GVF NOTES/ ITAÏ BEN YAACOV, E.H.

Contents

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1. Continuous logic

A language consists of function and relation symbols as usual. A structure is a set (or a family of sets in the many-sorted situation) with function symbols and terms treated as usual. But the interpretation of a relation symbol $R(x_1, \ldots, x_n)$ is a function $A^n \to \mathbb{R}$. In addition, we are given a map $h : A \to \mathbb{R}_{\geq 0}$, the *gauge*. This can be viewed as a continuous division into sorts; only sets of bounded 'gauge' are treated seriously, but no specific sharp bound is specified. We assume, for each function symbol F, that $h(F(a_1, \ldots, a_n)) \leq c_F(h(a_1), \ldots, h(a_n))$ for some continuous c_F; and $R(a_1, \ldots, a_n) \leq c_R(h(a_1), \ldots, h(a_n))$.

1.1. Ultraproducts.

. Let I be an index set, u an ultrafilter on I, L a language, A_i real-valued L-structures. We define the ultraproduct. The universe consists of the sequences $(a_i)_{i \in I}$ with $a_i \in A_i$ and with $h(a_i)$ bounded; *modulo* the equivalence relation: $a \sim_u b \iff \{i : a(i) = b(i) \in u\}$. The function symbols are interpreted pointwise. The relation symbols:

$$R(a_1, \ldots, a_n) = \lim_{i \to u} R(a_1(i), \ldots, a_n(i))$$

1.2. Formulas.

Formulas are formed using connectives and quantifiers. We denote their truth value (in some given structure) by $[\phi]$ and explain how to compute it.

Connectives: Any continuous function $c : \mathbb{R}^n \to \mathbb{R}$ can be viewed as a connective $\diamond_c$; the truth value $[\diamond_c(\phi_1, \ldots, \phi_n)]$ equals $c([\phi_1], \ldots, [\phi_n])$.

Quantifiers: let $c : \mathbb{R} \to \mathbb{R}$ be a continuous function with compact support. Then we have a formula $(Sup_c x)\phi$ whose interpretation is $sup_x c(h(x))[\phi]$.

Remark 1.3.
1. We could allow more general quantifiers for each continuous $c : \mathbb{R}^{n+1} \to \mathbb{R}$ whose support projects properly to $\mathbb{R}^n$; and form $(Sup_c x)\phi(x, y_1, \ldots, y_n)$ whose interpretation is $sup_x c(h(x), h(y_1), \ldots, h(y_n))[\phi]$.
2. We could more generally allow truth values in any space Y with a proper map $p : Y \to \mathbb{R}_{\geq 0}$; in this case, for each relation symbol R, a rule: $p([R(x)]) \leq c_R(h(x))$ is imposed. In particular if Y is compact, no gauge is needed.
3. In situations where we are concerned with effectivity, it makes sense to restrict the continuous functions c to some nice, uniformly dense on compacts class. For the moment, this is irrelevant.

Proposition 1.4 (Los)**.** *For any formula ϕ, any ultraproduct $I, u, A_i, a_i \in A_i$ with $h(a_i)$ bounded,*

$$\phi(a_1, \ldots, a_n) = \lim_{i \to u} \phi(a_1(i), \ldots, a_n(i))$$

2. Measure theory

We consider locally compact spaces X; a *measure* means: a regular Borel measure; it is not necessarily finite, but compact sets have finite measure. In fact μ may just be identified with the compatible collection of finite measures $\mu|U$, where U is a compact subset of X.

If $\pi : Y \to X$ is a proper map, and ν a measure on Y, then $\pi_*\nu$ is the measure on X with $(\pi_*\nu)(U) = \mu(\pi^{-1}(U))$.

Lemma 2.1. *Let Y be a locally compact space, G a finite group acting on Y, $X = Y/G$ the quotient, $\pi : Y \to X$ the natural map. Let μ be a measure on X. Then there exists a unique G-invariant measure ν on Y with $\pi_*\nu = \mu$.*

Proof. (1) This reduces to the case that Y is compact (lift $\mu|U$ for each compact subset U of X, note that they are compatible using the uniqueness statement.)

(2) In the compact case, we translate this to positive linear functionals on $C(X)$, using Riesz. We have a positive linear functional $\mu : C(X) \to \mathbb{R}$, and must show there is a unique extension to a positive linear functional $\nu : C(Y) \to \mathbb{R}$ which is invariant. Note that G acts on $C(Y)$, and let $tr(f) = \sum\{\sigma(f) : \sigma \in G\}$. Note that $tr(f) = F \circ \pi$ for a unique function F; since $tr(f) \in C(Y)$, we have $F \in C(X)$. Let us denote $Tr(f) = \frac{1}{G}F$.

