This text is based on classes given in Spring 2015 Jerusalem and in Spring 2016 in Paris on globally valued fields, aiming to prove the existential closure of $k(a)^{alg}$, and concentrating on the algebraic geometry needed for this.

§2 - 8 are essentially expository. I learned the material from (short initial sections of) [10], [18], [13], [19], [3] and [20]. In a number of cases, modifications of results there were needed; they probably can be found elsewhere in the literature. One critical statement that I did not find stated explicitly is the strong form of BDPP duality ([2]), where elements of the interior of the dual cone to the effective cone are shown to have $n-1$'st roots in some limit of blowups; the usual version in the literature presents them only as convex combinations of powers of ample divisors.

In these notes, we restrict attention to the function field case. Thus our globally valued fields should properly be called 'purely non-archimedean globally valued fields'. There are nevertheless some applications to number fields, see Corollary 1.8.

Globally valued fields are defined in §1 via globalizing measures (up to renormalization), and also via an explicit axiomatization in an appropriate language for real-valued logic. The equivalence of the two was proved in notes for the previous semester; in the function field case, it also follows from the geometric characterization of quantifier-free types in § 9.

We assume some basic algebraic geometry, including mainly blow-ups, Cartier divisors, Weil divisors; some of the material is reviewed in § 2.

§ 3 collects some results related to convex subsets of $\mathbb{R}^n$. The most important are Theorem 3.2, a multiplicative version of Legendre duality, and a finiteness statement for finitely generated semigroups or graded rings, due to Khovanskii and Okounkov. These convexity results will later be used to amplify a very modest geometric input to yield a great deal of information.

§ 4 presents the Hodge index theorem.

Volume is defined, via section growth, in § 5.

Okounkov bodies are presented in § 6, thereby giving another definition of volume for big divisors. They are used to prove log-concavity of volume for big divisors. § 7 proves the Fujita approximation theorem.

§ 8 presents the positive intersection product of [3].

The space of quantifier-free GVF types over a given constant field is described in § 9.

The existential closure of $k(x)^a$ is proved in § 10.

Notation for induced maps. Given a morphism of spaces or varieties $f : X \to U$, there are many induced morphisms on associated objects. They will all be denoted in the same way, namely f_* in the contravariant case and f^* if covariant. In case of ambiguity, we make the domain explicit as follows: $f_*|N_1(X)$ (a map from $N_1(X)$ to $N_1(U)$), or $f^*|N^1(U)$ (a map $N^1(U) \to N^1(X)$.)

A similar convention applies to various groups and spaces formed out of algebraic cycles. For instance consider a Weil divisor represented by an irreducible subvariety D of X. We will be interested in the image of D in various groups, such as $Pic(X)$ and $N_1(X) = \mathbb{R} \otimes Pic(X)/Pic^o(X)$. The class of D in any such group will be denoted $[D]$, but if we write $[D] = [D'] \in N_1(X)$ we mean that the equality holds specifically for the classes in $N_1(X)$.

1. Globally valued fields

A globally valued field is a field with a measure space of valuations and absolute values, satisfying the product formula. Here we consider only the geometric case and do not use absolute values. Our valuations will all be valued in $\mathbb{R}$.

Let K be a field. Let $\widehat{K}$ be the space of $\mathbb{R}$-valued valuations of K. The topology on $\widehat{K}$ is induced from the Tychonoff topology on the functions from $K \setminus (0)$ to $\mathbb{R}$.

If k is a subfield of K, let $\widehat{K/k}$ be the subspace of valuations trivial on k. When $K = k(X)$ for some variety X, this is the Berkovich space of X, from which the Berkovich spaces of all proper k-subvarieties have been removed.

v_{triv} denotes the trivial valuation.

Definition 1.1. A *purely non-archimedean GVF* is a field K along with a Baire (or regular Borel) measure μ on some Borel set of representatives for equivalence classes of nontrivial valuations; such that for any $f \in K \setminus (0)$, $v \mapsto v(f)$ is integrable, and

$$\int v(f) d\mu(v) = 0$$

In these notes, we will simply write GVF for purely non-archimedean GVF.

If K' is a subfield of K, we obtain a GVF structure on K' by taking the pushforward measure $\mu' = r_* \mu | (\widehat{K} \setminus \widehat{K/K'})$, where r is the restriction map from measures on $\widehat{K}$, nontrivial on K' to measures on $\widehat{K'}$.

If g is an L^∞ function on $\widehat{K}$ whose multiplicative inverse g^{-1} is also L^∞, we have a map $rn(g) : \widehat{K} \to \widehat{K}$ multiplying each valuation v by $g(v)$; we let $\mu^g = g^{-1} rn(g)_* \mu$. If μ concentrates on some set of representatives for the valuations up to equivalence, then μ^g corresponds on a different set, where each v is replaced by the equivalent valuation $g(v)v$ and given weight $1/g(v)$ of the μ-weight of v. We view (K, μ) and (K, μ^g) as 'the same' GVF structure. (In particular they give all formulas the same value.)

A *morphism* of globally valued fields is the composition of an inclusion with a renormalization.

1.2. A language for global fields. .

A real-valued language. A *formula $\phi(x_1, \dots, x_n)$, evaluated on a structure A*, gives a real-valued function on A^n.

Basic formulas:

Let Tr_n be the set of n-ary terms in the language $+, \min, \alpha \cdot x$ of divisible ordered Abelian groups (here $\alpha\cdot$ denotes scalar multiplication by the rational number α.) We will refer to these as $\mathbb{Q}$-*tropical polynomials*.

Each $t \in Tr_n$ defines a positively 1-homogeneous continuous function t on $\mathbb{R}^k$.

Let $S_n = \{x : \max_i |x_i| = 1\}$. Then $\{t|S_n : t \in Tr_n\}$ contains the constant function 1, is closed under addition, scalar multiplication and min, and separates points. By Lemma 3.13 these are uniformly dense in S_n.

For any $t \in Tr_n$, a symbol $R_t(x_1, \ldots, x_n)$ interpreted as: $x \mapsto \int t(vx_1, \ldots, vx_n)$ (the domain being $\{x : \Pi_{i=1}^k x_i \neq 0.\}$)

Example: $\int v(x)$. The main axiom, the (logarithmic) *product formula*, asserts that $\int v(x) = 0$ for any $x \neq 0$.

Example: $\int v(x)^+$ where $x^+ = \max(x, 0)$. By definition, this is the *height* of x, denoted $ht(x)$.

(In fact we will see later that the height suffices to generate the entire language, at least in the function field case.)

The analogue of quantifiers in real-valued logic is inf and sup operators. Let ψ be a continuous function on $\mathbb{R}$ with compact support. Let $\phi(x, y)$ be a formula. Then so is $\sup_x \psi(ht(x))\phi(x, y)$. Thus quantification is only available over bounded height subsets.

1.3. **Universal axioms.** GVF. (Function field case.)

We have an integral domain F, along with a function $F_t : (F \setminus (0))^n \to \mathbb{R}$ written as:

$$F_t(a_1, \ldots, a_n) = \int t(v(a_1), \ldots, v(a_n))dv$$

given for each n and each $t \in Tr_n$. The F_t are compatible with permutations of variables and dummy variables.

Let POS be the set of pairs (ϕ, t) of formulas $\phi(x_1, \ldots, x_n)$ in the language of rings, and $t \in Tr_n$, such that the theory of valued fields VF implies $(\forall x)(\phi(x) \implies t(v(x_1), \ldots, v(x_n)) \geq 0)$. The universal axioms are:

(1) (Linearity:) $F_{t_1+t_2} = F_{t_1} + F_{t_2}$.
(2) (Local-global principle for positivity) If $(\phi, t) \in POS$ and $\phi(a_1, \ldots, a_n)$ then $\int t(v(a_1), \ldots, v(a_n))dv \geq 0$.
(3) (Product formula) $F_x(a) = 0$ where x is the identity on $\mathbb{R}$.

Remark 1.4. If we wish to axiomatize GVF fields containing a given field k as a constant field, we may use the same axioms with respect to the collection POS_k, defined in the same way for the theory VF_k of valued fields containing k as a trivially valued field. To see the equivalence, a lemma of valued field theory is required; namely, if

$$VF_k \models (\forall x)(\phi(x) \implies t(v(x_1), \ldots, v(x_n)) \geq 0$$

, then for some rational $\alpha > 0$ and some $a_1, \ldots, a_n \in k$ we have: $VF_k \models (\forall x)(\phi(x) \implies \phi'(x, a_1, \ldots, a_n))$, and

$$VF \models (\forall x)(\phi'(x) \implies t(v(x_1), \ldots, v(x_n)) \geq -\alpha \min |v(a_1)|, \ldots, |v(a_n)|)$$

1.5. **Classical structures.** .

(a) $k(x)^{alg}$, $h(x) = \alpha$.

(b) $\mathbb{Q}^{alg}[r]$ ($h(2) = r$.) This is not a purely non-archimedean GVF for fixed r; but it is so asymptotically, i.e. any axiom is true for small enough r.

(c) $\mathcal{M}[l, r]$ value distribution theory. Not a GVF at all for fixed r, but is a purely non archimedean GVF for fixed r; but it is so asymptotically.

We elaborate a little on (c). Let $\mathcal{M}$ be a countably generated subfield of the field of meromorphic functions. Fix a function $\eta(r)$ (diverging to ∞; such as $\log(r)$ or r^d), and also an ultrafilter u on $\mathbb{R}^{>0}$, avoiding finite measure sets.

Let μ_r be the measure space on $\{a : 0 < |a| \leq r\}$ giving mass $\log(r/a)/\eta(r)$ to each point $0 < |a| < r$, and the uniform measure of mass $1/\eta(r)$ to the circle $|t| = r$. Define

$$v_a(f) = ord_a f \text{ for } |a| < r, \quad v_t(f) = -\log |f(t)|$$

$$ht_{\eta,u}(f) = \lim_{r \to u} \int \max(v_a f, 0) d\mu_r a$$

$$\mathcal{M}[\eta, u] = \{f \in \mathcal{M} : ht_{\eta,u}(f) < \infty\}$$

$$R_t(f_1, \ldots, f_n) := \lim_{r \to u} \int t(v_a f_1, \ldots, v_a f_n) d\mu_r a$$

.

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The product axiom is Jensen's formula:

$$\sum_{0<|a|<r} \log \frac{r}{a} ord_a(f) + \frac{1}{2\pi} \int_0^{2\pi} -\log |f(re^{i\theta})| \, d\theta = O(1)$$

We will see that in the language described above, $\mathcal{M}[\eta]^{alg}$ has the same *universal* theory as the ultraproduct of the $\mathbb{Q}^a[r]$, and also as $\mathbb{C}(t)^{alg}[1]$. This formalizes a part of Vojta's dictionary [**?**] between Nevanlinna theory and number theory, and it would be a the GVF language as currently developed does not permit discussing the support of a function so the most interesting parts of the dictionary are not as yet accessible.

Theorem 1.6. *$k(x)^{alg}[1]$ is existentially closed*

Remark 1.7. It follows from Theorem 1.6 that the asymptotic universal theory of $\mathbb{Q}^{alg}[r]$ is precisely GVF (+ char = 0 + h(2)=0.). Also it is existentially closed over any single parameter.

Let X be a projective variety over k. Let Y be a subscheme, cut out by specified homogeneous polynomials $g_1, \ldots, g_{k_Y}$, and let v be a valuation of L. For $x \in X(L)$ we define

$$\delta_v(x, Y) = \min_g v(g_i(\bar{x}))$$

where $\bar{x}$ is a representative of x in homogeneous coordinates, chosen so that the minimal valuation of a coordinate is 0.

(really $2^{-\delta}$ is a notion of distance from x to Y.). If L is a number field and $x \in X(L)$, let $\rho(x, Y)^L = \int \delta_v(x, Y) dv$ be the weighted sum of the local distances from x to Y.

Note that $\rho(x, Y)^L$ is the L-value of a certain quantifier-free formula $\phi_Y(x)$ in the language of GVF's.

Let $Y_0, \ldots, Y_m$ be subschemes of X. Let $\rho_Y(a) = (\rho_{Y_0} : \cdots : \rho_{Y_m}) \in \mathbb{P}^m(\mathbb{R})$.

Similarly let $\bar{i}(a) = (i(C, Y_0) : \cdots : i(C, Y_m))$.

Corollary 1.8. *Let $X \subset \mathbb{P}^n$ be a smooth projective variety over $\mathbb{Q}$. Let $Y_0, \ldots, Y_m$ be subschemes of X. Let $e \in \mathbb{P}^m(\mathbb{R})$. Assume there exist distinct $a_i \in V(\mathbb{Q})$ with $\rho_Y(a_i) \to e$ in $\mathbb{P}^m(\mathbb{R})$. Then there exists a curve C on V with $\bar{i}(C, Y_k)$ as close as one wishes to e.*

In fact the same conclusion is true if the a_i are only assumed to be of bounded degree in $V(\mathbb{Q}^a)$; or just, of height approaching infinity.

Proof. We may assume Y_0 is an ample divisor, and use height with respect to Y_0. Let $a_i \in V(\mathbb{Q}^a)$ $(i = 1, 2, \cdots)$ have height approaching ∞, such that $\rho_Y(a_i) \to e$. Choose $r_i = ht(2)/ht(a_i)$ so that $\mathbb{Q}^a[r_i]$ gives a_i height 1. Consider any non-principal ultrafilter u on the index set $\mathbb{N}$, and let (L, a) be the GVF ultraproduct of $(\mathbb{Q}^a[r_i], a_i)$. Then (L, a) is a purely non-archimedean GVF, and $\rho_Y(a) = \phi(a)^L = e$. Let $\epsilon > 0$ and fix some metric on $\mathbb{P}^m(\mathbb{R})$. Then by Theorem 1.6 there exists $a' \in K = k(t)^{alg}$ with $e' = \phi(a')^K$ satisfying $|e' - e| < \epsilon$. In fact $a' \in k(C)$ for some curve C, so a' corresponds to a morphism $g : C \to V$. We may choose a' so that g avoids any given proper subvariety of V. By computing the meaning of ϕ in $k(t)^{alg}$ we see that $\bar{i}(C, Y_k) = e'$. □

Remark 1.9. Conversely, if C is a curve on X defined over $\mathbb{Q}^a$, then for any sequence of distinct $a_i \in C(\mathbb{Q}^a)$ of bounded degree over $\mathbb{Q}$, $\rho_Y(a) \to i_Y(C)$. This follows upon taking normalized ultraproducts as above, from the uniqueness of the GVF structure on $k(C)$, Lemma 10.1.

By applying the above remark along with Corollary 1.8, we obtain a purely arithmetic corollary:

Corollary 1.10. *Let $X \subset \mathbb{P}^n$ be a smooth projective variety over $\mathbb{Q}$. Let $Y_0, \ldots, Y_m$ be subschemes of X. Let $e \in \mathbb{P}^(\mathbb{R})$. Assume there exist $a_i \in V(\mathbb{Q}^a)$ of height approaching ∞, with $\rho_Y(a_i) \to e$. Then there exists such a sequence a'_i of bounded degree over $\mathbb{Q}$, $[\mathbb{Q}[a'_i] : \mathbb{Q}] < d$.*

2. Divisors

Let X be an irreducible normal projective variety over a field k, $K = k(X)$ the function field.

Let $Div(X)$ be the group of Cartier divisors on X. If $j : X' \to X$ is a surjective morphism, pullback induces an injective map $Div(X) \to Div(X')$. By $Div(K)$ we denote the direct limit of the groups $Div(X')$ over all birational morphisms $X' \to X$.

A *global section* of a Cartier divisor D on X is a rational function $f \in k(X)$ such that for any open U, if D is represented by (d) on U, then $f = rd$ on U for some regular function r on U. Equivalently, $(f) + D$ is an effective Cartier divisor.

2.1. **Stable joins and meets of Cartier divisors.** Given effective Cartier divisors E, E_1, E_2 on X, represented by hypersurfaces with locally principal ideal sheaves I, I_1, I_2, we say that E is the *stable meet* of E_1, E_2 if I is the ideal sheaf $I_1 + I_2$ generated by $I_1 \cup I_2$. In general, E_1, E_2 may not have a stable meet on X, since $I_1 + I_2$ may not be locally principal. But if we pull back to divisors E_1', E_2' on the blowup X' at $I_1 + I_2$, then E_1', E_2' have a stable meet E', and this remains the case in any further pullback. We write in this situation: $E = E_1 \wedge E_2$.

A similar definition with sheaves of submodules of K extends to all Cartier divisors. It is moreover easy to see that $E = E_1 \wedge E_2$ iff $E + D = (E_1 + D) \wedge (E_2 + D)$, so one can reduce to the effective case.

Define $E = E_1 \vee E_2$ iff $-E = (-E_1) \wedge (-E_2)$; we then say that the *stable join* of E_1, E_2 exists and equals E.

It is clear that if E is the stable meet of E_1, E_2, then it is characterized as the greatest Cartier divisor below both E_1 and E_2; dually if E is the stable join of E_1, E_2, it is the smallest Cartier divisor above both E_1, E_2. Here we refer to the ordering: $X \geq Y$ iff there exists an effective Cartier divisor Z with $X = Y + Z$.

There need not exist any Cartier divisor E on X with $E = E_1 \wedge E_2$. For example, two distinct lines on $\mathbb{P}^2$ never have a stable meet, since their intersection is a point and not a divisor. In fact even a stable meet with 0 may not exist.

