac{\sqrt{-2c}-\sqrt{\sqrt{-c}-c}}{\sqrt{-2c}+\sqrt{\sqrt{-c}-c}}}\right)^4 = \left(\frac{\sqrt{-2c}}{\sqrt{\sqrt{-c}-c}}\right)^4 = \left(\frac{-2c}{\sqrt{-c}-c}\right)^2 \leq$$

The real function

$$g(x) = \frac{2x}{\sqrt{x}+x}$$

is increasing over $[0, \infty)$, so

$$\leq \frac{4}{\sqrt{2}+2} < \frac{20}{17}^2 < 1.18^2 < \sqrt{2}.$$

Finally

$$\left|\frac{\varphi'(\eta)}{\varphi'(\theta)}\right| < 2$$

as wished.

A.3 Computer Results

The data, which includes the values of the super-attracting parameters with order $p \leq 6$, was produced by using MATLA$B^{(c)}$. Algorithm *Multroot* written by Zhonggang Zeng was used in order to solve the polynomial equation in c, of order $2^{(p-1)} - 1$:

$$c_p(c) = 0.$$

Results for $p > 6$ were inaccurate. After having the parameters, Roprime - which is one over the conformal radius of the adequate component, was calculated using the formula 2.2 which appears in the introduction. Tchebychev's parameter is the first parameter in each group.

For cycle length 2

The parameter	Roprime	Sp	c1*c2*...*cp-1
-1	4	0.5	1

For cycle length 3

The parameter	Roprime	Sp	c1*c2*...*cp-1
-1.7549	105.07	0.60754	2.3247
-0.12256 + 0.74486i	10.587	0.76911	0.65587
-0.12256 - 0.74486i	10.587	0.76911	0.65587

For cycle length 4

The parameter	Roprime	Sp	c1*c2*...*cp-1
-1.9408	2016.9	0.64651	4.9368
-1.3107	16.955	0.35471	0.61109
-0.15652 + 1.0322i	235.77	0.83559	1.4847
-0.15652 - 1.0322i	235.77	0.83559	1.4847
0.28227 + 0.53006i	22.756	1.1823	0.38778
0.28227 - 0.53006i	22.756	1.1823	0.38778

For cycle length 5

The parameter	Roprime	Sp	c1*c2*...*cp-1
-1.9854	34371	0.66014	10.084
-1.8608	2562.9	0.60972	2.8653
-1.6254	492.75	0.61875	1.2472
-1.2564 + 0.38032i	694.09	0.67437	1.4178
-1.2564 - 0.38032i	694.09	0.67437	1.4178
-0.19804 + 1.1003i	2916.5	0.86055	2.5728
-0.19804 - 1.1003i	2916.5	0.86055	2.5728
-0.044212 + 0.98658i	575.44	0.85953	1.1435
-0.044212 - 0.98658i	575.44	0.85953	1.1435
-0.50434 + 0.56277i	25.78	1.019	0.22229
-0.50434 - 0.56277i	25.78	1.019	0.22229
0.35926 + 0.64251i	415.67	1.2	0.82252
0.35926 - 0.64251i	415.67	1.2	0.82252
0.37951 + 0.33493i	42.226	1.7287	0.21842
0.37951 - 0.33493i	42.226	1.7287	0.21842

For cycle length 6

The parameter	Roprime	Sp	c1*c2*...*cp-1
-1.9964	561040	0.66464	20.302
-1.9668	56173	0.64997	6.4961
-1.9073	17003	0.64257	3.5945
-1.7729	230.37	0.31561	0.597
-1.476	386.84	0.47223	0.63245
-1.2841 + 0.42727i	12483	0.69567	2.9601
-1.2841 - 0.42727i	12483	0.69567	2.9601
-1.138 + 0.24033i	37.74	0.51301	0.18953
-1.138 - 0.24033i	37.74	0.51301	0.18953
-0.59689 + 0.66298i	903.26	0.99614	0.6654
-0.59689 - 0.66298i	903.26	0.99614	0.6654
-0.21753 + 1.1145i	30321	0.87015	4.1249
-0.21753 - 1.1145i	30321	0.87015	4.1249
-0.1636 + 1.0978i	8484.6	0.86602	2.1872
-0.1636 - 1.0978i	8484.6	0.86602	2.1872
-0.01557 + 1.0205i	8926	0.89927	2.2015
-0.01557 - 1.0205i	8926	0.89927	2.2015
-0.11342 + 0.86057i	44.842	0.50873	0.20746
-0.11342 - 0.86057i	44.842	0.50873	0.20746
0.35989 + 0.68476i	3633.3	1.2051	1.2133
0.35989 - 0.68476i	3633.3	1.2051	1.2133
0.39653 + 0.60418i	1505.3	1.2564	0.76486
0.39653 - 0.60418i	1505.3	1.2564	0.76486
0.44333 + 0.37296i	665.64	1.6913	0.43838
0.44333 - 0.37296i	665.64	1.6913	0.43838
0.38901 + 0.21585i	70.802	2.4229	0.11945
0.38901 - 0.21585i	70.802	2.4229	0.11945

It's interesting to seek patterns in this table. For instance, the smallest Roprime for each p seems to be near and above p^2. The greatest is proved here to be attained for $c^{(p)}$ and proved to behave asymptotically as 16^p.

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