For any invariant extension ν, since $\nu(f) = \nu(\sigma(f))$ for $\sigma \in G$, we have:

$$\mu(F) = \nu(tr(f)) = |G|\nu(f)$$

On the other hand it is clear that $f \mapsto \mu(Tr(f))$ is a positive linear functional on $C(Y)$. □

The displayed formula in the proof extends to $\int Tr(f)(x)d\mu(x) = \int f(y)d\nu(y)$ for continuous functions f; or:

$$\int f(y)d\nu(y) = \int Tr(f)(x)d\mu(x) \tag{1}$$

Remark 2.2.
1. if N is a normal subgroup of G, $Y' = Y/N$ so that we have:
$$Y \to_{\pi_1} Y' \to_{\pi_2} Y/G = X$$
with $\pi = \pi_2\pi_1$, and ν' is the extension to Y' obtained in the same way, then $(\pi_1)_*\nu = \nu'$. This follows from uniqueness, since ν' is the unique G/N-invariant extension of μ, and clearly $(\pi_1)_*\nu$ is also G/N-invariant.
2. (1) and the lemma go through for G compact, replacing $\frac{1}{G}\sum_{g\in G} f(g(x))$ by the Haar integral.

Theorem 2.3. *(Caratheodory's extension theorem) Let R be a ring on a subset S of $\mathcal{P}(\Omega)$ and let μ be a pre-measure on R. Then there exists an extension μ' of μ to a measure on the σ-algebra generated by R.*

Remark 2.4. Note that if R is a semi-ring, then the following condition is sufficient in order for R to be a ring:

If (B_i) is an increasing sequence of elements in R and $\cup B_i \in R$ then (B_i) is eventually constant.

In particular, it follows from the compactness theorem that if R is a semi-ring consisting of definable subsets of some structure M then R is also a ring.

We will also need to know certain consequences of the stability of probability theory, viewed as a continuous-logic theory; see [3], [4]. Here is a statement in explicit terms of a key property of Morley sequences in stable theories, namely they are universal witness for dividing.

Lemma 2.5. *Let μ be a probability measure on X, ν a probability measure on Y. Let θ be a probability measure on on $X^{\mathbb{N}} \times Y$, projecting to the product measure $\mu^{\mathbb{N}}$, and also projecting to some fixed measure θ_2 on $X \times Y$ via any of the coordinate projections. Then $\theta_2 = \mu \times \nu$.*

Proof. (With thanks to David Kazhdan.) We have to show that $\theta_2(U \times V) = \mu(U)\nu(V)$, for any Borel $U \subset X, V \subset Y$. Fix V, and for any Borel $W \subset X^{\mathbb{N}}$, let $\mu'(W) = \frac{\theta(W \times V)}{\nu(V)}$. By Radon-Nykodim, $\mu'(W) = \int_W g d\mu^N$ for some $g \in L^1(X^{\mathbb{N}})$.Let π_i be the i'th projection, and let $h : X \to \mathbb{R}$ be any Borel function; we have $\int g(h \circ \pi_i) d\mu^{\mathbb{N}} = \int h \circ \pi_i d\mu' = (\int h d\mu)\nu(V)$, so this quantity does not depend on the projection i. We may assume X is compact and totally disconnected; then g can be approximated by a continuous step function g'; $\int |g - g'| < \epsilon$. Say $\int g' d\mu = 1 - \epsilon'$; so $|\epsilon'| \leq \epsilon$. Now g' depends on only finitely many coordinates. Take i bigger than any of them. Then since g' and $1_U \circ \pi_i$ depend on disjoint coordinates,

$$\frac{\theta_2(U \times V)}{\nu(V)} = \frac{\theta(\pi_i^{-1}(U) \times V)}{\nu(V)} = \int_{\pi_i^{-1}(U)} g d\mu^{\mathbb{N}} \sim \int_{\pi_i^{-1}(U)} g' d\mu^{\mathbb{N}} = \mu(U)(1-\epsilon') \sim \mu(U)$$

(Here $\sim$ denotes equality up to an error of ϵ.) Letting $\epsilon \to 0$ we obtain the required equality. □

We will also use a modularity - type phenomenon that belongs to the theory RCF of real closed fields, or more precisely to the randomization of RCF, or any o-minimal theory; for Abelian varieties of bad reduction, the relevant o-minimal theory will be that of the value group. We will see that this component induces modularity on certain global types; the results are due to Zhang, Bilu, Gubler, Chambert-Loir, Yamaki, Baker, Rumely, and related to Bogomolov's conjecture and its descendents. (We note that where there is no analogous o-minimal component, the conjecture remains open.)

Let μ be the uniform measure on the unit circle T of the complex plane (or on the circle group on $[0, 1]/\{0 = 1\}$.)

Lemma 2.6. *Let ν be a measure on $T^n \times Y$, $Y \subset \mathbb{R}^k$, projecting to ν_Y on Y. Given $m = (m_1, \ldots, m_n) \in \mathbb{Z}$, $x = (x_1, \ldots, x_n)$, let $h_m(x, y) = (x_1^{m_1} \cdot \cdots \cdot x_n^{m_n}, y)$. Assume $(h_m)_* \nu = \mu \times \nu_Y$, for any nonzero $m \in \mathbb{Z}^n$. Then $\nu = \mu^n \times \nu_Y$.*

Proof. This reduces - by conditioning on $y \in Y$ - to the case that Y is a point, so we consider this case. Let H be the space of functions on T^n spanned by the $h_m, m \in \mathbb{Z}^n$. It is closed under multiplication and complex conjugation, and separates points. By Stone-Weierstrass, H is uniformly dense in $C(T^n)$. So to show that $\mu^n = \nu$, it suffices to show that $\int h_m d\mu^n = \int h_m d\nu$ for each m. For $m = 0$, h_0 is constant and the statement is clear. For $m \neq 0$, letting z be the identity function on T, we have $h_m = z \circ h_m$ and so $\int h_m d\nu = \int z d((h_m)_* \nu) = \int z d\mu = 0 = \int h_m d\mu^n$. □

2.7. **Positive linear functionals.** If V is a real vector space, a *cone* is a subset closed under scalar multiplication by non-negative real numbers, and addition.