However, if E_1, E_2 are effective Cartier divisors, with ideal sheaves I_1, I_2, and $f : X' \to X$ is the result of blowing up the ideal $I = I_1 + I_2$, then it is clear that $f^*E_1 \wedge f^*E_2 = E$ where E is the exceptional divisor. By twisting with an appropriate very ample divisor, we can reduce to the effective case; so any two Cartier divisors do have a stable join, once pulled back to an appropriate blowup. Moreover once we have $E = E_1 \wedge E_2$, the same relation persists in pullbacks to variety X' dominating X. Thus going to the limit, we obtain a statement that is very essential to the theory of GVF's:

Proposition 2.2. *$Div(K)$ forms a lattice under stable meet and join.* □

2.3. **The Picard group.** $Pic(X)$ is the group of Cartier divisors on X, modulo the *principal divisors* (f). An element of $Pic(X)$ determines a line bundle on X. This shows that Pic is covariantly functorial; given a morphism $f : Y \to X$ of varieties, there is a natural map $f^* : Pic(X) \to Pic(Y)$.

2.4. **Intersections of subvarieties with divisors.** An important special case occurs when X is a subvariety of Y and f is the inclusion. In this case, we obtain a map from divisors on Y to divisors on X. If we further represent the divisor on X using subvarieties of X of codimension 1 and then view them as subvarieties of Y of codimension $1 + codim(X)$, this map becomes the operation of *intersecting a subvariety with a divisor.*

Specifically, a *very ample* divisor on X can be represented by a subvariety D on X that intersects Y transversally; further when $\dim(Y) \geq 2$, the intersection is an irreducible variety itself. So

$$[D] \cdot [Y] = [D \cap Y]$$

When $\dim(Y) = 1$, $D \cap Y$ is a finite set of isolated points, and we are interested in their number:

$$[D] \cdot [Y] = |D \cap Y| \in \mathbb{N}$$

2.5. **Weil divisors.** We will generally work with Cartier divisors, but sometimes the Weil viewpoint is useful.

An *integral Weil divisor* is a $\mathbb{Z}$-linear combination of irreducible hypersurfaces of X. More generally, the group $Z_k(X)$ of k-dimensional Weil cycles is the free Abelian group generated by the irreducible subvarieties of dimension k. $W_k(X)$ is the quotient by the k-dimensional divisors that are rationally equivalent to 0. $W^k(X) = W_{n-k}(X)$.

Principal Cartier divisors correspond to principal Weil divisors $(f) = \sum_{v_D(f)} D$. (Here v_D is the valuation associated with D; in an open affine intersecting D, D is the zero locus of a minimal prime ideal; localizing, we obtain a Dedekind domain.)

There exists a canonical biliear map $Pic(X) \times W^k(X) \to W^{k+1}(X)$, the intersection product.

2.6. **Global sections.** With each real Weil divisor $D = \sum \alpha_i D_i$ ($\alpha_i \in \mathbb{R}$) we have an associated k-subspace of K, namely $L(D) = L(X, D) := \{0\} \cup \{f \in K \setminus (0) : \bigwedge_i v_{D_i}(f) \geq -\alpha_i\}$. If D is a Cartier divisor, then $L(D) = \{0\} \cup \{f \in K : (f) + D \geq 0\}$. The elements of $L(D)$ can be viewed as rational sections of the associated line bundle, and called sections of D. When D is a Cartier divisor, with corresponding sheaf $\tilde{D}$, one can identify $L(D)$ with $H^0(X, \tilde{D})$.

2.7. **Intersections of curves with divisors.** The 0-dimensional cycles, up to algebraic equivalence, can be identified with the integers $\mathbb{Z}$ via the degree map. At any rate we have the map $\deg : W_0(X) \to \mathbb{Z}$, $\deg(\sum m_i c_i) = \sum m_i$ where c_i

are points of C. Composing with deg we obtain a numerical intersection product $W^1(X) \times W_1(X) \to \mathbb{Z}$.

The fact that the intersection pairing is well-defined amounts here to the *product formula*: when $D = (f)$ is a *principal* divisor, we have $D \cdot C = 0$

2.8. **Basic positivity property.** Let D be an effective Weil divisor and C a curve, not contained in the support of D. Then $D \cdot C \geq 0$, and equality holds iff $C \cap D = \emptyset$. (The intersection multiplicities are non-negative.)

Example 2.9. Let X be a smooth surface, $a \in X$, and $f : Y \to X$ the blowing up of X at a. Let $E = f^{-1}(0)$ the exceptional divisor. Let C be a a curve on X with a a nonsingular point of C. Then

(1) $f^*C = E + C'$, where C' is the strict transform of C.
(2) $E^2 < 0$, i.e. for the inclusion morphism $i : E \to Y$ we have $i^*E < 0$.

To see (2), lLet D be a divisor on X rationally equivalent to C, with $0 \notin D$. As $f^{-1}(D) \cap E = \emptyset$, $[f^*D] = [f^*C]$, and by (1) we have $(E + C') \cdot E = f^*(D) \cdot E = 0$, so $E^2 = -C' \cdot E$. But C', E intersect properly with nonempty intersection, so $C' \cdot E > 0$ and $E^2 < 0$.

2.10. **Numerical equivalence.** . We consider several subgroups on the group of Weil-divisors, described via their sets of generators. For us, numerical equivalence is the essential equivalence relation; we mention rational equivalence mostly in order to have a well-defined intersection product.

- Rational equivalence: $g : X \to \mathbb{P}^1$ a rational map; $g^{-1}(0) - g^{-1}(\infty)$. (We can view g as a morphism on $X' = X \smallsetminus W$ for some codimension 2 subvariety W; hypersurfaces of X and of X' can be identified.)
- Algebraic equivalence: $g : X \to C$ for some algebraic curve; or more generally, pushforwards of such under generically finite morphisms.
- Numerical equivalence; $Num_0 = \{D : (\forall C) D \cdot C = 0\}$

Example 2.11. Let X be a smooth surface, $a \in X$, and $f : Y \to X$ the blowing up of X at a. Let $E = f^{-1}(0)$ the exceptional divisor. Let C be a a curve on X with a a nonsingular point of C. Then

(1) $f^*C = E + C'$, where C' is the strict transform of C.
(2) $E^2 < 0$, i.e. for the inclusion morphism $i : E \to Y$ we have $i^*E < 0$.

To see (2), lLet D be a divisor on X rationally equivalent to C, with $0 \notin D$. As $f^{-1}(D) \cap E = \emptyset$, $[f^*D] = [f^*C]$, and by (1) we have $(E + C') \cdot E = f^*(D) \cdot E = 0$, so $E^2 = -C' \cdot E$. But C', E intersect properly with nonempty intersection, so $C' \cdot E > 0$ and $E^2 < 0$.

2.12. **The Néron-Severi group.** The Néron-Severi group is $W^1(X)$ modulo the subgroup of divisors algebraically equivalent to zero.

Theorem 2.13 (Lang-Néron, 1951)**.** *$NS(X)$ is finitely generated. In particular, $N^1(X) := NS(X)\otimes\mathbb{R}$ is a finite-dimensional real vector space.*

Note that the Mordell-Weil theorem on finite generation of the rational points of a simple non-isotrivial Abelian variety over a function field is a special case. For instance, if X is a surface admitting a morphism to $\mathbb{P}^1$, such that the generic fiber is an elliptic curve E_K, that does not descend to k, then any K-rational point of E_K has a Zariski closure over k which is a curve; this defines a homomorphsim from $E_K(K)$ to $NS(X)$ which is injective. Hence finite generation of $NS(X)$ implies finite generation of $E_K(K)$.

The Mordell-Weil theorem is usually proved using heights (in any book on diophantine geometry.)
Given this special case, the Néron-Severi theorem yields, with some care, to an induction on dimension.

Remark 2.14. A fourth equivalence relation is obtained by a homological representation. In characteristic zero, the finite dimensionality of $N^1(X)$ can also be seen via the representation in the homology group, along with a proof based on real geometry (e.g. o-minimality) of the finite topological type of $X(\mathbb{C})$.

Definition 2.15. The closure of the ample cone in $N^1(X)$ will be called the *nef* cone.

Remark 2.16. It is clear from the above definition that if D is nef, then $D\cdot Y \geq 0$ for any curve Y on X, and moreover $D^k \cdot Y \geq 0$ for any k-dimensional subvariety Y of X. In fact, the converse is also true: if $D \cdot Y \geq 0$ for any curve Y on X, then D is nef. This follows from Kleiman's theorem, and builds upon Nakai-Moishezon; see [10] for a very nice exposition. We will however simply use the characterization as the closure of the ample cone.

We will briefly use the partial ordering of $N^l_+(X')$ corresponding to the closed cone generated by classes of subvarieties of X' of codimension l. Multiplication by a nef divisor takes $N^l_+(X')$ to $N^{l+1}_+(X')$; it follows that if $x_i \leq y_i$ are both nef divisors, then $x_1 \cdot \ldots \cdot x_k \leq y_1 \cdot \ldots \cdot y_k$.

Lemma 2.17 ([3] Prop. 2.3)**.** *If $d_1, \ldots, d_k$ and $c_1, \ldots, c_k$ are nef, $k = \dim(X)$, $c_i \leq d_i$, then $c_1 \cdots c_k \leq d_1 \cdots d_k$.*

Proof. By continuity and multilinearity we may assume the d_i are ample, or even very ample. We may replace the c_i by d_i one by one; this reduces to the case $k = 1$; in this case, $d_1 - c_1$ is pseudo-effective, so $(d_1 - c_1)c_2 \cdots c_k \geq 0$. □

Let us also define other cones we will use later.

Notation 2.18. Let W be a subset of $\mathbb{R}^n$.

- The interior of a set W will be denoted W^o.

- $\mathbb{R}^{\geq 1}W := \{rw : r \in \mathbb{R}, r \geq 1, w \in W\}$.
- The smallest closed, convex set containing W will be denoted $[W]$.
- A function ϕ on a subset of a real vector space is *positively 1-homogeneous* if for all real $\alpha > 0$ and $x \in \mathrm{dom}(\phi)$ we have $\alpha x \in \mathrm{dom}(\phi)$ and $f(\alpha x) = \alpha f(x)$.

$N^1_+(X)$ = closure of the effective cone.
$N^1_+(X)^o$ = the big cone, interior of $N^1_+(X)$ = classes with positive volume.
$N_1^+(X)$ is defined to be the closed dual of $N^1_+(X)$.
$N_1^{eff}(X)$ is the cone in $N_1(X)$ generated by the classes of irreducible curves. It may be bigger than $N_1^+(X)$.

2.19. **Functoriality of the cycle groups.** Let $\pi : X \to Y$ be a morphism of varieties. If π is *dominant*, there is a natural pullback map of Cartier divisors. If π is not dominant, the trouble is that a Cartier divisor represented by (f) may not pull back to a Cartier divisor, since $f \circ \pi$ may be zero. But there is still a natural pullback of line bundles, or artier divisors up to principal divisors. This induces a pullback map $N^1(Y) \to N^1(X)$.

In the case of a generically finite morphism $\pi : X \to Y$, we can also define a pullback map $\pi^* : N_1(X) \to N_1(Y)$, by dualizing the pushforward map $\pi_* : N^1(X) \to N^1(Y)$. We have $\pi_*\pi^*(c) = \deg(\pi)c$.

When X, Y are smooth we can more generally define a pullback map $\pi^* : N^k(Y) \to N^k(X)$ by the formula:

$$\pi^*(u) = \pi_X((X \times u) \cdot \pi)$$

where $X \times u$ is the obvious class in $X \times Y$, and π here denotes the graph of π as a subvariety of $X \times Y$. Note that if $w \in N^l(X)$ is represented by W, and u is represented by a cycle $\sum \alpha_i D_i$ with $\pi^{-1}(D_i) \cap W = \emptyset$, then $\pi^*(w) = \emptyset$. (The intersection $(Y \times u) \times \pi$ is proper, and the intersection of that with $(X \times D_i)$ is empty, hence proper.)

2.20. **Weighted curves.** It will sometimes be convenient to speak of weighted curves. These are just formal pairs (α, C) with C a curve on X and $\alpha > 0$. The corresponding class in the cycle class group is $\alpha[C]$.

3. Convexity

Recall the notation of 2.18.

Lemma 3.1. *Let ϕ be a convex differentiable function on an open set $U \subset \mathbb{R}^n$. Let $F = d\phi : U \to (\mathbb{R}^n)^*$ the differential. Then F is increasing in the sense that for all $x_1, x_2 \in U$ we have*

$$(F(x_1) - F(x_2), x_1 - x_2) \geq 0$$

If $x_1 \neq x_2$ and ϕ is strictly convex on the line through x_1, x_2, then strict inequality holds.

Proof. By restricting to the line through x_1, x_2 we reduce to the one-dimensional case, where the statement is that a (strictly) convex differentiable function has a (strictly) increasing derivative. □

Theorem 3.2. *Let U be a (nonempty) open convex cone in $E = \mathbb{R}^n$. Let ϕ be a continuous real-valued function on E, differentiable on U. Let*

$$F = d\phi : U \to E^*$$

be the differential. Let

$$C_1 = \{u' \in E^* : (\forall u \in U)(u, u') \geq \phi(u)\}$$

and let C_1^o be the interior of C_1. Assume:

(1) *ϕ is concave on U.*
(2) *ϕ is positively 1-homogeneous*
(3) *$\phi > 0$ on U, and $\phi = 0$ on $E \setminus U$.*

Then

$$C_1^o \subset \mathbb{R}^{\geq 1} F(U) \subset C_1$$

Proof. We first note a few facts about F:

(1) $\phi(x + y) \geq \phi(x) + \phi(y)$
(2) ϕ is strictly increasing on U with respect to $<_U$.
(3) $F(U) \subseteq C_1$.
(4) Let $p \in cl(U)$, $p \neq 0$, and $l \in {C_1}^o$. Then $l(p) > 0$.

Proof of (1-4):

(1) This is immediate from concavity and 1-homogeneity.
(2) If $x, y \in U$ and $x <_U y$, then $y = x + z$ for some $z \in U$; so as $\phi(z) > 0$, $\phi(y) \geq \phi(x) + \phi(z) > \phi(x)$.
(3) For $u, u' \in U$ and $t > 0$ we have by (2) $t^{-1}(\phi(u + tu') - \phi(u)) \geq \phi(u')$. Thus

$$u' \cdot F(u) = \frac{d}{dt}\phi(u + tu')(u) = \lim_{t \to 0^+} t^{-1}(\phi(u + tu') - \phi(u)) \geq \phi(u')$$

Here is a more visual version of the proof of (3): if $c = F(u)$ with $u \in U$, then for appropriate b, the graph of $y = c \cdot x + b$ is tangent to the graph of ϕ at u, and so by concavity of ϕ lies above it, i.e. $cx + b \geq \phi(x)$ for $x \in U$. By homogeneity, for any $x \in U$ and $t > 0$ we have $ctx + b \geq \phi(tx) = t\phi(x)$, so $cx + b/t \leq \phi(x)$; letting $t \to \infty$ we find that $cx \geq \phi(x)$ on U, so $c \in C_1$.

(4) Let $p \in cl(U)$, $p \neq 0$. Then $l \cdot p \geq \phi \geq 0$ for $l \in C_1$. By Lemma 3.7, $l \cdot p > 0$ for $l \in {C_1}^o$.

Since $\mathbb{R}^{\geq 1}C_1 = C_1$, it follows from (3) that $\mathbb{R}^{\geq 1}F(U) \subseteq C_1$. It remains to show that ${C_1}^o \subseteq \mathbb{R}^{\geq 1}F(U)$.

Let $l \in {C_1}^o$. By (4) above, ϕ/l is continuous on $cl(U) \setminus (0)$, and invariant for the multiplicative action of $\mathbb{R}^{\geq 1}$. Let B be a compact ball in E, with boundary sphere S, and p a point of $S \cap cl(U)$ with $(\phi/l)(p) = \gamma$ maximal; we have $\gamma \leq 1$ since $\phi \leq l$ on U and hence on $cl(U)$; also, $\phi > 0$ on U implies $l > 0$ on U so $\phi/l > 0$ on U, and thus $0 < \gamma$ since $S \cap U$ is nonempty. We have $\phi \leq \gamma l$ on $S \cap U$ and hence by $\mathbb{R}^{>0}$-invariance, on U.

Now $\phi(p) = \gamma l(p)$; as $l(p) \neq 0$ and $\gamma > 0$ we have $\phi(p) > 0$, so $p \in U$, and p is a local minimum of $\gamma l - \phi$. Hence $\phi, \gamma l$ are tangent at p, i.e. $d(\gamma l - \phi(p)) = 0$ so $\gamma l = d\phi(p) = F(p) \in F(U)$. Thus $l = \gamma^{-1}(\gamma l) \in \mathbb{R}^{\geq 1}F(U)$.

Thus $C_1^o \subset \mathbb{R}^{\geq 1}F(U) \subset C_1$. □

Corollary 3.3. *With the hypotheses of Theorem 3.2, let C be the closed dual cone:*

$$C = \{u' \in E^* : (\forall u \in U)(u, u') \geq 0\}$$

and let C^o be the interior of C. Then $C = \mathbb{R}^{>0}C_1$. Thus

$$C^o \subset \mathbb{R}^{>0}F(U) \subset C$$

Proof. By Lemma 3.7, If $l \in C^o$ then $l > 0$ on $\bar{U}$; in this case by compactness of the sphere, $\kappa l > \phi$ on $\bar{U}$ for some $\kappa > 1$; so $\kappa l \in C_1^o \subset \mathbb{R}^{\geq 1}F(U)$. Thus $l \in \mathbb{R}^{>0}F(U)$. The other inclusion is immediate from Theorem 3.2. □

Remark 3.4.

(1) We saw in (3) of the proof of Theorem 3.2 that $v \cdot F(u) \geq \phi(v)$. It follows from the 1-homogeneity of ϕ that equality holds for $u = v$:

$$u \cdot F(u) = \frac{d}{dt}\phi(u + tu) = \phi(u)$$

(2) Let $a \in U^o$. Then $F(a)$ is the unique element $w \in C_1$ satisfying $a \cdot w = \phi(a)$. Indeed assume w is such an element. For $v \in E$ and for sufficiently small real t we have $a + tv \in U^o$, so

$$(a + tv) \cdot w \geq \phi(a + tv) = \phi(a) + tv \cdot F(a) + o(t)$$

Since $a \cdot w = \phi(a)$ this gives for small $t > 0$ $t(v \cdot w) \geq tv \cdot F(a)$, and for small $t < 0$ the opposite inequality. We thus obtain $v \cdot w = v \cdot F(a)$, and thus $F(a) = w$. If for $b \in E^*$ we define $\phi^*(b) = \inf\{(u, b)/\phi(u) : u \in U\}$, so that $b \in C_1$ iff $\phi^*(b) \geq 1$, then $a \cdot b \geq \phi(a)\phi^*(b)$, and for fixed $a \in U^o$, equality holds on a unique ray of b.

(3) We will apply the theorem with $\phi = \mathrm{vol}^{1/n}$ for a certain function vol, and set $\psi = \frac{1}{n}d\mathrm{vol}$. Then (1) translates to $u \cdot \psi(u) = \mathrm{vol}(u)$; in the setting of [2] and [3] this is the 'orthogonality relation'.

(4) Similarly, (2) yields that $\psi(a)$ is the unique element b of C_1 such that

$$a \cdot b = \mathrm{vol}\,(a)$$

Example 3.5. Let C be a compact convex subset of $\mathbb{R}^n$. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then $L|C : C \to L(C)$ need not be an open map.

.

Proof. L will be the projection from $\mathbb{R}^3$ to $\mathbb{R}^2$, $L(x, y, z) = (x, y)$. Let

$$C = \{(x, y, z) \in [0, 1]^3 : x^2 \leq y, x^4 \leq zy^2\}$$

Let $p = (0, 0, 0)$. Then p projects to $(0, 0)$, but a neighborhood of p will have no images strictly on the curve $y = x^2$ with $x > 0$. Convexity can be checked by intersecting with lines l. If $L(l)$ passes through $(0, 0)$, i.e. it has the equation $y = mx$, $m \geq 1$, we have above l the set $y^2 \leq m^2 z \leq 1$ which is clearly convex. Away from $(0, 0)$, differentiating $x^4 y^{-2}$ twice we see that it is concave, so the set of points above it is convex.

□

Lemma 3.6. *Let U be an open subset of $\mathbb{R}^n$, and let C be a dense convex subset of U. Then $C = U$. In particular, an open convex set is equal to the interior of its closure.*

Proof. Say $0 \in U$; we have to show that $0 \in C$. Let $(a_i)_{i=1}^{2^n}$ be points of C in each of the 2^n quadrants; this is possible by density. Then 0 lies in the convex hull of the points a_i. Indeed otherwise there is a hyperplane $c \cdot x \sum_{k=1}^n c_k x_k = 0$ through 0, such that all the points a_i lie to one side, i.e. satisfy say $c \cdot x_i < 0$. However there is some i in the same quadrant as c, i.e. $(a_i)_k > 0$ iff $c_k < 0$. Then $\sum_k c_k (a_i)_k > 0$, a contradiction.

For the last statement, let C be open and convex, and let U be the interior of the closure of C; then C is dense in U, so equals U. □

Lemma 3.7. *Let D be a closed cone in $E = \mathbb{R}^n$, C the closed dual cone in the dual space E', i.e.*

$$c \in C \iff (\forall d \in D) c \cdot d \geq 0$$

Let C^o be the interior of C. Then

$$c \in C^o \iff (\forall d \in D \smallsetminus (0)) c \cdot d > 0$$

Proof. Let $c \in C^o$ and $0 \neq d \in D$. As $d \neq 0$, there exists $a \in E'$ with $a \cdot d > 0$. As $c \in C^o$, for small enough $\epsilon > 0$ we have $c - \epsilon a \in C$; so $(c - \epsilon a) \cdot d \geq 0$, or $c \cdot d \geq \epsilon a \cdot d > 0$.

Conversely, assume $a \in E'$ and $a \cdot b > 0$ for all $b \in D$. Fix some Euclidean structure on E, let S be the unit sphere and $P = S \cap U$. Then P is compact, so for some $\epsilon > 0$ we have $a \cdot b > \epsilon$ for all $b \in P$. Similarly there exists a neighborhood U of 0 in E such that $c \cdot b < \epsilon$ for all $c \in U, b \in P$. Then clearly $(a + c) \cdot b > 0$ for all $c \in U$, so $a \in C^o$. □

Lemma 3.8. *Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a surjective linear map; let C be an open convex cone in $\mathbb{R}^n$. Then $L(C) = \mathbb{R}^m$ if $C \cap \ker L \neq \emptyset$. Otherwise, $L(C)$ is an open convex cone in $\mathbb{R}^m$; and $L(cl(C)) = cl(L(C))$.*

Proof. Easy, using Lemma 3.7. □

Lemma 3.9. *Let C be a closed convex subset of $\mathbb{R}^n$. Then the boundary of C has Lebesgue measure zero. If C has nonempty interior and finite measure, it is compact.*

Proof. For the first point, we may assume C is bounded, since the boundary of C is contained in the union of the boundaries of $C \cap B_n$, where B_n is the ball of radius n. Also, we may assume $0 \in C$. Then for any $0 < \alpha < 1$ we have $\alpha C \subset C^o$, i.e. αC is contained in the interior of C. As $\mathrm{vol}\,(\alpha C) = \alpha^n \mathrm{vol}\,(C)$, we see that $\mathrm{vol}\,\partial C \leq (1 - \alpha^n)\mathrm{vol}\,(C)$. Letting $\alpha \to 1$ we see that $\mathrm{vol}\,\partial C = 0$.

Assume now C has non-empty interior; so it contains $a + B_\epsilon$ for some a and some open ball B_ϵ of 0. If $a_1, a_2, \ldots$ are points of C then C contains $(a_i + a)/2 + B_{\epsilon/2}$; if the volume of C is finite then two of these balls must intersect, so $d(a_i, a_j) < \epsilon$ for some i, j. It follows that C is bounded. □

Theorem 3.10 (Khovanskii). *Let F be a finite subset of a $\mathbb{Q}$-vector space B. Let S, A, C be respectively the subsemigroup, subgroup, and rational convex cone generated by F; thus C be the rational convex cone generated by S, i.e.*

$$S = \{\sum m_i a_i : m_i \in \mathbb{N}, a_i \in F\}$$

$$A = \{\sum m_i a_i : m_i \in \mathbb{Z}, a_i \in F\}$$

$$C = \{\sum m_i a_i : m_i \in \mathbb{Q}^{>0}, a_i \in F\}$$

Then for some $s \in S$,

$$A \cap (s + C) = s + A \cap C \subseteq S \subseteq A \cap C$$

Proof. We may assume B is also generated by F. Let $E = A \otimes \mathbb{R} = B \otimes_{\mathbb{Q}} \mathbb{R}$. Then A is a discrete subgroup of the finite-dimensional $\mathbb{R}$-space E. Let $Y = \{\sum_{i=1}^m r_i f_i : -1 \leq r_i \leq 0\}$. Then Y is compact. As A is discrete, $A \cap Y$ is a finite set; we can find $s \in S$ such that $(A \cap Y) + s \subset S$. Let $a \in A \cap C$; so $a = \sum \alpha_i f_i$ for some $\alpha_i \geq 0$. Let $[\alpha_i]$ be the least integer $\geq \alpha_i$, and $r_i = \alpha_i - [\alpha_i]$. Then $\sum r_i f_i \in A \cap Y$; so $s + \sum r_i f_i \in S$. Certainly $\sum [\alpha_i] f_i \in S$. So $s + a \in S$. □

We remark that if we identify A with $\mathbb{Z}^n$, that S is definable in Pressburger arithmetic; hence globally it is defined by a combination of inequalities and congruences.

Proposition 3.11. *Let Λ be a finitely generated, torsion free Abelian group. For $k \in \mathbb{N}$ let S_k be subsets of A with $0 \in S_1$ and $S_k + S_l \subset S_{k+l}$. Let $S = \cup_{k=1}^{\infty} S_k$. Let $C_1 = [\cup_k S_k/k]$ be the closed convex hull of $\{s/k : s \in S_k\}$ in $E = \Lambda \otimes \mathbb{R}$. If K is a compact subset of C_1^o, then for all large enough m,*

$$K \cap \frac{\Lambda}{m} = K \cap \frac{S_m}{m}$$

Proof. Let $\bar{S} = \{(n, x) : x \in S_n\}$; we view $\bar{S}$ as a subset of $\mathbb{R} \times E$. It is clear that $\bar{S}$ forms a semigroup of $\mathbb{Z} \times \Lambda$. The group generated by $\bar{S}$ includes $\mathbb{Z} \times 0$ (since $0 \in S_1$), and maps onto Λ, so it equals $\mathbb{Z} \times \Lambda$. Since $\mathbb{Z} \times \Lambda$ is a Noetherian $\mathbb{Z}$-module, it is generated by a finite set F; let W be the subsemigroup of $\bar{S}$ generated by F. Let C be the convex cone generated by F. By enlarging F we may arrange that $\{1\} \times K \subset C$. (The interior C_1^o of C_1 equals the union of the interiors of the convex hulls $[F']$ of finite subsets F' of $\cup_k S_k/k$, using Lemma 3.6; as K is compact, $\{1\} \times K \subset [F']^o$ for some such F'.) By Theorem 3.10, there exists $w \in W$ with

$$(\mathbb{Z} \times \Lambda) \cap C + w \subseteq W$$

Let $K' = \{1\} \times K$. As K' is compact and $K' \subset C$, there exists a symmetric neighborhood U of 0 in $\mathbb{R} \times E$ such that $K' + U \subset C$. Let m be large enough so that $w/m \in U$. If $b \in K \cap \Lambda/m$, then $(1, b) - w/m \in C$, so $(m, mb) \in w + C$ and hence $(m, mb) \in W \subset \bar{S}$. So $mb \in S_m$ and thus $b \in S_m/m$. □

Lemma 3.12. *Let C be a nonempty open cone in $\mathbb{R}^n$, with closure $\bar{C}$. Then $a \in \bar{C}$ iff $a + C \subseteq C$.*

Proof. Pick $c \in C$. If $a + C \subseteq C$, pick $c \in C$; then $a + tc \in C$ for all $t \in (0, 1)$, so $a \in \bar{C}$. Conversely, if $a \in \bar{C}$, let U be a neighborhood of 0 with $c + \bar{U} \subset C$; then $a + c + \bar{U} \subset \bar{C}$; so $a + c$ lies in the interior C of $\bar{C}$. □

Lemma 3.13 (Tropical Stone-Weierstrass)**.** *Let Z be a compact Hausdorff topological space. Let A be a subset of the space of continuous functions $Z \to \mathbb{R}$ containing the constant function 1, closed under addition, scalar multiplication and* min*, and separating points. Then A is dense in $C(Z)$ in the uniform topology.*

Proof. Let B be the uniform closure of A. If $a \in A$ and $p : \mathbb{R} \to \mathbb{R}$ is a convex function of the form $p(x) = \max_{k=1}^{n}(a_k x + b_k)$, it is clear that $p \circ a = \max_{k=1}^{n}(a_k a + b_k)$ lies in A. It follows that B has the same closure property.

On any compact subset W of $\mathbb{R}$, let $s_W : W \to \mathbb{R}$ be the function $s_W(x) = x^2$; then s_W is the uniform limit of a sequence of piecewise linear maps p_W^j, obtained as the maxima of of certain affine functions $ax + b$. If $g \in B$, letting $W = g(Z)$, we see that $p_W^j \circ g \in B$ and hence $g^2 = lim_j p_W^j \circ g$ lies in B. Writing $xy = ((x+y)^2 - x^2 - y^2)/2$, we see that B is closed under multiplication. By the usual Stone-Weierstrass, $B = C(Z)$. □

4. Higher-order hyperbolic spaces and Hodge index theorem

We will encounter a generalization of the basic linear algebra of Hilbert spaces to higher order maps; we provide here an elementary setting for this.

Definition 4.1. An n-th order hyperbolic space is $\mathbb{R}$- vector space V, an open cone A and a symmetric map $\cdot : A^n \to \mathbb{R}$ denoted $(a_1, \dots, a_n)$, such that:

- $(a_1, \cdots, a_n) > 0$
- Positive homogeneity in each variable: for fixed $a_2, \dots, a_n$, and $t > 0$, we have $(ta_1, a_2, \dots, a_n) = t(a_1, \dots, a_n)$.
- For fixed $a_2, \dots, a_n$, the map $x \mapsto (x, a_2, \dots, a_n)$ is concave.
- For any $a_1, \dots, a_{n-2}, b, c \in A$, we have:

$$(a_1, \dots, a_{n-2}, b, c)^2 \geq (a_1, \dots, a_{n-2}, b, b)(a_1, \dots, a_{n-2}, c, c)$$

Example 4.2 (The bilinear, order 2 case)**.** (1) Consider a real vector space V with a symmetric bilinear form of signature $(1, -1, \cdots, -1)$, or more generally $(1, -1, \cdots, -1, 0, \cdots, 0)$ (with precisely one 1 and at least one -1.) The *future cone* A is one of the two components of $\{x : (x, x) > 0\}$; say the one with $x_1 > 0$.

On A, Cauchy-Scwhartz takes a 'reverse' form, $(x, y)^2 \geq (x, x)(y, y)$; whereas at any fixed time, i.e. on the orthogonal complement to any $a \in A$, we have the usual Cauchy-Scwhartz inequality.

To see that $(x, y)^2 \geq (x, x)(y, y)$ whenever $(x, x) > 0$: we may assume $(y, y) > 0$ as well; renormalizing we may take $(x, x) = 1$, and $y = x + n$ with $(x, n) = 0$; as the orthogonal space to x is negative definite, we have $(n, n) \leq 0$. Now $(x, y)^2 = 1 = (x, x)$ while $(y, y) = 1 + (n, n) \leq 1$.

(2) Conversely, let $(V, A, \cdot)$ be hyperbolic of order 2, with $\cdot$ bilinear. Pick $a \in A$ with $|a| = 1$. Consider

$$H = \{b : (a, b) = 0\}$$

Then H is negative-semidefinite: otherwise we can find $b \in H$ with $(b, b) > 0$ and (replacing b by a small positive scalar multiple, so that b is close to 0) with $a + b \in A$, yet $(a, a+b)^2 = 1 < 1 + (b, b) = (a, a)(a + b, a + b)$.

(3) Going back and forth, from (2) to (1) and back, we see that in (2) we may take any $a \in H$ with $(a, a) > 0$; the orthogonal complement is still negative-semidefinite.

(4) It is sometimes convenient to formulate Cauchy-Scwharz for two classes x, y, assuming $(a, a) > 0$ and $y \cdot a = 0$, but without any assumption on x.

Then by considering $x' = x - \alpha a$, where $a \cdot x = \alpha a^2$, one obtains

$$(x,y)^2 \le (x^2 - \alpha^2 a^2) y^2$$

(5) A nondegenerate symmetric bilinear map (,) is hyperbolic iff the function $|x| = (x,x)^{1/2}$ is concave on $A = \{x : (x,x) > 0\}$. Indeed assuming concavity, we have $(a+b, a+b)^{1/2} \ge (a,a)^{1/2} + (b,b)^{1/2}$, or squaring, $(a,a) + 2(a,b) + (b,b) \ge (a,a) + 2(a,a)^{1/2}(b,b)^{1/2} + (b,b)$ which yields $(a,b)^2 \ge (a,a)(b,b)$

Lemma 4.3. *For $x_1, \ldots, x_n \in A$ we have*

$$(x_1, \ldots, x_n) \ge |x_1| \cdots |x_n|$$

More generally, for $0 < k < n$,

$$(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n)^{n-k} \ge (x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+1}) \cdot \ldots \cdot (x_1, \ldots, x_k, x_n, \ldots, x_n)$$

Proof. By induction on $n-k$, being given the case $n-k=2$. Let us consider for example the case $n=3$. First we have:

$$(a,b,b)^2 \ge (b,b,b)(a,a,b)$$

so

$$(a,b,b)^4 \ge (b,b,b)^2 (a,a,b)^2$$

Plugging in the dual of the first equation, i.e. $(a,a,b)^2 \ge (a,a,a)(b,b,a)$, and dividing by (a,b,b) we obtain:

$$(a,b,b)^3 \ge (b,b,b)^2 (a,a,a)$$

Now this in turn can be plugged into the relation

$$(a,b,c)^2 \ge (a,a,c)(b,b,c)$$

to give $(a,b,c)^6 \ge (a,a,a)^2 (b,b,b)^2 (c,c,c)^2$ and taking square roots we obtain the required formula.