The *core* of a cone P is the set $P \cap -P$; it is a subspace of V.

If P is a cone, a *positive* linear map with respect to P is a linear map $h : V \to \mathbb{R}$ with $h(x) \geq 0$ for $x \in P$. (Hence h vanishes on the core.)

Lemma 2.8. *[mriesz] Let V be a real vector space. Let U be a subspace of V, and let $h : U \to \mathbb{R}$ be a linear map, positive with respect to $P \cap U$. Assume:*

(1) *V is finite-dimensional, and P is closed* or
(2) *for any $v \in V$, for some $w \in U$, $w - v \in P$.*

Then h extends to a positive linear map on (V, P).

Proof. (1) The finite dimensional case. Using induction, we may assume $\dim(V/U) = 1$. So $U = \ker \lambda$ for some $\lambda \in V^*$. Consider first the case that λ or $-\lambda$ are positive. We may assume λ is. Let H be any linear map extending h. We claim that for any sufficiently large real c, the map $H(x) + c\lambda(x)$ is positive for P. Consider $g(x) = (H(x), \lambda(x)) : V \to \mathbb{R}^2$; let (u, v) be coordinates for the target $\mathbb{R}^2$. We have to show that $u + cv$ is non-negative on $g(P)$, i.e. that $g(P)$ lies (on or) above the line $u + cv = 0$. Now since λ is positive, $g(P)$ lies above the line $v = 0$. Moreover $g(P)$ is a closed cone (let S be the unit ball of V with respect to some metric; then $P \cap S$ and hence $g(P) \cap S$ are compact, so $\mathbb{R}^{\geq 0}(g(P) \cap S) = g(P)$ is closed. It is here that we use the finite dimensionality of V.) When $\lambda(x) = 0$ with $x \in P$, we have $x \in U$, so $H(x) = h(x) \geq 0$. Thus $g(P)$ intersects the $v = 0$-axis in the $u \geq 0$ ray (the intersection is either the $u \geq 0$-ray or the origin). The only closed cones with this property in $\mathbb{R}^2$ are bounded between two rays $v = \beta u \geq 0$ (with $\beta \geq 0$) and another ray $v = \alpha u, u \leq 0$, with $\alpha < 0$. It is clear that this cone lies above the line $u + cv = 0$ if $c^{-1} < -\alpha$.

The remaining case is that neither $-\lambda$ nor $-\lambda$ are positive; so there exist $a, a' \in P$ be such that $\lambda(a) < 0 < \lambda(a')$. Let $e \in V \setminus U$. Any linear extension of h to V has the form $x + \alpha e \mapsto h(x) + \alpha\beta$, for some $\beta \in \mathbb{R}$. We seek β such that for any $\alpha > 0$, whenever $x + \alpha e \in C$, we have $h(x) + \alpha\beta \geq 0$, or $\beta \geq -h(x)/\alpha$; and whenever $x' - \alpha' e \in C$, we have $h(x') - \alpha'\beta \geq 0$, or $\beta \leq h(x')/\alpha'$. Note that a, a' show that such pairs (x, α) and (x', α') do exist. Clearly such a β can be found provided $h(x')/\alpha' \geq -h(x)/\alpha$ for all such (x, α) and (x', α'). Now $x + \alpha e, x' - \alpha' e' \in C$, so $\alpha^{-1}x + e, (\alpha')^{-1}x' - e \in C$, and $\alpha^{-1}x + \alpha^{-1}x' \in C$. Thus $h(\alpha^{-1}x + (\alpha')^{-1}x') \geq 0$, i.e. $\alpha^{-1}h(x) + (\alpha')^{-1}h(x') \geq 0$, or $h(x')/\alpha' \geq -h(x)/\alpha$.

(2) This is the M. Riesz extension theorem; see Wikipedia entry or [15] Theorem 2.6.2. □

3. Valued fields

3.1. **Generically stable types.**

3.2. **Comparison of $\widehat{V}$ with V^{an}.**

Let F be a field valued in $\mathbb{R}$. We define a structure F^c in the ACVF geometric sorts, extending F but contained in the continuous-logic definable closure of F. Namely, let F_{max} be a maximally completely extension of F^{alg}, with the same residue field as F^{alg}, and with value group $\mathbb{R}$; this field is unique up to F-isomorphism. Let F^c be the fixed substructure of $Aut(F_{max}/F)$. Note that any intersection of elements $\Lambda_n \in L_n(F)$ is an element of $L_n(F^c)$.

Lemma 3.3. *Let $F = F^{alg}$ be $\mathbb{R}$-valued. The natural map $\widehat{V}(F^c) \to V^{an}_F$ is an isomorphism.*

Proof. Injectivity of the map $p \mapsto p|F^c$ is a completely general fact about stably dominated types ([**?**]), and one sees easily that $p|F \vdash p|F^c$ sot hat $p \mapsto p|F$ is also injective. To prove surjectivity, let q be an element of V^{an}, represented by a valued field extension $L = F(b)$, $b \in V(F)$. We may assume V is affine, and consider $H = H_d$ as above. Then $\{h \in H(F) : vh(b) \geq 0\}$ generates a lattice $\Lambda_d \in L_{H_d}$, with corresponding linear seminorm l_d on H_d We can functorially define p_L by the formulas: $vh(x) = l_d(h)$. The problem is to show consistency: that for any $h_1, \ldots, h_k \in H_d(L)$ there exists x with $vh_i(x) = l_d(h_i)$. Taking an ultrapower (F^*, b) of (F, b) in the sense of continuous logic, it is clear that $b \models p_{F^*}$. (In fact any L with value group $\mathbb{R}$ embeds in such an ultrapower, so p_L is consistent in this case.) It follows that p_L is consistent whenever F^* is an elementary extension of L. But by Robinson's quantifier elimination for ACVF, any $L' \geq F$ admits a common elementary extension with F^*; so $p_{L'}$ is consistent. □

In this section we work locally, in a valued field (K, v), with valuation ring $\mathcal{O}$; in the applications v takes values in $\mathbb{R}$.

3.4. **The linear part of a generically stable type.**

Let p be a generically stable type on $\mathbb{A}^n$. Some notation:

- $K[X]_1$= space of linear polynomials on $\mathbb{A}^n$. $\mathcal{O}[X]_1$ is the *standard lattice in H_d*, consisting of linear polynomials with $\mathcal{O}$-coefficients.

- $J_\infty(p)$= the elements of $K[X]_1$ vanishing generically on p.
- $V_1(p) = K[X]_1/J_\infty(p)$ = linear polynomials, modulo those vanishing on p. We define the image of $\mathcal{O}[X]_1$ to be the *standard lattice in* $V_1(p)$.
- $J_1(p)$ is the lattice in $V_1(p)$ consisting of images of linear polynomials, whose value at a generic element of p lies in $\mathcal{O}$.
- $V_1^*(p)$= dual space to $V_1(p)$. It can be viewed as a subspace of $\mathbb{A}^n$, namely the linear-Zariski closure of p.
- $d_1(p) = \dim V_1(p) = \dim V_1^*(p)$.
- $J_1^*(p)$ = dual lattice to $J_1(p)$, i.e $J_1^*(p) = \{a \in V_1^*(p) : (\forall b \in J_1(p))(a,b) \in \mathcal{O}\}$.

Lemma 3.5. $V_1^*(p)$ *coincides with the vector space generated by all realizations of* p*, or by* n *generic independent such realizations.* $J_1^*(p)$ *coincides with the* $\mathcal{O}$*-module generated by all realizations of* p*, and also with the* $\mathcal{O}$*-module generated by any* $d_1(p)$ *mutually generic realizations of* p*.*

Proof. Say p is F-definable. let $(a_1, a_2, \cdots) \models p^n|F$. Let d be maximal such that $a_1, \ldots, a_d$ are K- linearly independent. Then for any $k > d$, $a_k \models p|\{a_1, \ldots, a_d\}$ and so a_k depends K-linearly on $a_1, \ldots, a_d$. Thus a generic realization of p lies in the space generated by $a_1, \ldots, a_d$, and hence any realization of p does. So the vector space W generated by all realizations of p is also generated by $a_1, \ldots, a_d$; hence the space $J_\infty(p)$ of linear polynomial vanishing generically on p is just $W^\perp$, so $V_1(p) = K[X]_1/W^\perp = W^*$, hence $W = V_1(p)^*$ and $d = \dim(W) = \dim(V_1(p)) = d_1(p)$.

Next let Λ be the $\mathcal{O}$-module generated by $a_1, \ldots, a_d$. So $\Lambda \leq J_1^*(p)$, equivalently $J_1(p) \leq \Lambda^*$ (where $\Lambda^* \leq J_1(p)$ is the dual lattice to Λ), and it suffices to show that $\Lambda^* \leq J_1(p)$. Now Λ^* is generated as an $\mathcal{O}$-module by the dual basis to the basis $(a_1, \ldots, a_d)$. of $V_1(p)^*$. Fix some identification of $V_1(p)^*$ with K^d. By Cramer's rule, the dual basis is generated by the functionals $b_i : x \mapsto c_i \det(a_1, \cdots, a_i/x, \cdots, ad)$ where a_i is replaced by x, and where c_i is an appropriate scalar. By generic stability of p, one can find a scalar $c_i' \in F$ (or at any rate such that $(a_1, \ldots, a_d) \models p^d|F$) with $v(c_i) = v(c_i')$. Let $b_i' = c_i' \det(a_1, \cdots, a_i/x, \cdots, ad)$. So $a_i \models p|F(b_i')$, and $b_i'(a_i) = c_i'/c_i \in \mathcal{O}$, so $b_i' \in J_1(p)$. Also, up to multiplication by the units c_i/c_i', (b_i') is the dual basis to $(a_1, \ldots, a_d)$, and hence generates Λ^* as an $\mathcal{O}$-module. This shows that $\Lambda^* \leq J_1(p)$ as required. □