(See [18].)

□

4.4. **log-concavity.** Let f be a function on a cone in a real vector space, into the positive reals, and assume $f(tx) = tf(x)$ for $t > 0$. Then f is concave iff $\log f$ is concave. The left to right implication is clear since log is concave. For the other, assume $\log f$ is concave; then whenever $0 < \alpha, \beta$ and $\alpha + \beta = 1$, we have $\log f(\alpha x + \beta y) \ge \alpha \log f(x) + \beta \log f(y)$; in particular when $f(x) = f(y)$ this implies that $f(\alpha x + \beta y) \ge f(x)$. By positive homogeneity of f, for any $\gamma > 0$ we have $f(\gamma \alpha x + \gamma \beta y) \ge f(\gamma x) = \gamma f(x)$; in other words for any $0 < \alpha, \beta$, if $f(x) = f(y)$ we have $f(\alpha x + \beta y) \ge (\alpha + \beta) f(x)$. Given arbitrary x, y, find γ such that $\gamma f(x) = f(y)$; let $x' = \gamma x$; let $\alpha' = \alpha / \gamma$; then $f(\alpha x + \beta y) = f(\alpha' x' + \beta y) \ge (\alpha' + \beta) f(y) = \alpha f(x) + \beta f(y)$.

The volume function on A**.** We define vol : $A \to \mathbb{R}$ by $\mathrm{vol}\,(a) = (a, \dots, a)$.

Lemma 4.5. $\mathrm{vol}^{1/n}$ *is concave on* A.

Proof. Using concavity in each variable, we have

$$\mathrm{vol}\,(a+b) = (a+b, \cdots, a+b) \geq \sum_k \binom{n}{k} (a, \cdots, a, b, \cdots, b)$$

By the main statement of Lemma 4.3, applied term-by-term, we obtain:

$$\mathrm{vol}\,(a+b) \geq \sum_k \binom{n}{k} \mathrm{vol}^{k/n}(a) \mathrm{vol}^{(n-k)/n}(b) = (\mathrm{vol}^{1/n}(a) + \mathrm{vol}^{1/n}(b))^n$$

or

$$\mathrm{vol}^{1/n}(a+b) \geq \mathrm{vol}^{1/n}(a) + \mathrm{vol}^{1/n}(b)$$

Using the positively homogeneity of $\mathrm{vol}^{1/n}$, this gives concavity of $\mathrm{vol}^{1/n}$ □

Thus $-\log \mathrm{vol}$ is convex on A.

Example 4.6. Let $\mathbb{R}^n$ have basis $e_1, \dots, e_n$. Consider the symmetric n-multilinear form with $(e_1, \cdots, e_n) = 1$, and $e_i^2 = 0$ (meaning $(e_i, e_i, x_3, \dots, x_n) = 0$ for all $x_3, \dots, x_n$.) Let $A = \{\sum \alpha_i e_i : \alpha_1, \dots, \alpha_n > 0\}$. This is a generalized hyperbolic space; if we fix any $a_1, \dots, a_{n-2} \in A$, then $(x, y) \mapsto (a_1, \dots, a_{n-2}, x, y)$ has a kernel of dimension $n-2$ and signature $0, \cdots, 0, 1, -1$. Note that if $c = \sum \alpha_i e_i \in A$, then $\frac{c^n}{n!}$ is the Euclidean volume of the rectangle with sides $(\alpha_1, \dots, \alpha_n)$. Thus in this case Lemma **??** gives the concavity of $\mathrm{vol}^{1/n}$ for rectangles. This is a special case of Brunn-Minkowski.

4.7. **The Hodge index theorem.**

Theorem 4.8. *Let* X *be a projective variety of dimension* n. *Then* $N^1(X)$ *with intersection product, and the ample cone, is an order-n multilinear hyperblic space.*

Proof. Let $a_1, \dots, a_{n-2}, a_{n-1} = b, a_n = c$ be ample divisors. We have to show that $\mathrm{vol}\,(a_1) = a_1 \cdot \dots \cdot a_1 > 0$ (this is clear), and that

$$(a_1 \cdot \ldots \cdot a_{n-2} \cdot b \cdot c)^2 \geq (a_1, \dots, a_{n-2}, b, b)(a_1, \dots, a_{n-2}, c, c)$$

Now by Bertini, a_i can be represented by an irreducible hypersurface D_i, such that $D_1 \cap \cdots \cap D_k$ is a transversal intersection, resulting in an irreducible subvariety of X (for $k \leq n-2$.) Let $S = D_1 \cap \cdots \cap D_{n-2}$. Then S is a surface. Let $b' = D_{n-1}|S, c' = D_n|S$. Then $(a_1, \dots, a_{n-2}, b, c) = b' \cdot_S c'$ and similarly for (b, b) and (c, c). Thus the inequality reduces to the case of a *surface*. In this case, as noted above, it suffices to show that the signature is $(1, -1, \cdots, -1)$.

If $\widetilde{S} \to S$ is a resolution, then $N^1(X)$ embeds into $N^1(\widetilde{S})$ in a way that respects the intersection form, and maps an ample class to an ample class. Thus it suffices to show that $\widetilde{S}$ has hyperbolic signature.

This is the classical Hodge index theorem for surfaces, cf. [14]. If one uses the Okounkov approach, Corollary 6.8, this requires only the 2-dimensional Brunn-Minkowski. □

Remark 4.9. When X is a smooth projective variety of dimension n, $V = N^1(X)$, $A =$ the ample cone, the n-fold intersection pairing on divisors is *nondegenerate*, i.e. whenever $v \in V$ and $(a_1, \dots, a_{n-1}, v) = 0$ for all $a_1, \dots, a_{n-1} \in A$, then $v = 0$.

Note that the pairing $N^1(X) \times N_1(X) \to \mathbb{R}$ is nondegenerate by definition; thus $\dim(N^1(X)) = \dim N_1(X)$, and the content of the above statement is that every class in $N_1(X)$ is in the image of $\cdot^{n-1}$.

This can be seen as a consequence of the Hard Lefschetz theorem for algebraic classes, asserting that even for a fixed ample a, the map $(u, v) \mapsto (a, \cdots, a, u, v)$ is nondegenerate. This can be interpreted as the Hessian of volume. We will see a cone-adapted version of this further down, see Theorem **??**.

Example 4.10. Let S be a smooth projective surface admitting morphisms $\pi_i : S \to C_i$ into curves, mapping onto $C_1 \times C_2$. Let P_i be the pullback of a point under π_i; then $A = P_1 + P_2$ is ample on S. The Hodge index theorem includes, and is easily seen to be equivalent to, the negative definiteness of the intersection form on V, the orthogonal space to P_1 and P_2. For any divisor D, we have $D^* := D - (D \cdot P_1)P_2 - (D \cdot P_2)P_1$ in V. Thus the Hodge index theorem asserts that $(D^*, D^*) \leq 0$. Explicitly, this reads:

$$(D, D) \leq 2(D \cdot P_1)(D \cdot P_2)$$

This is the Castelnuovo inequality [7]. Weil developed the foundations of algebraic geometry between 1940-1948 precisely in order to provide a statement and proof of this inequality valid in positive characteristic.

Example 4.11. Here is Weil's 1941 deduction of the Riemann hypothesis for curves from Castelnuovo, i.e. from the Hodge index theorem for a product of curves.

By Example 4.2 we have

$$(\Phi^* \cdot \Delta^*)^2 \leq (\Phi^*, \Phi^*)(\Delta^*, \Delta^*)$$

where we take $\Phi = \Phi_q$ to be the graph of the Frobenius morphism on $C = C_1 = C_2$, and Δ the diagonal. Here (Δ^*, Δ^*) does not depend on q;$\Phi \cdot \Delta$ is the number of points of $C(\mathbb{F}_q)$ that we seek; all other terms are easily evaluated, yielding Weil's theorem, $|C(\mathbb{F}_q)| = q + O(q^{1/2})$.

Lemma 4.12. *Let S be a irreducible projective surface. Let $\widetilde{S} \to S$ be a surjective, generically finite morphism, with $\widetilde{S}$ smooth. Let e be an irreducible curve on S, and let $A, D \in N^1(S)$. Assume $D \cdot A = 0$ while $(A, A) = 1$. Then*

$$(e \cdot D)^2 \leq (\tilde{e}^2 - (A \cdot e)^2)D^2$$

where $\tilde{e}$ is any weighted curve on $\widetilde{S}$ lifting e. In particular if $D^2 = 0$ then $e \cdot D = 0$.

Proof. If we replace D, A by their pullbacks, and e by $\tilde{e}$, the same conditions hold and all the quantities in the displayed inequality are the same, by the projection formula ([10] 1.9,1.10). Thus we may assume S is smooth and connected, and $e = \tilde{e}$. By Remark 4.2 (4) we have $(e \cdot D)^2 \leq (e^2 - (A \cdot e)^2)D^2$. □

The material from here to the end of the section is adapted from [20] (part of Theorem 1.1; they acknowledge an idea of Blocki's.) We begin with a *relative* Hodge index theorem. (In case $\pi^*(c)$ is represented by a subvariety of X, it is easy to deduce it from the Hodge index theorem on a plane section of that subvariety. For a surface mapping to a curve, let l be an ample divisor with $(f, l) = 1$, and note that $v_i \mapsto v_i - (v_i, l)f$ is an isometry into the space orthogonal to l, with kernel $\mathbb{R}f$. But we give a direct proof.)

Proposition 4.13. *Let X be a smooth variety with a morphism $\pi : X \to U$, of relative dimension n. Let D be an effective divisor of U; and let $c \in N_1(U)$ be a class represented by an effective curve C disjoint from any specified subvariety of U of codimension > 1. V be the subspace of $N^1(X)$ generated by the classes of the irreducible components of $\pi^{-1}(D)$. Then for nef $a_1, \ldots, a_{n-1}$, the pairing $V^2 \to \mathbb{R}$,*

$$(v, v') = a_1 \cdot \ldots \cdot a_{n-1} \cdot v \cdot v' \cdot \pi^*(c)$$

is negative semidefinite.

Proof. By approximation, we may take the a_i to be ample, with rational coefficients; multiplying by an integer, we may take them integral, and very ample. Let $v_1, \ldots, v_k$ be the distinct irreducible components of $\pi^{-1}(D)$. Let $D' = \sum \beta_j D_j$ be a divisor on U, numerically equivalent to D, such that $D_j \cap D$ has codimension ≥ 2 in U for any j. (We use smoothness of U to move D, to a not necessarily effective divisor.) Choose a representative C of c which is disjoint from $D \cap D_j$, for any j. Let α_i be the multiplicity of v_i in $\pi^{-1}(D)$; then $f = \sum \alpha_i v_i$ is numerically equivalent to $\pi^{-1}(D')$. Now $v_i \cap \pi^{-1}(D_j) \cap \pi^{-1}(C) = \emptyset$ for each j. Thus (see § 2.19) $v_i \cdot \pi^*(D') \cdot \pi^*(c) = 0$; so $v_i \cdot f \cdot \pi^*(c) = 0$, hence $< v_i, f >= 0$. For $i \neq j$, v_i meets v_j properly in an effective cycle; thus $a_1 \cdot \cdots \cdot a_{n-1} \cdot v_i \cdot v_j$ is effective, hence so is $\pi_*(a_1 \cdot \cdots \cdot a_{n-1} \cdot v_i \cdot v_j)$; by the moveability property of c, it follows that $c \cdot \pi_*(a_1 \cdot \cdots \cdot a_{n-1} \cdot v_i \cdot v_j) \geq 0$; so $< v_i, v_j >= a_1 \cdot \cdots \cdot a_{n-1} \cdot v_i \cdot v_j \cdot \pi^*(c) \geq 0$. Thus Lemma 4.14 applies. □

Lemma 4.14. *Let V be a vector space over $\mathbb{R}$ with a symmetric bilinear form. Assume V is generated by vectors $v_1, \ldots, v_n$,*

$$f = \sum_{i=1}^{n} \alpha_i v_i$$

for some $\alpha_1, \ldots, \alpha_n > 0$*, and:*

$$v_j \cdot f = 0, v_i \cdot v_j \geq 0$$

for $i \neq j \leq n$*. Then* V *is negative semidefinite. The kernel* $\{x : (\forall y) x \cdot y = 0\}$ *is generated by the vectors* $\sum_{i \in c} \alpha_i v_i$*, where* c *is a connected component of the graph whose edges are the pairs* (i, j) *with* $v_i \cdot v_j \neq 0$*.*

Proof. Replacing v_i by $\alpha_i v_i$, we may assume $f = \sum_i v_i$. Let $b_i \in \mathbb{R}, i = 1, \ldots, n$, $g = \sum b_i v_i$; we have to show that $g^2 \leq 0$. As $v_j \cdot f = 0$ we have also $b_j^2 v_j (\sum_i v_i) = 0$; summing over j we find that

$$\sum_{i,j} b_j^2 v_i v_j = 0$$

As $v_i v_j \geq 0$ for $i \neq j$, we have:

$$g^2 = -\frac{1}{2} \sum_{i,j=1}^{l} (b_i - b_j)^2 v_i v_j = -\sum_{i \neq j} (b_i - b_j)^2 v_i v_j \leq 0$$

If equality holds, then $b_i = b_j$ whenever $v_i v_j \neq 0$, i.e. the coefficients are constant on each connected component. (And the converse is clear.) □

Remark 4.15. This also shows (assuming $\pi^{-1}(D)$ is connected) that there are no unexpected linear relations among the classes of the v_i in $N^1(X)$. In fact there are no unexpected linear inequalities among the classes $[v_i]$. Suppose $[\sum \alpha_i v_i] = [e]$ with e effective on X, and some α_i negative. We may assume e does not include any of the v_i as components, or else we may subtract. Let $u = \sum \max(0, \alpha_i) v_i$. Then $(u, u) < 0$ by negative definiteness, while (u,e) and for j with $\alpha_j < 0$ we have$(u, v_j) \geq 0$ since the intersection is proper. This is a contradiction unless there are no positive entries, and in that case the sum is 0.

4.16. **Calabi uniqueness.**

Lemma 4.17. *Let* W *be a real vector space endowed with a symmetric, nondegenerate* $n+1$*-multilinear map* $\cdot$*. Let* V *be a subspace of* W*,* A *a subset of a coset of* V*. For* $a = (a_1, \ldots, a_{n-1}) \in A$*, let* $b_a(v, v') = a_1 \cdot \ldots \cdot a_{n-1} \cdot v \cdot v'$*. Assume each* b_a *is negative semidefinite on* V *(*$a \in A$*.)*

Let $a_1, a_2 \in A$*, and assume* $v \cdot a_1^n = v \cdot a_2^n$ *for all* $v \in V$ *(or even just for* $v = a_1 - a_2$*). Then* $a_1 - a_2$ *lies in the kernel of* b_a *for any* $a \in A^{n-1}$*.*

Proof. Let $c = a_1 - a_2$. Consider first the case $n = 2$.

We assume $ca_1^2 = ca_2^2$. So

$$0 = c(a_1^2 - a_2^2) = c^2(a_1 + a_2)$$

But $c^2 a_i \leq 0$, so $c^2 a_1 = c^2 a_2 = 0$.

By Cauchy-Schwarz (for b_{a_i}), we have $vca_i = 0$ for any $v \in V$.

Hence $c^2v = c(a_1 - a_2)v = 0$ for any $v \in V$. As $c^2a_1 = 0$, we have $c^2x = 0$ for $x \in a_1 + V$ and in particular $c^2a = 0$ for all $a \in A$. Again by negative semidefiniteness, c lies in the kernel of b_a.

Now assume $n > 2$. We have by assumption:

$$ca_1^n = ca_2^n$$

So

$$0 = c(a_1^n - a_2^n) = c(a_1 - a_2)(a_1^{n-1} + \cdots a_1^i a_2^{n-1-i} + \cdots a_2^{n-1})$$

But $c^2a_1^ia_2^{n-1-i} \leq 0$; so each term in the sum must vanish, in particular $c^2a_i^{n-1} = 0$.

By Cauchy-Schwarz we have $vca_i^{n-1} = 0$ for any $v \in V$. As $a_ica_i^{n-1} = 0$, we hve $eca_i^{n-1} = 0$ for any $e \in A$. Thus fixing any $e \in A$, the $n-1$-multilinear map $x_1 \cdot \ldots \cdot x_{n-1} \cdot e_n$ satisfies the same conditions. By induction, c lies in the kernel of $b_{e_1,\ldots,e_{n-1},e}$ for any $e_1, \ldots, e_{n-1} \in A$, which finishes the proof.

□

5. Volume via section growth

So far, we defined volume *for ample divisors* via the intersection product. Our official definition of volume will instead use the asymptotic dimension of the space of sections of multiples of the divisor. This definition is also more general; an equally general definition in terms of positive intersections will be considered later (§ 8.3), see [18], 11.4.10, 11.4.11. We begin by relating the section growth with the intersection theory definitions for ample divisors.

If X is 0-dimensional scheme, let $|X|$ be the sum over all points of X of the dimension of the local ring.

If X is closed subvariety of $\mathbb{P}^N$, of dimension d, we define

$$\deg(X) = |X \cap L_1 \cap \cdots \cap L_{N-d}|$$

where L_i are hyperplanes chosen generically enough, so that $\dim(X \cap L_1 \cap \cdots \cap L_{N-d}) = 0$. (It will follow from Lemma 5.1 that the choice of the L_i is irrelevant.)