3.6. **valuative volumes and mixed volumes.** Let $\mathbb{S}_n$ be the space of lattices on K^n. Given lattices $L_1, \ldots, L_n \in S_n$, we define $mvol(L_1, \ldots, L_n)$ as follows: choose a generic element $(a_1, \ldots, a_n)$ of $L_1 \times \cdots \times L_n$, and take the valuative volume of the lattice it generates. We also write this as $[L_1, \ldots, L_n]$.

Note that $[L, \ldots, L] = v\mathrm{vol}\,(L)$ is the valuative volume of L itself.

If there are fewer lattices than the dimension, write e.g. $[L_1, L_2, L_3]$ for $[L_1, L_2, L_3, \mathcal{O}^N, \ldots, \mathcal{O}^n]$, filling in the other coordinates with the standard lattice. Note (as the valuative volume of a product is the sum of the volumes) that $[L_1 \times L_1', L_2 \times L_2', L_3 \times L_3'] = [L_1, L_2, L_3] + [L_1', L_2', L_3']$.

Let $\mathcal{P}(\mathbb{S}_n)$ be the space of probability measures on $\mathbb{S}_n$. (More precisely, on generically stable types of elements of $\mathbb{S}_n$ over some valued field F.) If $l_1, \ldots, l_n \in \mathcal{P}(\mathbb{S}_n)$, write $[l_1, \ldots, l_n]$ for the expected value of $[L_1, \ldots, L_n]$, where L_i varies randomly and independently through l_i. Note that $[l, \ldots, l]$ is still the expected mixed volume of n independently chosen lattices, along l.

Lemma 3.7. $[l_1, \ldots, l_n] \leq \frac{1}{n}\Sigma_{i=1}^n [l_i, \ldots, l_i]$

The fundamental case here is $n = 2$:

$$2[l_1, l_2] \leq [l_1, l_1] + [l_2, l_2] \tag{2}$$

The deterministic case can be easily seen: by twisting we may assume $[L_1, L_2] = 0$, so that (2) asserts that $[l_1, l_1] + [l_2, l_2] \geq 0$. Indeed if $[L_1, L_2] \geq 0$, then L_1 is contained in the dual lattice to L_2, (with respect to $\det(\cdot, \cdot, a_3, \cdots, a_n), a_3, \ldots, a_n \in \mathcal{O}^{n-2}$ generic.) and as valuative volume is monotone decreasing with increasing lattices, $v\mathrm{vol}\,(L_1) \geq v\mathrm{vol}\,\widehat{L_2} = -v\mathrm{vol}\,(L_2)$.

However (2) does not follow from the deterministic one by simple summation, since the expectations are with respect to different distributions; $[l_1, l_1]$ denotes not refer to the expected volume of L, but to the expected mixed volume for two independent choices. See Itaï's notes for all this.

Curiously, if we write $(p, q) = exp(-[p, q])$, the result states: $(p, q)^2 \geq (p, p)(q, q)$, looking like a *reverse* Cauchy-Schwartz. This is the form Cauchy-Schwartz takes for timelike vectors in the hyperbolic signature $(1, -1, \cdots, -1)$.

Fixing lattices $L_3, \ldots, L_n$ one obtains: $2[l_1, l_2, L_3, \ldots, L_n] \leq [l_1, l_1, L_3, \ldots, L_n] + [l_2, l_2, L_3, \ldots, L_n]$. This is equivalent to (2) for the images of l_1, l_2 in $K^n/(v_3, \ldots, v_n)$, where the v_i are mutually generics elements of L_i for $i \geq 3$. Now averaging over $L_3, \ldots, L_n$, we obtain: $2[l_1, l_2, l_3, \cdots, l_n] \leq [l_1, l_1, l_3, \cdots, l_n] + [l_2, l_2, l_3, \cdots, l_n]$. From this Lemma 3.7 follows algebraically by induction.

Corollary 3.8. *The mixed volume* $[l, \cdots, l]$ *is a convex function on* $\mathcal{P}(\mathbb{S}_n)$.

Proof. Let $p, q, l \in \mathcal{P}(\mathbb{S}_n)$ with $l = \frac{1}{2}(p + q)$. To choose a random lattice in l amounts to randomly choosing an element $r \in \{p, q\}$, then randomly choosing $L \in r$. Thus n random choices in l amounts to a choice of a subset of $\{1, \ldots, n\}$, say with i elements, and then i random choices along p and $n - i$ choices along q. Denoting $[p, \cdots, p, q, \cdots, q]$ by $p^i q^{n-i}$ if p occurs i times and q occurs $n - i$

times, we have:

$$[l, \cdots, l] = 2^{-n}\Sigma_{i=0}^{n}\binom{n}{i}p^i q^{n-i}$$

By Lemma 3.7,

$$p^i q^{n-i} \leq \frac{i}{n}p^n + \frac{n-i}{n}q^n$$

Using that $\binom{n}{i}\frac{i}{n} = \binom{n-1}{i-1}$ and $\sum_{i=0}^{n-1}\binom{n-1}{i-1} = 2^{n-1}$, and similarly for q, we conclude that $l^n \leq \frac{1}{2}(p^n + q^n)$, as required.