On the other hand, let $A = \oplus H_n$ be the graded ring $k[X_0, \ldots, X_N]$; H_n the homogeneous polynomials of degree n. For a homogeneous ideal I, say $vol(I) = v$ if $\dim(H_n/H_n \cap I) = vm^d/d! + O(m^{d-1})$.

Lemma 5.1. *Let $X = Proj(A/I)$. Then $\deg(X), \mathrm{vol}\,(I)$ are both defined and are equal. In fact, for large enough n, $\dim(H_n/H_n \cap I) = p(n)$ for some polynomial $p(n)$ of degree d, with leading term equal to $\deg(X)n^d/d!$.*

Proof. The case $\dim(X) = 0$ follows easily from the definitions. Assume the result known below $\dim(X)$, and choose any $x \in H_1$ such that x is not a 0-divisor in A/I. Let $I_n = I \cap H_n$, $I' = I + xA$, $I'_n = I' \cap H_n$, $X' = Proj(I')$. Note that $(I + xA) \cap H_n = I_n + xH_{n-1}$ since the ideal I is homogeneous. Thus

$$\dim(I'_n/I_n) = \dim(I_n + xH_{n-1}/I_n) = \dim(xH_{n-1}/I_n \cap xH_{n-1}) = \dim(H_{n-1}/I_{n-1})$$

using in the last step the assumption that x is not a 0-divisor in A/I. So

$$\dim H_n/I'_n = \dim H_n/I_n - \dim H_{n-1}/I_{n-1}$$

Let $f(n) = \dim H_n/I_n$. Then $\dim H_n/I'_n = f(n) - f(n-1)$. By induction, as $\dim(X') < \dim(X)$, for large n we have $\dim H_n/I'_n = q(n)$ with q a polynomial in n of degree $d-1$, and leading term $\deg(X') = \deg(X)$. We can find a polynomial $p(n)$ of degree d, with the same leading term $\deg(X)$, and with $p(n) - p(n-1) = q(n)$. Then $p-f$ is eventually constant; adjusting by this constant we may assume $p-f$ is eventually 0, i.e. $p(n) = f(n)$ for large n. □

Let X be a variety of dimension d. We define the volume of an integral divisor D to be

$$vol(D) = d! \limsup_m \dim L(X, D^m)/m^d$$

In particular $\mathrm{vol}\,(D) = v$ if $\dim H^0(X, D^m) = v\binom{m}{d} + O(m^{d-1})$.

D is *big* if it has nonzero volume.

Corollary 5.2. *If D is very ample, then* $\mathrm{vol}\,(D) = D^d$. *Hence for any D', $L(X, mD') = O(m^{\dim(X)})$.*

Proof. Embed X into projective space $\mathbb{P}^N$ via D. Then D^d equals the intersection number of X with d generically chosen hyperplanes on $\mathbb{P}^N$, and the statement follows from Lemma 5.1. The second statement follows by choosing an ample $D \geq D'$. □

Proposition 5.3 ([18] 2.2.15)**.** *Let A, B be ample Cartier divisors on X. Then*

$$\mathrm{vol}\,(A - B) \geq \mathrm{vol}\,(A) - n(A^{n-1}B)$$

Proof. We may assume A, B are very ample. Fix a countable base field of definition k_0 for the data. Using Bertini, let $E = E_1, E_2, \ldots$ be irreducible hypersurfaces of X, E_i defined over some extension k_i of k_0, such that (k_i, X, E_i) are all isomorphic over k_0, and E_i represents B as a Weil divisor. Let $A_i = A|E_i$; then the $n-1$-fold intersection product of A_i on E_i equals $A^{n-1} \cdot B$; by Lemma 5.1 applied to $A|E$ on E,

$$h^0(E_i, mA_i) = h^0(E, mA|E) = A^{n-1} \cdot Bm^{n-1}/(n-1)! + O(m^{n-2})$$

Again by Lemma 5.1 for A on X,

$$h^0(X, m(A-B)) = A^n m^n/n! + O(m^{n-1})$$

We have an exact sequence

$$0 \to H^0(X, mA - mB) \to H^0(X, mA) \to \oplus_{i \leq m} H^0(E_i, mA_i)$$

Thus

$$h^0(X, m(A-B)) \geq h^0(X, mA) - \sum_{i \leq m} h^0(E_i, mD|E_i) = (\mathrm{vol}\,(A) - n(A^{n-1}B))\frac{m^n}{n!} + O(m^{n-1})$$

□

Remark 5.4 (Continuity of volume)**.** If we assume only that B is ample, with B represented by E as above, the proof of Proposition 5.3 still yields the inequality in the form: $\mathrm{vol}\,(A-B) \geq \mathrm{vol}\,(A) - n\mathrm{vol}_E(A|E)$. Thus if $\beta = 1/b$, we have $|\mathrm{vol}\,(A-\beta B) - \mathrm{vol}\,(A)| = b^{-n}|\mathrm{vol}\,(bA-B) - \mathrm{vol}\,(bA)| \leq b^{-n}(nb^{n-1}\mathrm{vol}_E(A|E)) = nb^{-1}\mathrm{vol}_E(A|E)$, so $|\mathrm{vol}\,(A-\beta B) - \mathrm{vol}\,(A)| \leq n\beta\mathrm{vol}_E(A|E)$. By choosing n linearly independent very ample divisors, we see in this way that the volume function is continuous on $N^1(X)$; see [18] 2.2.44.

Remark 5.5. From the continuity of volume Remark 5.4, it is clear e.g. that Proposition 5.3 is valid for *nef* and not only ample divisors.

Definition 5.6. The open cone of elements of $N^1(X)$ of positive volume is called the *big cone.*

A big integral divisor clearly has an effective multiple, so the big cone is contained in the interior of the pseudo-effective cone. Conversely if D lies in the interior of the effective cone, equivalently by Lemma 3.6 of the pseudo-effective cone, it can be written as ample + effective, so being $\geq$ than an ample divisor D' it has all the sections of D' and so is big. We denote the big cone, interior of the pesudo-effective cone, by $N^1_+(X)^o$. In particular ample classes lie in $N^1_+(X)^o$.

The following corollary of Proposition 5.3, a lower bound on volume near a nef divisor, is used in [3] to prove differentiability of volume on the big cone. The point is that the constants in front of the quadratic error term t^2 do not change upon blowing up X or diminishing β.

Corollary 5.7. [3] *Let X be a projective n-dimensional variety. Let $\beta \in N^1(X)$ be nef, and $\gamma \in N^1(X)$. Let ω be nef and big, and assume $\beta \leq \omega$ and $\omega \pm \gamma$ is nef. Then for any $t \in [-1, 1]$ we have*

$$\mathrm{vol}\,(\beta + t\gamma) \geq \beta^n + nt\beta^{n-1}\cdot\gamma - 8^n\omega^n t^2$$

Proof. As the inequality for t and γ is the same as for $-t$ and $-\gamma$, we may assume $t \in [0, 1]$. Note first that with $\gamma' = \omega - \gamma$, we have $\gamma' \leq 2\omega$, so

$$\beta^i\gamma^{n-i} = \beta^i(\omega-\gamma')^{n-i} \geq \beta^i(\omega - 2\omega)^{n-i} \geq -\beta^i\omega^{n-i} \geq -\omega^n$$

Thus (using $|t| \leq 1$)

$$(\beta + t\gamma)^n - \beta^n - nt\beta^{n-1}\cdot\gamma = \sum_{i\geq 2}\binom{n}{i}t^i\beta^i\gamma^{n-i} \geq -\omega^n t^2 \sum_{i\geq 2}\binom{n}{i} \geq -\omega^n 2^n t^2$$

Now let $A = \beta + t(\gamma + \omega)$ and $B = t\omega$. Then $A \leq 3\omega$ so

$$(A-B)^n = A^n - nA^{n-1}B + \sum_{i\geq 2} \pm \binom{n}{i} t^i A^{n-i}\omega^i \leq A^n - nA^{n-1}B + t^2(3\omega)^n 2^n$$

But $(A - B) = \beta + t\gamma$, so combining the last two displayed equations,

$$A^n - nA^{n-1}B \geq \beta^n + nt\beta^{n-1}\gamma - 8^n\omega^n t^2$$

By Proposition 5.3 and Remark 5.5, we have

$$\mathrm{vol}\,(\beta + t\gamma) = \mathrm{vol}\,(A - B) \geq \beta^n + nt\beta^{n-1}\gamma - 8^n\omega^n t^2$$

□

We include here a basic lemma on the effect of morphisms on the pseudo-effective cone of divisors, and its dual.

Lemma 5.8. *Let $\pi : X \to U$ be a surjective morphism of projective varieties, U normal. Then*

(1) $(\pi^*)^{-1}N^1_+X = N^1_+U$.
(2) *If* $\dim(X) > \dim(U)$, *then* $\pi^*(N^1_+(U))$ *lies on the boundary of* N^1_+X*; and only then.*
(3) $\pi_*N_1^+(X) = N_1^+(U)$, *and* $\pi_*N_1^+(X)^o = N_1^+(U)^o$.
(4) $\pi_*N_1^{eff}(X) = N_1^{eff}(U)$, *and* $\pi_*N_1^{eff}(X)^o = N_1^{eff}(U)^o$.
(5) *If U is normal and D is a Cartier divisor on U, then* $\mathcal{O}(X, \pi^*D) \cong \mathcal{O}(U, D)$.

Proof. We first prove (4). If C is an irreducible curve on U, then it is the image under π of an irreducible curve C' on X, and we have $[C] = d\pi_*[C']$ for an appropriate $d > 0$. It follows that $\pi_*N_1^{eff}(X) = N_1^{eff}(U)$. In particular $\pi_* : N_1(X) \to N_1(U)$ is surjective, and hence (being linear) open. In particular $\pi_*N_1^{eff}(X)^o \subset N_1^{eff}(U)^o$. By continuity $(\pi_*)^{-1}N_1^{eff}(U)^o \subset \pi_*N_1^{eff}(X)^o$ and so $\pi_*N_1^{eff}(X)^o = N_1^{eff}(U)^o$.

As for (5), consider a rational rational function h on X which is regular on $X \smallsetminus \pi^*D$. Let h be a section of π^*D. Then h has no zeroes or poles on a fiber of $X \to U$ above $U \smallsetminus D$, so it must be constant on the fibers of π above $U \smallsetminus supp(D)$. So h equals a rational function on U. Moreover $\pi^*(D + (h)) \geq 0$ so $D + (h) \geq 0$ - any poles of $D + (h)$ as a Weil divisor would show up in the pullback.

Towards (1), consider Cartier divisors D on U (with integer coefficients). If D is effective, it is clear that π^*D is effective. Going to rational coefficients and by continuity of π, it follows that $(\pi^*)^{-1}N^1_+X \subseteq N^1_+U$.

If $D \in N^1(U) \smallsetminus N^1_+U$, then D is represented say as $\sum \alpha_i D_i$ with $\alpha_i \in \mathbb{R}$ and D_i a Cartier divisor, and (by Corollary 8.8) there exists a curve C on U with proper intersection with each D_i, and with $\sum \alpha_i D_i \cdot C < 0$. By the proof of (4) we may take $C = \pi_*C'$ for some curve C' on X. It follows that C' is not contained in any $D_i' = \pi^*D_i$, and that $\sum \alpha_i D_i' \cdot C' = \sum \alpha_i D_i \cdot C_i < 0$. Thus $\pi^*D \notin N^1_+X$. This finishes the proof of (1).

For (2), suppose $\pi^*(N^1_+(U))$ meets the interior of N^1_+X. Then by density there exists a $\mathbb{Q}$-Cartier divisor D with $\pi^*(D)$ big, and multiplying by an integer we may assume D is a Cartier divisor. But by (5), $\mathcal{O}(X, \pi^* D) \cong \mathcal{O}(U, D)$. Thus $\pi^* D^n$ has a dimension $O(n^{\dim(U)})$, and cannnot be big if $\dim(U) < \dim(X)$. Conversely, if $\dim(X) = \dim(U)$ then the pullback of an ample divisor of U is big on X, and thus lies in the interior of $N^1_+ X$ and not on the boundary. This proves (2).

For (3), Let $c \in N_1^+(X)$. Then for any $d \in N^1_+(U)$, we have $\pi^*(d) \in N^1_+(X)$, so $\pi_* c \cdot d = \pi_*(c \cdot \pi^*(d)) \geq 0$. Thus $\pi_* N_1^+(X) \subseteq N_1^+(U)$. By the M. Riesz theorem, since $(\pi^*)^{-1} N^1_+ X = N^1_+ U$, any positive linear map on $N^1_+ U$ extends to one on $N^1_+ X$; in other words, $\pi_* N_1^+(X) = N_1^+(U)$. The rest of (3) follows from Lemma 3.8 and Lemma 3.6.

□

6. Okounkov bodies

6.1. **Valuations of maximal rank.** Let K/k be a field extension with $tr.deg._k K = d$. A *maximal rank* valuation v on K/k is a valuation on K/k with value group $\Lambda = v(K^*)$ of rank d; i.e. $E_v := \mathbb{R} \otimes \Lambda$ is a d-dimensional real vector space. Later, we will use the Lebesgue (or Haar) measure on E_v, normalized so that Λ has covolume 1.

When K/k is finitely generated, a maximal rank valuation necessarily has finitely generated value group, isomorphic to $\mathbb{Z}^d$. It determines a strongly stably dominated type over $v(K^*)$, in the sense of [15]. We will not need these facts here.

We note however that any field extension K/k of finite transcendence degree d admits a maximal rank valuation: we can embed K into the algebraic closure of $k[X_1, \dots, X_d]$, and extend the lexicographic valuation $v(\sum \alpha_\nu X^\nu) = \min\{\nu : \alpha_\nu \neq 0\}$. This gives a valuation on K with value group a finite extension of $\mathbb{Z}^d$, so it must also be isomorphic to $\mathbb{Z}^d$.

Lemma 6.2. *Let v be a valuation on K/k. Let H be a finite-dimensional k-subspace of K, of dimension e. Then $v(H \smallsetminus (0))$ is a set of cardinality at most e. If v has residue field k, in particular if $k = k^{alg}$ and v has maximal rank, then $|v(H \smallsetminus (0))| = e$.*

Proof. For $\gamma \in v(H \smallsetminus (0))$, let $H_\gamma = \{h \in H : v(h) \geq \gamma\}$. Then H_γ is a nonzero subspace of H; if $\gamma' < \gamma \in v(H \smallsetminus (0))$ then $H_{\gamma'}$ contains H_γ properly. Thus $\gamma \mapsto \dim H_\gamma$ is an injective map from $v(H \smallsetminus (0))$ to $\{1, \dots, e\}$.

Now assume v has residue field k. Pick $\gamma \in \Gamma$, and let γ' be the least element of $v(H)$ greater than γ. Pick $h_\gamma \in H$ with $v(h_\gamma) = \gamma$. Define $f : H_\gamma \to k$ by $f(x) = \mathrm{res}(x/h_\gamma)$. Then f is k-linear, with kernel $H_{\gamma'}$. This shows that $H_\gamma / H_{\gamma'}$ is 1-dimensional. So $e = \dim(H)$. □

6.3. **Okounkov bodies ([19]).** To define Okounkov bodies, we will need the following data: a function field K/k, i.e. a finitely generated extension field of

k, of transcendence degree d; a maximal rank valuation v on K/k, and a k-subalgebra S of K generating K as a field, along with a filtration $S = \cup_n S_n$, with $1 \in S_1$ and $\dim(S_n) = O(n^d)$ (i.e. for some $C \in \mathbb{R}$, for $n \geq 1$ we have $\dim(S_n) \leq Cn^d$.)

We will view v as auxiliary and omit it from the notation; we thus write $Ok(S)$.

Typically, S_n will be $L(X, nD) \cong H^0(X, nD)$, where X is a k- variety, D an effective divisor on X (so $1 \in S_1$.). In this case we write $Ok(X, D)$ for the Okounkov body.

Definition 6.4. Let $V_n = \{v(x) : x \in S_n \setminus (0)\}$, $V = \cup_n V_n$. $Ok(S)$ is the closed convex hull in E_v of $\cup_n \frac{V_n}{n}$. We will also write $Ok(V)$ for the same set.

We could also define $Ok(S)$ to be the closure of $\cup_n \frac{V_n}{n}$, since this set is closed under the operations $(u, v) \mapsto \alpha u + (1 - \alpha)v$ for rational $\alpha \in (0, 1)$.

Since S generates K, there exist $x_1, \ldots, x_n$ with $v(x_i)$ linearly independent; the convex hull $[0, v(x_1), \cdots, v(x_n)]$ of $0 = v(1), v(x_1), \ldots, v(x_n)$ is an n-simplex, so $Ok(S)$ has nonempty interior S^o.

We let μ_n be the counting measure on $\frac{V_n}{n}$, normalized so that each point has measure n^{-d}. This can be compared to μ'_n, the (infinite) counting measure on all of Λ/n, with the same normalization.

If S' is a k-subalgebra of S, $S'_n := S_n \cap S'$, then $V'_n \subset V_n$ and $Ok(S') \subset Ok(S)$.

Claim . Let $C \subset Ok(S)^o$ be compact. Then there exists a finitely generated subalgebra S' of S, with $C \subset Ok(S')^o$; in fact there exists a finitely generated subsemigroup V' of $V = v(S \setminus (0))$ with $C \subset Ok(V')^0$. Moreover, for large enough n,

$$C \cap \frac{\Lambda}{n} \subset V'_n$$

Proof. Immediate from Proposition 3.11. □

Theorem 6.5. *The measures μ_n converge weakly to Lebesgue measure μ on $Ok(S)$. We have $\mu(Ok(S)) = \mathrm{vol}\,(D)/\dim(X)!$.*

Proof. Let μ'_n be the counting measure on Λ/n, normalized so that each point has measure n^{-d}. Then $\mu_n \leq \mu'_n$, and μ'_n tends weakly to μ on any compact set. This shows that any limit of the μ_n is bounded below μ.

Note that $\mu(\partial Ok(S)) = 0$ (Lemma 3.9). Thus it suffices to show that $\mu_n \to \mu$ on any compact subset C of the interior $Ok(S)^o$.

But on C, for large n, by Lemma 6.3, $\mu'_n = \mu_n$. As μ'_n tends weakly to μ, so does μ_n.

Let $v_n = \mu_n(Ok(S))$; so $v_n \to \mu(Ok(S))$; and we have by definition $\mathrm{vol}\,(D) = (\dim(X)!) \limsup v_n$.

□

Corollary 6.6. *$Ok(S)$ is compact.*

Proof. By Theorem 6.5, and since μ_n has bounded total mass, $Ok(S)$ has finite Lebesgue measure. Compactness follows from Lemma 3.9. □

Corollary 6.7 (bigness, volume properties)**.** *Let D be an integral divisor on X such that $tr.deg.(\oplus_n H^0(X, nD)) = \dim(X)$. Then $\operatorname{vol}(D) > 0$, and this volume is attained as a limit, not just a lim sup.*

Proof. As the μ_n tend weakly to a limit, namely to Lebesgue measure on $Ok(S)$, the total volume $v_n = \mu_n(Ok(S))$ tends to a limit $\alpha > 0$; we have by definition $\operatorname{vol}(D) = (\dim(X)!) \limsup v_n = (\dim(X)!) \lim v_n$. □

Here is Okounkov's proof of log-concavity of volume, in the setting of big divisors. (Another proof that will be accessible a little later reduces to the ample case Lemma 4.5 by Fujita approximation; but our proof of Fujita approximation will also use Okounkov bodies.) As shown in [3], this latter proof lends itself to an analysis of the case of equality (sharp log-concavity). It would be interesting to determine when two divisors have the same Okounkov bodies, or translates thereof, and thus deduce the sharp log-concavity using Okounkov bodies.

Corollary 6.8 (log concavity of volume)**.** $\operatorname{vol}^{1/n}$ *is concave: for D_1, D_2 with* $\operatorname{vol}^{1/n}(D_i) > 0$ *we have* $\operatorname{vol}^{1/n}(D_1 + D_2) \geq \operatorname{vol}^{1/n}(D_1) + \operatorname{vol}^{1/n}(D_2)$

Proof. $Ok(X, D)$ contains the Minkowski sum $Ok(X, D_1) + Ok(X, D_2)$. Denoting these bodies by O, O_1, O_2 we have by Brunn-Minkowski ([13] 3.1):

$$\mu(O)^{1/n} \geq \mu(O_1)^{1/n} + \mu(O_2)^{1/n}$$

where μ is Lebesgue measure, normalized so that the group Γ generate by $v(\oplus_m L(mD))$ has covolume 1. In this case the corresponding groups Γ_i of D_i have covolume ≥ 1, so if we let μ_i be the normalization giving O_i covolume 1 we have $\mu_i = c_i \mu$ with $c_i < 1$. Thus

$$\mu(O)^{1/n} \geq \mu_1(O_1)^{1/n} + \mu_2(O_2)^{1/n}$$

and we are done by Theorem 6.5. □

7. Fujita approximation

7.1. **Global generation.**

Definition 7.2. A Cartier divisor D is *globally generated* or *base-point free* or simply *free* if the global sections $L(X, D)$ if for any point p there exist f, r, d as above on a neighborhood U of p, such that in addition $r(p) \neq 0$.

A $\mathbb{Q}$-divisor D is said to be semi-free if mD is globally generated, for some $m \in \mathbb{N}$.

Since a one-dimensional vector space is generated by any subset containing a nonzero element, a Cartier divisor D is free iff associated invertible sheaf is generated by its global sections.

Remark 7.3. If D is semi-free, then D is nef, i.e. $D^k \cdot Y \geq 0$ for any k-dimensional subvariety Y. In particular, $D \cdot Y \geq 0$ for every curve Y. To see this we may assume D is free. Given a variety Y on X, we can find a global section s of D that does not vanish on Y; then $D \cdot Y$ is represented by the zero scheme of s on Y. so it is a positive sum of effective $[Y']$ of dimension $k-1$.

Remarks 7.4. Let D be a free Cartier divisor on X.

(1) Any pullback g^*D via $g : X' \to X$ is free on X'.
(2) If D, D' represent the same line bundle, then D' is free. (Global generation is a property of the line bundle.)
(3) We have a natural morphism $f : X \to PH^0(X, D)$ (Grothendieck convention.) We have $D = f^*\mathcal{O}[1]$ (the global sections agree by definition, and both line bundles are generated by them.)
(4) D is ample iff $D \cdot C > 0$ for any curve C on X. Indeed, if $D \cdot C = f^*\mathcal{O}[1] \cdot C = \mathcal{O}[1] \cdot f_*C$ so f_*C cannot be a point. Thus f does not collapse any curve to a point, so it is finite-to-one. By a theorem of Serre, $f^*\mathcal{O}[1]$ is ample.
(5) D has at least one section, so it is effective.
(6) If D' is an ample Cartier divisor, then $D + D'$ is ample. (This follows immediately from (4), aplied to $D + D'$.)

Lemma 7.5 (Corollary 1.32 of [10])**.** *Assume D is a nef Cartier divisor with* $\mathrm{vol}\,(D) > 0$*. Then there exist ample Cartier divisors D_t (say for t a small positive rational) such that $D_t \leq D$ and $D_t \to D$ numerically. In particular, for any $\epsilon > 0$ there exists an ample $A \leq D$ with* $\mathrm{vol}\,(A) \geq (1-\epsilon)\mathrm{vol}\,(D)$*.*

Proof. Let H be any ample divisor, represented by a hypersurface E. As $\dim(E) = n-1$, by Corollary 5.2 we have $h^0(mD|E) = O(m^{n-1})$, while by definition $h^0(mD)$ grows like m^n. The exact sequence

$$0 \to L(X, mD - H) \to L(X, mD) \to L(E, mD|E)$$

shows that for large enough m, $L(mD-H) \neq 0$. Thus $mD = H + E'$ with $E' \geq 0$. For any $t \in (0,1)$, let $D_t = \frac{t}{m}H + (1-t)D$. But D_t is ample (the boundary of the ample cone stabilizers the ample cone.) Now $D = D_t + \frac{t}{m}E'$ so $D_t \leq D$, and $D_t \to D$ as $t \to 0$; by continuity of volume at D, $\mathrm{vol}\,(D_t) \to \mathrm{vol}\,(D)$. □

Remark 7.6. This gives a quick way of seeing using Definition 2.15 that the pullback of the nef cone is contained in the nef cone: the pullback of an ample is free, of positive volume, and so can be approximated by ample divisors.

Let $D \geq E \geq 0$ be Cartier divisors.

Lemma 7.7. *(cf.* [3] *Lemma 2.6). Let C, D be Cartier divisors on X, and assume their stable join $C \vee D$ exists. If C, D are free, then so is $C \vee D$.*

Proof. By assumption, C, D are generated by global sections; we have to show the same for $C \vee D$. In other words given a point $p \in X$, we have to find a global section of $C \vee D$ that does not vanish at p. Let f be a global section of C not vanishing at p, and g a global section of D not vanishing at p. On some neighborhood U of p, C is represented by (f^{-1}) and D by (g^{-1}), while $C \vee D$ is represented by h^{-1} say. Then $-C$ is represented by (f), $-D$ by (g), and $-(C \vee D) = (-C) \wedge (-D)$ by (h); so we have $(h) = (f, g)$. Thus $f = rh$, $g = sh$ with r, s regular at p. Not both r, s can be 0, since otherwise $(h) = \mathcal{M}_p(h)$ where $\mathcal{M}_p$ is the maximal ideal of the local ring $\mathcal{O}_p$, contradicting Nakayama. Now both f, g are global sections of $C \vee D$, and at least one of them fails to vanish at p. □

Lemma 7.8 ([3] Lemma 2.6)**.** *D_1, D_2 be $\mathbb{Q}$-Cartier divisors, and assume $D = D_1 \vee D_2$ is the stable join of D_1, D_2. If D_1, D_2 are semi-free, then so is D. If D_1, D_2 are nef, then so is D.*

Proof. For semi-free this is immediate from Lemma 7.7. Assume D_1, D_2 are nef. Let A be a very ample Cartier divisor. Then for any rational $t > 0$, $D_i + tA$ is ample, hence semi-free. Moreover $D + tA$ is the free join of $D_1 + tA$ and $D_2 + tA$. Thus $D + tA$ is semi-free, hence nef. Letting $t \to 0$ we see that D is nef. □

Lemma 7.9. *Let D be an effective Cartier divisor on X, and let $f_1, \dots, f_n \in L(D)$ be sections of D. Then there exists a blow-up $\pi : X' \to X$ and a free divisor D' on X' with $D' \leq \pi^* D$, such that each $f_i \circ \pi$ is a section of D'.*

Proof. First we may find a birational morphism $\pi^* : X^* \to X$, such that each rational function f_i extends to a morphism $f_i^* : X^* \to \mathbb{P}^1$. Let $N_i = (f_i^*)^{-1}(\infty)$. Clearly $f_i \circ \pi^*$ is a section of N_i. The Cartier divisor N_i is represented by f_i^{-1} away from $f_i^{-1}(0)$, and by 1 away from $f_i^{-1}(\infty)$; as $D \geq 0$ and $(f_i) + D \geq 0$, we have $N_i \leq \pi^* D$. Being the pullback of the free divisor ∞ of $\mathbb{P}^1$, N_i is free.

Passing to a further blowup X', we may assume the N_i have a stable join N (§ 2.1). Each $N_i \leq \pi^* D$, so $N \leq \pi^* D$. Being the stable join of free divisors, N is also free, by Lemma 7.7. □

Theorem 7.10 (Fujita approximation)**.** *Let $D = \sum \alpha_i D_i$ with D_i an effective Cartier divisor, and $\alpha_i \in \mathbb{R}, \alpha_i > 0$. Assume* $\mathrm{vol}\,(D) > 0$*, and let and let $\epsilon > 0$. Then there exists a birational morphism $\phi : X' \to X$, $D' = \phi^* D$, with $D' \geq N$ for some ample $\mathbb{Q}$-divisor N, satisfying* $\mathrm{vol}\,(N) \geq (1 - \epsilon)\mathrm{vol}\,(D)$.

Proof. Note that there exists a very ample A with $D \leq A$; it follows that $\dim L(X, kD) \leq \dim L(X, kA) \leq O(k^d)$, $d = \dim(X)$.

Let $K = k(X)$, and $S_k = L(X, kD)$. Let C be a compact subset of $Ok(X, D)^o = Ok(S)^o$ with Lebesgue measure $\mu(C) > (1 - \epsilon/2)\mu(Ok(X, D))$. Let $f_1 \in L(X, k_1 D), \dots, f_r \in L(X, k_r D)$ be elements, such that the multiplicative semigroup S' they generate in $\oplus_k L(X, kD)$ has Okounkov body containing C (see Claim before Theorem 6.5).

If we replace each f_i by $f_i^m \in L(X, mk_iD)$, the Okounkov body does not change; hence it does not change if we just change f_1 by $f_1^{m_i}$. In this way, taking $m_i = \Pi_{j \neq i} k_j$, we may take all k_i to be equal to the same integer k.

By Lemma 7.9, there exists a blow-up $\pi : X' \to X$ and a free divisor D' on X' with $D' \leq \pi^*(kD)$, such that each $f_i \circ \pi$ is a section of D'. Let $N = D'/k$; then N is semi-free, and $N \leq \pi^*(D)$.

Each $f_i \in L(X, kN)$. So $S' \subset \oplus L(X, N)$, hence $C \subset Ok(X, N)$, and thus $\mathrm{vol}\,(N) \geq (1 - \epsilon/2)\mathrm{vol}\,(D)$.

As $\mathrm{vol}\,(D) > 0$, 'semi-free' can be replaced by 'ample' using Lemma 7.5. □

Theorem 7.10 was first proved in [11] (developing, according to [18], ideas of Demailly on positive intersections.)

It shows in particular that volume is a numerical invariant.

8. Positive intersection product and differentiability of volume

The Zariski-Riemann approach of [3] fits very well indeed with the model theory of GVF's (indeed the description of quantifier-free types led us to the same formalism). Nevertheless we wish to remain for a while longer in the finite-dimensional setting, and will describe the main results of [3] without this formalism. The positive intersection product here is the restriction of the one in [3] to domain $N^1(X)$ and range $N_1(X)$; moreover we consider only the $n-1$'st power.

8.1. **Definition of ψ, the $n-1$'st positive intersection power.** For $x \in N^1_+(X)^o$, define:

$$A^{nef}(x) = \{(f, X', y) : f : X' \to X \text{ birational}, y \in N^1(X') \text{ a nef } \mathbb{Q}\text{-Cartier divisor }, y \leq f^*(x)\}$$

$$W_x = \{f_* y^{n-1} : (f, X', y) \in A^{nef}(x)\}$$

W_x is precompact. To see this pick an ample $a \geq x$. Also pick ample divisors b_j such that $x \mapsto (x \cdot b_1, \ldots, x \cdot b_m)$ is an isomorphism $N_1(X) \to \mathbb{R}^m$. Any nef $y \leq f^*(x)$ satisfies $y \leq f^*(a)$, $y^{n-1} \leq f^*(a)^{n-1}$, and $f_* y^{n-1} \cdot b_i \leq a^{n-1} \cdot b_i$. (We used Lemma 2.17.) Thus the image of W_x is bounded.

Secondly, W_x is *directed* with respect to the partial ordering of $N_1(X)$ given by the closed cone generated by classes of curves on X. In fact $A^{nef}(x)$ is directed, in the following sense: if y_1, y_2 are nef $\mathbb{Q}$-Cartier divisors below $f^*(x)$, let D' represent $f^*(x)$; and represent y_1, y_2 by $\mathbb{Q}$-Cartier divisors below D'. By pulling back to a further blowup we may assume their stable join y exists. Then $y \leq D'$, and by Lemma 7.8, y is nef.

Thus we can define $\psi : N^1_+(X)^o \to N_1(X)$ by:

$$\psi(x) = \sup W_x$$

Remarks 8.2. (1) The definition of ψ would not change if in the definition of W_x we replaced *nef* by *semi-free* or by *ample*. Indeed, since x has positive volume, by Theorem 7.10 some $y_0 \in A^{nef}(x)$ has positive volume. Whenever $(f, X', y) \in A^{nef}(x)$ lies above y_0, it has positive volume, and so can be approximated by an ample divisor, as any big, nef divisor is approximated by smaller amples by Lemma 7.5 .

(2) .

8.3. **A second definition.** We also give a definition that does not require blowing up, and instead uses a naive intersection theory on X; compare [18], 11.4.10, 11.4.11 (where the n-fold intersection is treated.) Let D be an effective Cartier divisor, with at least one nonzero section. Let k be an integer (in practice we will use only $k = \dim(X) - 1$. Let X^- be a Zariski open subset of X. Let $f_1, \ldots, f_k$ be k mutually generic sections of $L(X, D)$, let Z be their common zero set *on* X', viewed as a subscheme of X'; let $[Z]$ be the sum of components of Z of codimension k, counted with multiplicity. Each irreducible subvariety of X' appearing in $[Z]$ can be identified with its Zariski closure in X, so we can also view $[Z]$ as a cycle in $N^k(X)$; denote $\langle D; X' \rangle^k = [Z]$.

Claim . For some $z \in N^k(X)$, for all Zariski open X' disjoint from the support of D, we have $\langle D; X' \rangle^k = z$.

Let B_D be the base locus of D: the set of points at which every section of D vanishes. Let W be a subvariety of X. If W is not contained in B_D, then a generic section f of $L(X, D)$ does not vanish on W, so cutting W with the zero set of f reduces dimension. It follows that as soon as $X' \cap B_D = \emptyset$, all components of Z have codimension at least k.

Moreover if we replace X' by $X'' = X' \setminus W$, where W is some hypersurface of X, then for generically chosen $f_1, \ldots, f_k$, no k-dimensional component of Z will be contained in W; thus $\langle D; X' \rangle^k = \langle D; X'' \rangle^k$.

We can now define $\langle D \rangle^k_{naive}$ to be equal to $\langle D; X' \rangle^k$, for any sufficiently small open $X' \subset X$. We will assume in particular (the base field being perfect) that X' is smooth, i.e. disjoint from the singular locus of X.