□

3.9. **Minkowski's mixed volumes.** This paragraph is only given as background, and will not be used directly. We show here that valuative mixed volumes (at least over valued fields of equal characteristic zero) can be viewed as a limiting non-archimedean version of Minkowski's mixed volumes.

For a model of ACVF, we will use an ultraproduct K of $K_\nu = (\mathbb{C}, +, \cdot, v)$ over $\nu \in \mathbb{N}$, with $v^{K_\nu}(x) = -1/\nu \log |x|$. In the ultraproduct we retain only elements with $v(x) \geq -n$ for some $N \in \mathbb{N}$, modulo elements of infinite $v(x)$. Thus all standard complex numbers have valuation 0. Let $I_\nu = [-e^\nu, e^\nu]^n$ be the renormalized cube; the ultraproduct of the I_ν is contained in the $\mathcal{O}^n$, and is equal to it up to $(\mathcal{O}^*)^n$.

Any lattice $L = M\mathcal{O}^n$ in n-space K^n is generated by $M[-1,1]^n$ for some matrix M, an ultraproduct of a sequence M_ν. We view L as the limit of the polyhedra $M_\nu[-1,1]^n$. When given $L_1, \ldots, L_n$ lattices in K^n, we can choose $M_1^\nu, \ldots, M_n^\nu$ to be in general position. In particular (using the definitions in [17]), this sequence of n polytopes is developed. (Every proper face is locked; otherwise it is a sum of faces of dimension ≥ 1 so - as no directions cancel - it has dimension n, but it has dimension $< n$.) Also, it is easy to see from the definition that the combinatorial coefficients of such polytopes (of bounded number and combinatorial type) are bounded (by n alone, so independently of ν.) Let $V(, \cdots,)$ denote mixed volume, and let $\mathbf{v}(, \cdots,) = \frac{-1}{\nu}\log V(, \cdots,)$. Thus by [17] §4 Theorem 2 (or Gelfond, referenced there) we have

$$V(M_1^\nu[-1,1]^n, \ldots, M_n^\nu[-1,1]^n) = \sum b_\mu \det(A_1, \ldots, A_n)$$

where $A_1, \ldots, A_n$ range over all possible choices μ of vertices A_i of $M_n^\nu[-1,1]^n$, and $b_\nu = (-1)^n C_{A_1+\cdots+A_n}/n!$ is a bounded coefficient. Thus taking $1/\nu \log$ and limits along the ultrafilter we find:

$$\lim_\nu \frac{-1}{\nu} \log V(M_1^\nu[-1,1]^n, \ldots, M_n^\nu[-1,1]^n) \geq \min v \det(A_1, \ldots, A_n)$$

where now $A_1, \ldots, A_n$ is a choice of vertices of the nonstandard polytope in the ultraproduct. As the minimum of $v\det(v_1, \ldots, v_n)$ is obtained at generic points $v_i \in L_i$, we have $\min v \det(A_1, \ldots, A_n) \geq [L_1, \ldots, L_n]$.

On the other hand, mixed volume is known to be increasing with set inclusion in each variable. One can find a line segment in $M_1^\nu[-1,1]^n$, from the origin to a vector v_i, such that in the ultraproduct we obtain a generic element $(A_1,\dots,A_n) \in L_1 \times \cdots \times L_n$. The mixed volume $V([0,v_1],\dots,[0,v_n])$ is easily computed to be $\det(v_1,\dots,v_n)$. Thus $[L_1,\dots,L_n] = \lim_\nu \frac{-1}{\nu}\log\det(v_1,\dots,v_n) \geq \mathbf{v}(M_1^\nu[-1,1]^n,\dots,M_n^\nu[-1,1]^n)$ This expresses $Mdet$ in terms of nonstandard mixed volumes:

$$[L_1,\dots,L_n] = \lim_\nu \frac{-1}{\nu}\log V(M_1^\nu[-1,1]^n,\dots,M_n^\nu[-1,1]^n)$$

For classical mixed volumes we have indeed the Alexandrov-Fenchel inequality, [16] 3.1.2. : $-\log V(K,L,K_3,...,K_n) \leq (1/2)(-\log V(K,K,K_2,...) - \log V(L,L,K_2,...))$, i.e.

$$[K,L,K_3,\dots,K_n] \leq (1/2)([K,K,K_2,\dots,K_n] + [L,L,K_2,\dots,K_n]$$

The deterministic case of the valuative mixed volume inequality Proposition 3.7 follows by taking limits. It is natural to ask about a more general inequality, from which the full Proposition 3.7 follows.

4. Multiply valued fields

For a field F, let VAL_F be the set of real-valued valuations of F, in the following sense: $v(a) = -\log|a|$ for a function $|\cdot| : F \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ satisfying:

(1) $|xy| = |x||y|$
(2) $|x+y| \leq |x| + |y|$
(3) $|1| = 1, |0| = |0|$.

Let $Val_F \subset VAL_F$ be the the subset avoiding the infinite values $\pm\infty$ on $F \setminus (0)$.

A valuation $v = -\log|\cdot|$ is *archimedean* if $v(2) < 0$ (in particular $2 \neq 0$.). If $v(2) \geq 0$ it can be shown that

$$v(x+y) \geq \min(v(x), v(y))$$

If $v \in VAL_F$ and $0 \leq r \leq 1$ then $rv \in VAL_F$. If v is non-archimedean, then $rv \in VAL_F$ for all real $r \geq 0$. For $r > 0$, the valuations v, rv give isomorphic valued field structures.

Let Tr_n be the set of n-ary terms in the language $+, \min, 0, \alpha \cdot x$ of divisible ordered Abelian groups (here $\alpha\cdot$ denotes scalar multiplication by the rational number α.) We will refer to these as $\mathbb{Q}$-*tropical polynomials* or tropical polynomials for short. We define $\mathbb{Z}$-tropical polynomials to be the terms generated by $+, \min, 0$ alone; but we will not use them much at present. The $\mathbb{Q}$-tropical polynomials are the tropical analogues of polynomials with coefficients $x^\alpha, \alpha \in \mathbb{Q}$ (Laurent-Puiseux.) Note that we have acess to $\max(x,y) = -\min(-x,-y)$, $|x| = \max(x,-x)$, and to $x - y$ as well.

Definition 4.1. A *multiply valued field* is an integral domain F, along with a function $F_t : (F \setminus (0))^n \to \mathbb{R}$ written as:

$$F_t(a_1, \dots, a_n) = \int t(v(a_1), \dots, v(a_n))dv$$

given for each n and each $t \in Tr_n$. We assume the F_t are compatible with permutations of variables and dummy variables, and:

Linearity: $F_{t_1+t_2} = F_{t_1} + F_{t_2}$.

Positivity: If $t(v(a_1), \dots, v(a_n)) \geq 0$ for every $v \in VAL_F$, then $\int t(v(a_1), \dots, v(a_n)) \geq 0$. $(\phi, t) \in POS$ and $\phi(a_1, \dots, a_n)$ then $\int t(v(a_1), \dots, v(a_n))dv \geq 0$.

This is analogous to Boolean-valued valued fields; instead of the truth value of a valuation-theoretic formula, we give the average value of a valuation-theoretic term. Compatibility with the valued field axioms follows from the positivity axioms.

Lemma 4.2. *The positivity condition is equivalent to: if $t(v(a_1), \dots, v(a_n)) \geq 0$ for every $v \in VAL_F$, then $\int t(v(a_1), \dots, v(a_n)) \geq 0$.*

Proof. It suffices to consider valuations and absolute values valued in $\mathbb{R}$ (without $-\infty$, and with the value ∞ taken only by 0.) Given $a_1, \dots, a_n$, it suffices to consider such valuations on the field generated by $a_1, \dots, a_n$; this is by extension of valuations, where we note that one can always extend to an $\mathbb{R}$-valued valuation or $\mathbb{C}$-valued absolute value. Write proof. □

Let $\mathbb{R}^{>0}$ be the multiplicative group of positive reals. Let S_{n-1} be the unit sphere for the sup norm on $\mathbb{R}^n$. Note that each element of Tr_n defines a continuous function $\mathbb{R}^n \to \mathbb{R}$, with $f(ax) = af(x)$ for $a \geq 0$ (positive homogeneity, in particular$\mathbb{R}^{>0}$-equivariance), and $f(x) \leq m \max |x_i|$ for some m. If t_0 is positive on S_{n-1} (for instance, $t_0(x_1, \dots, x_n) = \max_i |x_i| = \max\{x_i, -x_i\}$) and $t \in Tr_n$ then t/t_0 is a well-defined, continuous function $S_{n-1} \to \mathbb{R}$.

Lemma 4.3. *Let $t_0(x) = \max |x_i|$. Then the functions $t|S_n$, $t \in Tr_{n+1}$ are uniformly dense in $C(S_n)$. Hence the F_t can be defined for any 1-homogeneous continuous t, so that if $t_k \in T_{n+1}$ and $t_k|S_n \to t|S_n$ then $F_{t_k} \to F_t$ on S_n.*

Proof. The density follows from the tropical Stone Weierstrass theorem; the constant function is available as $t_0|S_n$. □

Let POS be the set of pairs (ϕ, t) of formulas $\phi(x_1, \dots, x_n)$ in the language of rings, and tTr_n, such that $VF \models (\forall x)(\phi(x) \implies t(v(x_1), \dots, v(x_n)) \geq 0)$.

Remark 4.4. (1) If t is any $\mathbb{Q}$-tropical polynomial, then $t = t^+ - t^-$ with t^+, t^- nowhere-negative tropical polynomials; and we have $F_t = F_{t^+} - F_{t^-}$. Thus there is no harm in restricting the basic relations F_t to non-negative t.