Note also that if $g_1, \ldots, g_k \in L(X, D)$ have common zero set Z' on X' of codimension k, then $[Z'] \leq [Z] = \langle D; X' \rangle^k$ (the inequality holds with respect to the effective cone in $N^k(X)$.) In other words the generic choice of sections maximizes the intersection on X'. The reason is that $(f_1, \ldots, f_k)$ specializes to $(g_1, \ldots, g_k)$; by smoothness of X', every component of Z' is the specialization of some component of Z, and is covered with appropriate multiplicity. Some components of Z may specialize into $X \setminus X'$ and not be counted in Z'; but at any rate we have the inequality. We may summarize this symbolically in the formula:

$$\langle D; X' \rangle^k = \inf_{X'} \sup_{f_1,\dots,f_k \in L(D)} [Z_{X'}(f_1) \cap \cdots \cap Z_{X'}(f_k)]_{n-k}$$

If $D \le D'$, let us compare $\langle D \rangle^k_{naive}$ with $\langle D' \rangle^k_{naive}$. Choose a sufficiently small open X', and generic $f_1, \dots, f_k \in L(X, D)$ and $g_1, \dots, g_k \in L(X, D')$. Then by the above specialization argument, the generic choice in D' gives a bigger or equal result. Thus $\langle D; X' \rangle^k_{naive} \le < D'; X' >^k_{naive}$, and

$$\langle D \rangle^k_{naive} \le \langle D' \rangle^k_{naive}$$

Next we compare $\langle D \rangle^k_{naive}$ to $\langle lD \rangle^k_{naive}$. Again choose a sufficiently small open X', and generic $f_1, \dots, f_k \in L(X, D)$ and $g_1, \dots, g_k \in L(X', D)$. Then $(g_1, \dots, g_k)$ specializes to $(f_1^l, \dots, f_k^l)$. It follows as above that $l^n \langle D; X' \rangle^k_{naive} \le \langle lD; X' \rangle^k_{naive}$ and so (using linearity of intersection with Cartier divisors)

$$l^k \langle D \rangle^k_{naive} \le \langle lD \rangle^k_{naive}$$

Finally we define

$$\langle D \rangle^k = \limsup_m \frac{1}{m^k} \langle mD \rangle^k_{naive}$$

(in fact it follows easily from the above that this is also the lim inf and hence the limit.)

Clearly $\langle lD \rangle^k = l^k \langle D \rangle$.

Lemma 8.4. *Let D be a big Cartier divisor. Then $\langle D \rangle^{n-1} = \psi(D)$.*

Proof. We first show that $\langle D \rangle^{n-1} \le \psi(D)$, say in the sense that $\langle D \rangle^{n-1} \cdot A \le \psi(D) \cdot A$ for every very ample A. It suffices to show that $\langle D; X' \rangle^{(n-1)} \le \psi(D)$. Let $f_1, \dots, f_{(n-1)}$ be $(n-1)$ mutually generic sections of $L(X, D)$. By Lemma 7.9 there exists a blow-up $\pi : X' \to X$ and a free divisor D' on X' with $D' \le \pi^* D$, such that each $f_i' = f_i \circ \pi$ is a section of D'; moreover by the proof, such that $(f_i)^+$ is nef. So $\langle D; X' \rangle^{(n-1)} \cdot A \le (f_1)^+ \cdot \ldots \cdot (f_{(n-1)})^+ \cdot \pi^* A \le (D')^{n-1} \cdot \pi^* A \le \psi(D) \cdot A$. (As $(f_i)^+$ and D' are nef, and $(f_i)^+ \le D'$.)

Conversely for any blow-up $\pi : X' \to X$ and ample $\mathbb{Q}$-Cartier divisor $B \le \pi^* D$, we have to show that

$$\pi_*(B^{n-1}) \cdot A \le \langle D \rangle^{n-1} \cdot A = \limsup_m \frac{1}{m^n} \langle mD \rangle^k_{naive}$$

It suffices to show for each m that $\pi_*(B^{n-1}) \cdot A \le \frac{1}{m^n} \langle mD \rangle^k_{naive}$; replacing D by mD and B by mB, it suffices to show that $\pi_*(B^{n-1}) \cdot A \le \langle D \rangle^k_{naive}$. Once more multiplying B and D by a large integer, we may assume B is a very ample Cartier divisor. Let E be a proper subvariety of X' containing the exceptional divisor of π, as well as the pullback $\pi^{-1} B_D$. We have $\pi_*(B^{n-1}) \cdot A = (f_1)^+ \cdot \ldots (f_{n-1})^+ \cdot (g)^+$ for sufficiently generic sections f_i of B and g of A. But the f_i, viewed as rational functions are also sections of $\pi^* D$ and hence of D. Moreover, $(f_1)^+ \cdot \ldots (f_{n-1})^+ \cap (g)^+ \cap E = \emptyset$. Thus if Z is the common zero set of $f_1, \dots, f_{n-1}$

on X, away from B_D, then $Z \cap (g)^+ = \pi_*(B^{n-1}) \cdot A$. It follows that the common zero set of generic sections $F_1, \ldots, F_{n-1}$ of D, away from B_D, is no smaller, and so $\langle D; X' \setminus C_B \rangle^k \geq \pi_*(B^{n-1}) \cdot A$. Hence finally $\langle D \rangle^k_{naive} \geq (f_1)^+ \cdot \ldots \cdot (f_{n-1})^+ \cdot (g)^+$. □

Lemma 8.5. *ψ is homogeneous of degree $n-1$, monotone and continuous on $N^1_+(X)^o$.*

Proof. Monotonicity is immediate from the definition of ψ, and homogeneity is also easy.

Now any function v, monotone with respect to the ordering of a cone E, and homogeneous of degree $d \geq 1$, must be continuous on the interior of E. For let $e \in E^o$ and $\epsilon > 0$. Then $\epsilon a \in E^0$, so there exists a symmetric neighborhood $N = N_\epsilon$ of 0 with $\epsilon a + N \subset E^0$, hence $\epsilon a \geq N$. Thus $(1+\epsilon)a \geq a + N \geq (1-\epsilon)a$ so by homogeneity $(1+\epsilon)^d f(a) \geq f(a+N) \geq (1-\epsilon)^d f(a)$; hence for any linear functional L with $L(E) \geq 0$ we have $(1+\epsilon)^d Lf(a) \geq Lf(a+N) \geq (1-\epsilon)^d Lf(a)$; letting $\epsilon \to 0$ we see that Lf is continuous at a. As such L's span the dual space, f is continuous. □

Here is the main theorem of [3].

Theorem 8.6. *The volume function* vol *is differentiable on the big cone of $N^1(X)$; and $d\mathrm{vol} = n\psi$.*

Here we use the identification $N_1 = N^1(X)^*$ given by the intersection pairing. Explicitly, if $\alpha \in N^1(X)$ is big and $\gamma \in N^1(X)$ is arbitrary, the statement is that

$$\frac{d}{dt}|_{t=0} \mathrm{vol}\,(\alpha + t\gamma) = n\psi(\alpha) \cdot \gamma$$

:

Proof. Fix an ample ω such that $\alpha \leq \omega$ and $\omega \pm \gamma$ is ample. Consider birational morphisms $f : X' \to X$; let $\omega' = f^*\omega$, $\alpha' = f^*\alpha$, $\gamma' = f^*\gamma$. Note that ω' and $\omega' \pm \gamma'$ remain nef. Recall Corollary 5.7: if $\beta \in N^1(X')$ is nef, for any $t \in [-1, 1]$ we have $\mathrm{vol}\,(\beta + t\gamma') \geq \beta^n + nt\beta^{n-1} \cdot \gamma' - 8^n \omega^n t^2$. When $\beta \leq \alpha'$ it follows that $\mathrm{vol}\,(\alpha' + t\gamma') \geq \beta^n + nt\beta^{n-1} \cdot \gamma' - 8^n \omega^n t^2$. Taking the supremum over all X' and all nef $\beta \leq \alpha'$, we find, using Theorem 7.10 and the definition of ψ:

$$\mathrm{vol}\,(\alpha + t\gamma) \geq \mathrm{vol}\,(\alpha) + nt\psi(\alpha) \cdot \gamma - 8^n \omega^n t^2$$

Applying this to $\alpha^* = \alpha + t\gamma$ (t small enough so that this is still big), $\gamma^* = -\gamma$, so $\alpha = \alpha^* + t\gamma^*$,

$$\mathrm{vol}\,(\alpha) \geq \mathrm{vol}\,(\alpha + t\gamma) - nt\psi(\alpha^*) \cdot \gamma - 8^n \omega^n t^2$$

As ψ is continuous at α, this implies that $\frac{\mathrm{vol}\,(\alpha+t\gamma) - \mathrm{vol}\,(\alpha)}{t}$ is bounded on both sides by functions that approach $n\psi(\alpha)$ as $t \to 0$. □

Let $N_1^+(X)$ be the closed dual cone to $N_+^1(X)$; thus $c \in N_1^+(X)$ iff for every effective $e \in N^1(X)$ we have $c \cdot e \geq 0$.

Call a class in $N_1(X)$ 'strongly movable' if it is a pushforward, of the intersection product of $n-1$ ample rational classes in some blowup of X; if these are the same ample class, call it 'very strongly movable'. In either case, note that such a class is a positive scalar multiple of the class of a curve, a complete intersection of very ample divisors.

Theorem 8.7. *Let* $\dim(X) = n > 1$. *Then*

$$N_1^+(X)^o \subseteq \psi(N_+^1(X)^o) \subseteq N_1^+(X)$$

Proof. By definition of ψ as a positive intersection product, any element of $\psi(N_+^1(X)^o)$ is the supremum of a family of very strongly movable classes, and in particular the limit of a sequence of such classes. Now if a is ample in X' and $f : X' \to X$ is a birational morphism, then for any effective e on X we have $f_*(a^{n-1}) \cdot e = a^{n-1} \cdot f^*(e) \geq 0$. Thus any very strongly movable class lies in $N_1^+(X)$ and so $\psi(N_+^1(X)^o) \subseteq N_1^+(X)$.

To show the inclusion $N_1^+(X)^o \subseteq \psi(N_+^1(X)^o)$ we verify the conditions of Theorem 3.2, with $U = N_+^1(X)^o$ the big cone, $\phi(x) = \mathrm{vol}\,(x)^{1/n}$, $F = \mathrm{vol}^{-\frac{n-1}{n}}\psi$, $C = N_1^+(X)$. The volume function is continuous and non-negative and hence so is ϕ. The concavity (1) follows from Corollary 6.8 (or from the nef case along with Theorem **??**, by continuity.)

The positive homogeneity (2) is clear, and (3) comes from the definition of the big cone.

Note that $\mathbb{R}^{\geq 1}F(U) \subseteq \mathbb{R}^{>0}\psi(U) = \psi(U)$ (this last equality, by the $n-1$-positive homogeneity of ψ.) By Corollary 3.3, $N_1^+(X)^o \subseteq \psi(U)$.

□

Corollary 8.8. [2] *The dual of the pseudo-efffective cone, $N_1^+(X)$, is the closed convex cone generated by the (very) strongly movable classes, or by all movable classes.*

For our purposes it is important to know that the image of ψ is dense, without taking convex closure.

Corollary 8.9. *Let $c \in N_1(X)$ satisfy: $(c, e) \geq 0$ for all $e \in N_+^1(X)$. Then c can be approximated by classes $\alpha[C]$ with C an irreducible curve, and $\alpha \in \mathbb{Q}, \alpha > 0$. Moreover, C lies in an algebraic family of curves, whose generic members are not contained in any given hypersurface of X. Indeed $[C]$ can be taken to have the form $f_*(D_1 \cap \cdots \cap D_{n-1})$ where D_i are Bertini / Weil representatives of an ample class on X', $f : X' \to X$ birational.*

Remarks 8.10. (1) ψ is injective on $Nef^1(X) \cap N_+^1(X)^o$.

(2) If $x, y \in Nef^1(X) \cap N_+^1(X)^o$ and $\psi(x), \psi(y)$ are proportional then so are x, y.

(3) ψ is a homeomorphism on the ample cone, and continuous on the nef cone.

Proof. (1,2) $\mathrm{vol}^{1/n}$ is strictly concave on $Nef^1(X) \cap N^1_+(X)^o$ by [3] Corollary E. Hence $\log \mathrm{vol}$ is also strictly concave there. By Lemma 3.1, the differential $-n\mathrm{vol}^{-1}\psi$ is therefore strictly increasing, and hence injective.

If $\psi(x), \psi(y)$ are proportional, say $\psi(x) = c\psi(y)$. Note $c > 0$ since $N_1^+(X)^o$ is a strict cone. Let $x' = \lambda x$ with $\lambda = c\mathrm{vol}(y)/\mathrm{vol}(x)$. Then $\mathrm{vol}(x')^{-1}\psi(x') = \lambda^{-1}\mathrm{vol}(x)^{-1}\psi(x) = \mathrm{vol}(y)^{-1}\psi(y)$. By injectivity of $\mathrm{vol}^{-1}\psi$ we have $y = \lambda x$. In case $c = 1$, by homogeneity of ψ we have $\lambda^{n-1} = 1$ so $\lambda = 1$, and $x = y$.

(3) In fact ψ is polynomial on the nef cone, and injective on the ample cone (by (1)).

□

9. Quantifier-free types over a one-dimensional base

We will work over a constant field $k = k^{alg}$, and describe the quantifier-free GVF types over k geometrically. We will see later how to deduce a description of types over the non-constant GVF $k(t)^{alg}$.

When $X = SpecA$ is affine, the Berkovich space of X over k can be defined as the set of all $\mathbb{R}$-valued semi-valuations of A; these are just the $\mathbb{R}$-valued valuations of the fraction field of A/P, for any prime ideal P of A. The topology on $\widehat{X}$ is induced from the Tychonoff topology on the functions from $K$to $\mathbb{R} \cup \{\infty\}$. For general X, the Berkovich space $\widehat{X}$ can be defined by glueing over an affine covering.

Let X be an irreducible, normal projective variety over k, $K = k(X)$. Then $\widehat{X}$ is compact.

Let $\widehat{K}$ be the space of $\mathbb{R}$-valued valuations of K/k. So $\widehat{K} = \widehat{X} \setminus \cup_Y \widehat{Y}$, the union ranging over proper subvarieties of X.

We define a pairing $\beta : \widehat{X} \times Pic(X) \to \mathbb{R}_\infty$ as follows: given $v \in hX$ and a Cartier divisor D pick a Zariski open set U of X such that $\mathcal{O}_X(U) \subset \mathcal{O}_v$ and D is represented on U by $f \in K$; and let $\beta(v, f) = v(f)$. It is easy to check that this is well-defined and linear in the $Pic(X)$- coordinate. We also have $\beta(v, (f)) = v(f)$ where (f) is the principal Cartier divisor represented everywhere by f.

This restricts to

$\beta : \widehat{K} \times Pic(X) \to \mathbb{R}$

Extend it to an $\mathbb{R}$-linear map

$$\beta : \widehat{K} \times Pic_{\mathbb{R}}(X) \to \mathbb{R}$$

Let $\mathcal{I}_X$ be the set of birational morphisms $\pi : Y \to X$ (over k). Given π, π' there exists at most one $j_{Y',Y} : Y' \to Y$ with $\pi' = \pi \circ f$; in this case write $\pi \leq \pi'$; we obtain a directed partially ordered set. Let $\widehat{Pic}(K)$ be the direct limit of

$Pic_{\mathbb{R}}(Y)$ over $\mathcal{I}_X$. By going to the limit we obtain:

$$\beta : \widehat{K} \times \widehat{Pic_{\mathbb{R}}}(K) \to \mathbb{R}$$

Finally, $C(\widehat{K})$ be the space of continuous functions $\widehat{K} \to \mathbb{R}$, and define $\beta_* : Pic_{\mathbb{R}}(K) \to C(\widehat{K})$ by $\beta_*(D)(v) = \beta(v, D)$.

Lemma 9.1. *The image $\beta_*(Pic_{\mathbb{R}}(K))$ separates points on $\widehat{K}$, and is closed under addition, scalar multiplication and* min. *It separates points on $\widehat{K}$.*

Proof. Closure under addition and scalar multiplication follows from linearity of β in the second variable.

Let $v \neq v' \in \widehat{X}$. Then $v(f) \neq v'(f)$ for some $f \in K$. So $\beta(v, (f)) \neq \beta(v', (f))$ for the associated principal divisor (f). Thus β_* separates points.

Closure under min follows from the existence of stable meets, in an appropriate blowup. By adding a large ample divisor, it suffices to prove this for two effective divisors; we can assume they come from divisors W_1, W_2 on a blowup Y of X, with ideal sheaves I_1, I_2. Further by blowing up $I_1 + I_2$ we may assume it is principal, and hence determines a Cartier divisor W. In this case, it is easy to see that $\beta_*(W) = \min \beta_*(W_1), \beta_*(W_2)$, as required.

□

If $Z \subset \widehat{X}$ is compact, let $C(Z)$ be the space of continuous functions $Z \to \mathbb{R}$, and define β_Z to be the composition of β_* with the restriction function $C(\widehat{K}) \to C(Z)$. We endow $C(Z)$ with the topology of uniform convergence.

Lemma 9.2. *Let $Z \subset \widehat{K}$ be compact, and assume $v(f_0) = 1$ for some f_0 and for all $v \in Z$. Then $\beta_Z(Pic_{\mathbb{R}}(K))$ is dense in $C(Z)$.*

Proof. By Lemma 3.13, it suffices to show that the image of β_Z separates points on Z, and is closed under addition, scalar multiplication and min, as well as containing the constant function 1. The constants function 1 is provided by $v(f_0)$. The rest was proved in Lemma 9.1. □

Remark 9.3. We can similarly define $\beta : \widehat{X} \times Pic_{\mathbb{R}}(K) \to \mathbb{R}_\infty$, and hence $\beta_* : Pic_{\mathbb{R}}(K) \to C(\widehat{X}, \mathbb{R}_\infty)$. Let $Z \subset \widehat{X}$ be compact, and $\beta_Z(y) = \beta_*(y)|Z$. Let $B = \beta_* : Pic_{\mathbb{R}}(K) \cap C(Z)$, be the set of functions of the form $\beta_*(y)|Z$ that take only real values, and $\bar{B}$ the uniform closure of B. Then $\bar{B} = C(Z)$, provided that $1 \in \bar{B}$.

The proof is the same as of 9.2. For separation of points, if $z \neq z' \in Z$, we find $y \in Pic_{\mathbb{R}}(K)$ as above with $\beta(z, y) \neq \beta(z', y)$, then replace y by $\max(-r, \min(\beta(y, r)))$ where $-r \leq \beta(z, y), \beta(z', y) \leq r$.

Let M_K be the set of regular Borel measures μ on $\widehat{K}$, such that for any $f \in K$, $v \mapsto v(f)$ is integrable. It follows that for any $D \in \widehat{Pic}(K)$, $v \mapsto \beta(v, D)$ is μ

integrable. Thus we can naturally extend β to a map

$$\beta : M_K \times \widehat{Pic}(K) \to \mathbb{R}$$

Corollary 9.4. *Let $\mu, \mu' \in M_K$ and suppose they define different GVF structures on K, i.e. μ, μ' are distinct even up to renormalization. Then for some $D \in \widehat{Pic}(K)$, $\beta(\mu, D) \neq \beta(\mu', D)$.*

Proof. We may assume k is countable [1]. We have $\widehat{K} \smallsetminus \{v_{triv}\} = \cup_{f \in K \smallsetminus (0)} W_f$ where $W_f = \{v : v(f) > 0\}$. Thus for some f, the measures μ, μ' are distinct (even up to normalization) when restricted to $W = \{v : v(f) > 0\}$. By renormalizing we may assume μ, μ' concentrate on $\{v : v(f) = 1\}$. Moreover these are regular measures, approximated by their values on compact sets, hence μ, μ' have distinct restrictions to some compact $Z \subset \{v : v(f) = 1\}$. Now the statement follows from Lemma 9.2 (with $f_0 = f$.) □

Proposition 9.5. *Let $\mu \in M_K$ satisfy the product formula:*

$$\textit{for all } f \in K \smallsetminus (0), \quad \int v(f) d\mu(v) = 0$$

Then $\beta(\mu, \ ) : Pic_{\mathbb{R}}(Y) \to \mathbb{R}$ factors through the quotient map $Pic_{\mathbb{R}}(Y) \to NS(Y)$.

Proof. We have a normal, irreducible variety Y, and a homomorphism $\xi = \beta(\mu, \ ) : Pic(Y) \to \mathbb{R}$. Let H be an ample divisor on Y. Let $Pic^0(Y)$ be the group of divisors algebraically equivalent to 0. It is known ([16]) that $Pic^0(Y)$ is an algebraic variety, in particular constructible, and that for $a \in Pic^0(Y)$, for large enough m, the line bundle corresponding to $mH - a$ has a nonzero section. So $-a + mH \geq 0$. Since $Pic^0(Y)$ is constructible, by compactness, there exists m such that for all $a \in Pic^0(Y)$, $-a + mH \geq 0$. So $\xi(a) \leq m\xi(H)$. Thus $\xi | Pic^0(Y)$ is a homomorphism into $\mathbb{R}$ with bounded image; so it must be zero. Since $\mathbb{R}$ is torsion-free, if $mb \in Pic^0(Y)$ then $\xi(b) = 0$. Hence ξ vanishes on the divisors that have integral multiples algebraically equivalent to zero. But by [16], Theorem 6.3, these are the same as the divisors numerically equivalent to 0. So ξ induces a homomorphism $NS(Y)_{\mathbb{R}} \to \mathbb{R}$. and factors through that homomorphism. □

Note that the homomorphism $\beta_\mu : Pic_{\mathbb{R}}(Y) \to \mathbb{R}$ obtained above is order-preserving with respect to the effective cone; i.e. an effective divisor has non-negative image. This is because by definition, $\beta(v, D) = v(g) \geq 0$ where D is represented by (g) on some affine open set Y' with $\mathcal{O}_{Y'} \subset \mathcal{O}_v$, and as D is effective, $g \in \mathcal{O}_{Y'}$ so $v(g) \geq 0$; and because μ is a non-negative measure. hence the induced homomorphism $NS(Y)_{\mathbb{R}} \to \mathbb{R}$ is order-preserving with respect to the to the pseudo-effective cone.

[1]For this we use the equivalence of GVF structures and globalizing measures up to renormalization

Let S_K be the space of GVF structures on K that are trivial on k; it can be viewed as the space of quantifier-free GVF types whose restriction to ACF is the generic type of X. If we pick any transcendence basis $f_1, \dots, f_n$ for K over k, and $r \in \mathbb{R}$, then the subspace of types with $ht(f_i) \leq r$ is compact. Each element p of S_K is induced by a measure $\mu \in M_K$ satisfying the product formula: this can be proved using a homogeneous variant of the Riesz representation theorem, but will also follow from the proof of Theorem 9.6 below.

It is clear that $\beta(\mu, D)$ depends only on p and not on the choice of μ; for instance this may be cheked for very ample D, with a section f, so $\beta(\mu, D) = \int \min(v(f), 0) d\mu(v)$. We thus write $\beta(p, D)$ for $\beta(\mu, D)$.

Let $N^1(K)$ be the direct limit of the $\mathbb{R}$-spaces $N^1(Y)$ along $\mathcal{I}_X$; let $N_1(K)$ be the dual space, the inverse limit of the spaces $N_1(Y)$. (Cf. [3].) Note that when $j : Y' \to Y$ is birational, $j^* N_1^+(Y) \subset N_1^+(Y')$. Let $N_1^+(K)$ be the inverse limit of the cones $N_1(Y)^+$, dual to the effective cones of $N^1(Y)$.

We define a map $\alpha : S_K \to N_1^+(K)$ via the pairing $\beta : S_K \times N^1(K) \to \mathbb{R}$. So $\alpha = \lim \alpha_Y$ where Y is similarly defined via $\beta : S_Y \times N^1(Y) \to \mathbb{R}$, with Y a birational extension of X.

Theorem 9.6. $\alpha : S_K \to N_1^+(K)$ *is a homeomorphism.*

Proof. Injectivity: Let p, p' be distinct elements of S_K. They are represented by measures μ, μ' that are distinct even up to renormalization. By Lemma 9.4, for some D we have $\beta(\mu, D) \neq \beta(\mu', D)$. Thus $\alpha(p) \neq \alpha(p')$.

Continuity: It suffices to prove that $\alpha : S_X \to N_1(Y)$ is continuous. Now the topology on $N_1(Y)$ is generated by $O_{\beta,\gamma,D} = \{a : \beta < |a \cdot D| < \gamma\}$, where β, γ are reals and D is a very ample divisor on Y. We can find a rational function $f \in K$ such that $(f)_\infty = D$. So $\alpha^{-1}(O_{\beta,\gamma,D})$ is cut out by $\beta < ht(f) < \gamma$.

A closed map:

Let $f_1, \dots, f_n$ be a transcendence basis for K/k; let A be an ample divisor, such that each f_i has poles at most on A. Let D_i be the divisor of poles of f_i. Let C be a closed subset of S_X. For $t > 0$ let $R_t = \{b \in N_1^+(K) : (b, D_i) \leq t\}$. Then R_t is contained in the interior of $R_{t'}$ for $t < t'$, and $\cup_t R_t = N_1^+(K)$. Thus to show that a set is closed, it suffices to show that the intersection with each R_t is closed. As $\alpha^{-1}(R_t)$ is closed, to show that α is a closed map, it suffices to show that $\alpha(C)$ is closed when C is a closed subset of $\alpha^{-1} R_t$ for some t. But such a set is compact as $ht(f_i)$ is bounded on it.

In particular, the image of α is closed. Hence to show that α is surjective, it suffices to prove that α is dense. For this, it suffices to show that each α_Y is surjective. This follows from Lemma 9.9. ☐

Remark 9.7. The fact that continuity was proved using the open sets

$$\beta < ht(f) < \gamma$$

alone shows that the height, along with the field operations, generates the entire language of GVFs.

9.8. **Discrete globalizing measures.** Let X be a smooth projective variety. Let $\mathcal{W}$ be the set of irreducible hypersurfaces of X.

Let a be a class in $N_1(X)^+$.

We let μ_a be the measure, supported on divisorial valuations v_D of K for each irreducible hypersurface D of X, and with $\mu_a(v_D) = a \cdot [D]$.

The same definition extends additively to Cartier divisors, in such a way that principle divisors map to 0. This amounts to the fact that a rational function on a curve has as many zeroes as poles.

From this it is clear that for $f \in K$, letting $(f) = \sum m_i D_i$ be the divisor of zeroes and poles of f, we have $\sum m_i \mu(D_i) = 0$. Indeed let $p \in C_t \cap D_i$; if D_i is cut out by (g) locally at p then $(g^{m_i}) = (f)$ on this local ring, so $m_i i(C_t, D_i)$ is the order of zero (or pole) of f at p on C_t, and the sum of these is zero. Thus μ_a is a GVF measure. In particular, it defines an element of S_X, depending only on the numerical equivalence class of C, that we denote $\delta_X([C])$. This gives a map

$$\delta_X : N_1^+(X) \to S_K$$

Lemma 9.9. $\alpha_X \circ \delta_X = Id_{N_1^+(X)}$.

Proof. This amounts to the relations: $v_D(D) = 1$, $v_D(D') = 0$ when $D, D' \in \mathcal{W}$ are distinct, and v_D is the valuation corresponding to D, evaluated on D, D' as Cartier divisors. Both are straightforward. □

Remark 9.10. It is not true that $\delta_X \circ \alpha_X$ is the identity on S_K; but $\delta_X \circ \alpha_X(p)$ and p do give the same value to a formula $\int t(vf_1, \ldots, vf_m)dv$ if $f_1, \ldots, f_m$ extend to morphisms $X \to \mathbb{P}^1$, t is a term formed out of $+, -, \min$, and for any minimum occuring in $t(vf_1, \ldots, vf_m)$, the corresponding minimum of Cartier divisors $(f_1), \ldots, (f_m)$ is is a stable meet on X.

In particular, the same height is computed for rational functions f extending to a morphism $X \to \mathbb{P}^1$.

This makes it possible to define α_K as the limit of α_Y over all blowups Y, and prove injectivity in a more explicit way.

Remark 9.11. Let $a \in N_1^+(X)$; assume $a \cdot b = 0$ for some effective b with nonzero section growth. Then there exists a nonconstant rational function g on X such that $F(g)$ is a constant subfield of the GVF $F(X)_a$. The converse is also true.

Proof. By assumption there exists a nonconstant rational function g with poles only on (a representative of) b. So the height of g in $F(X)_a$ is zero. □

9.12. **Quantifier-free types over the function field of a curve.** Here the picture is very similar. Let k be a constant field, and let $k_1 = k(U)$ for a curve U. Let $K = k_1(X)$ be a finitely generated field extension of k_1. Let μ_1 be a nontrivial GVF structure on k_1/k. Then $N_1^+(k_1) = \mathbb{R}^{\geq 0}$, with μ_1 corresponding to 1; and we have a natural map $\pi_* : N_1^+(K) \to N_1^+(k_1)$; it is obtained as the limit of the projections $\pi_* : N_1(X) \to N_1(U)$ at the level of varieties. Let $N_1^+(K/\mu_1)$ be the pullback of 1. We have an $\mathbb{R}^{>0}$-action on $N_1(X)$ and on $N_1(U)$ respecting $\pi_*|N_1(X)$; so $N_1^+(K/\mu_1)$ can also be identified with the set of elements of $N_1^+(K)$ with nontrivial images in $N_1^+(U)$, modulo the $\mathbb{R}^{>0}$-action.

Letting S_X denote the space of quantifier-free types over (k_1, μ_1), we immediately obtain from Theorem 9.6, by restriction:

Theorem 9.13. *$\alpha : S_X \to N_1^+(K/\mu_1)$ is a homeomorphism.*

10. Existential closedness of $k(x)^{alg}[r]$

GVF's and their ultraproducts.

Let $k = k^{alg}$.

Let μ be a globalizing measure for $k(V)$ over k. Let D be an ample divisor on V.

Lemma 10.1 (Artin-Whaples)**.** *Let C be a curve of genus g over k. Let $x \in k(C) \setminus k$, and let $r > 0$. Then $k(C)$ admits a unique GVF structure over k with $ht(x) = r$.*

Proof. The nontrivial valuations of $k(C)$ over k can be identified with the points of $C(k)$. They form a discrete set; the compactification includes also the trivial valuation, which plays no role here. We have to show that if μ is a measure on $C(k)$ satisfying the product formula, then μ gives equal weight to each element of $k(C)$. Let a, b be two points. Then $(n+g)a - nb$ is effective, so $(n+g)\mu(a) \geq n\mu(b)$. Thus $(1 + g/n)\mu(a) \geq \mu(b)$. Letting $n \to \infty$, $\mu(a) \geq \mu(b)$. So $\mu(a) = \mu(b)$ for all $a, b \in k(C)$. □

A c-twisting automorphism σ of K is an isomorphism from K to K', where K' is the renormalization of K moving from μ to $c\mu$.

Corollary 10.2. *There is a unique GVF structure $k(x)^{alg}[r]$ on $k(x)^{alg}$ over k, with $ht(x) = r$. Moreover if F is a finitely generated extension field of k of transcendence degree 1, and $g, g' : F \to k(x)^{alg}$ are two field embeddings, and $a \in F \setminus k$, there exists an $ht(f'(a))/ht(f(a))$-twisting automorphism σ of $k(x)^{alg}$ with $g' = \sigma \circ g$.*

Remark 10.3. If $r/r' \in \mathbb{Q}$ then $k(x)^{alg}[r] \cong k(x)^{alg}[r']$. Let $m \in \mathbb{N}$ and let σ be an automorphism of the field $k(x)^{alg}$ with $\sigma(x) = x^m$. Then σ is m-twisting so it gives an isomorphism $k(x)^{alg}[r] \to k(x)^{alg}[mr]$. Similarly find an m'-twisting σ', with $m/m' = r/r'$. Then $(\sigma')^{-1}(\sigma)$ is an isomorphism $k(x)^{alg}[r] \to k(x)^{alg}[r']$.

Corollary 10.4. *Let K^* be an ultrapower of $K = k(x)^{alg}$. Let F be finitely generated over k, of transcendence degree 1, and let $f, f' : F \to K^*$ be two embeddings, with $ht(f(a)) = ht(f'(a)) > 0$ for some $a \in F$. Then there exists an automorphism σ^* of K^* with $\sigma \circ f = f'$.*

This also holds if K^* is an ultraproduct of GVF fields $k_i(x)^{alg}$ of increasing positive characteristics, and F is finitely generated over $\mathbb{Q}$, of transcendence degre 1.

Proof. F is the field of fractions of a finitely generated k-algebra D, with $a \in D$. $f|D, f'|D$ can be represented as ultrapowers of homomorphisms $f_i, f_i' : F' \to K$. We have $ht(f_i(a)) \to ht(f(a))$ and $ht(f_i'(a)) \to ht(f'(a))$ along the ultrafilter u. So $r_i = ht(f_i(a))/ht(f_i'(a))$ approaches 1 along u. Note that f_i, f_i' extend to field embeddings (first on $k(a)$ since $f_i(a), f_i'(a) \notin k = k^{alg}$; then on F since it is a finite extension of $k(a)$.) Using Corollary 10.2 let σ_i be an r_i-twisting automorphism of K with $f_i' = \sigma \circ f_i$. Let σ be the ultraproduct of the σ_i. Then $f' = \sigma \circ f$ and σ is a 1-twisting automorphism, i.e. simply an automorphism. The statement with ultraproducts is proved similarly. □

Proof of Theorem 1.6. We first show that any quantifier-free type q over k can be approximately realized in $K = k(t)^{alg}[1]$. Consider first a smooth projective variety X over k, and a family of curves C_t on X as in § 9.8, with class $[C] \in N_1^+(X)$. We have the corresponding quantifier-free type $\delta_X(C)$. Now the inclusion of C_t in X corresponds to an element $a_t \in X(k(C_t))$. It is easy to see that any approximation to $\delta_X(C)$ involving finitely many divisors is realized by a_t, for almost all t (namely as soon as C_t is not contained in the support of these finitely many divisors.) Thus $\delta_X([C])$ is approximately realized in K. Since $K \cong K[m]$ for integral m by Remark 10.3, we aso have $\delta_X(\frac{1}{m}[C])$ approximately realized in K. Now by Corollary 8.9, Theorem 9.6 and Lemma 8.7, the types $\delta_Y(\frac{1}{m}[C])$ for Y a blow-up of X, and C as above on Y, are dense on S_X. Thus all types in S_X are approximately realized in K.

Now we must also consider types over K and not just over k. Let $K \leq L$, L finitely generated over K, with a GVF structure. We have to find a K- embedding of GVF's $L \to K^*$, where K^* is an ultrapower of K. By the above we do have a k-embedding of GVFs $h : L \to K^*$. Let g be the inclusion of K in K^*, and $g' = h|K$. Using Lemma 10.4, find $\sigma \in Aut(K^*)$ with $\sigma \circ g' = g$. Then $\sigma \circ h$ is an embedding of L to K^* over K, as required.

□

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