(2) MVF's do not form an elementary class in the sense of real-valued logic; but they can be extended to such a class, where the value ∞ is permitted for the basic relations. The linearity and positivity axioms can be expressed in real-valued logic, i.e. the family of multiply valued rings is closed under ultraproducts. (If $\phi(x)$ is a quantifier-free formula in the language of rings, such that the theory VF of valued fields implies: $(\forall x)(\phi(x) \implies t(v(x)) \geq 0$, then $(\forall x)(\phi(x) \implies t(v(x)) \geq 0)$ is an axiom of multiply valued rings.)
(3) The notion of height is fundamentally global, and there is no good analog for MVF. In particular $\int |v(a)|, \int |v(b)| \leq 1$ while $\int |v(a+b)| = \infty$ is possible.

Definition 4.5. A *globally valued field* is a multiply valued field K satisfying the *product formula* $\int v(a) = 0$, for any $a \in K \setminus (0)$.

4.6. **Topology on VAL_F.** We topologize VAL_F as a subspace of the space of functions $F \to [-\infty, \infty]$; so it is compact.

We have 0_F, the trivial valuation on F, namely the constant function 0. More generally let $TRIV_F$ be the elements of VAL_F that take only the values $-\infty, 0, \infty$. Let $\Omega_F = VAL_F \setminus TRIV_F$. So Ω_F is an open subspace of VAL_F, hence locally compact.

4.7. **Admissible measures.**

Definition 4.8. An admissible measure on F is a regular Borel measure μ on Ω_F such that for any $a \in F$, the function $v \mapsto v(a)$ is integrable. In particular, $\{v : v(a) = \pm\infty\}$ has measure zero, and $\{v : |v(a)| > \epsilon\}$ has finite measure for $\epsilon > 0$. We further demand that μ be supported on a set S with at most one point from each isomorphism class of valuations.

The last requirement is not really restrictive; it can be obtained by an appropriate renormalization.

An admissible μ can be viewed as a measure $\bar{\mu}$ on the set $\overline{\Omega}$ of isomorphism classes of valuations, along with a measurable section $s : \overline{\Omega} \to \Omega$ (so that $\mu = s_*\bar{\mu}$.)

Definition 4.9. A globalizing measure on F is a measure μ on Ω_F such that for any $a \in F$, the function $v \mapsto v(a)$ is integrable, and

$$\int v(a) d\mu(v) = 0 \tag{3}$$

Admissible measures are M-fields in the sense of [6] (with the measure carried by a specific topological space.)

Remark 4.10. (1) The possibility of $|x| = \infty$ is a technicality that can in practice be ignored in the global case (see Remark 4.18). Note that $\{x : |x| < \infty\}$ is a valuation ring; an absolute value is induced on the residue field $\bar{F}$, satisfying $|x| = 0$ iff $u = 0$; and $|x| = |\bar{x}|$ where for $|x| < \infty$, $\bar{x}$ is the residue of $|x|$.

4.11. **Induced structure on a subfield.** Let μ be a globalizing measure on Ω_K, and let F be a subfield of K. Let $r : \Omega_{K;F} \to \Omega_F$ be the restriction map. Then $r_*(\mu|\Omega_{K;F})$ is a globalizing measure on F (called the induced globalizing measure.)

4.12. **Renormalization.** Two valuations are *equivalent* if they differ by a scalar multiple. Let $\overline{\Omega}_F$ be the set of nontrivial valuations on F up to equivalence. Let $\pi : \Omega \to \overline{\Omega}$ be the natural quotient map.

If μ is an admissible measure, let $\overline{\mu}$ be the pushforward measure to $\overline{\Omega}_F$; thus $\overline{\mu}(U) = \mu(\pi^{-1}(U))$.

We say μ *can be renormalized* to μ' if $\overline{\mu} = \overline{\mu'}$, and for almost all $y \in \overline{\Omega}$, if a point is chosen in $\pi^{-1}(y)$ so that $\pi^{-1}(y)$ is identified with $\mathbb{R}^{\geq 0}$, then the conditional measures satisfy $\int x d\mu_y = \int x d\mu'_y$.

Let $\overline{v}$ denote the image of v in $\overline{\Omega}$. For any $0 \neq a \in F$, let $\overline{\Omega}(a) = \{\overline{v} : 0 < v(a) < \infty\}$. We may identify $\overline{\Omega}(a)$ with the compact space $\{v \in \Omega : v(a) = 1\}$.

Proposition 4.13. *Let F be a field. There is a 1-1 correspondence between:*

1. *Admissible measures on Ω_F, up to renormalization.*
2. *Multiply valued structures on F.*
3. *For any $0 \neq a \in F$, a finite measure μ_a on $\overline{\Omega}$, supported on $\overline{\Omega}(a)$; such that for $0 \neq a, b \in F$, $\mu_b|\overline{\Omega}(a) \cap \overline{\Omega}(b) = v(a)/v(b)\mu_a|\overline{\Omega}(a) \cap \overline{\Omega}(b)$.*

Note in (3) that $v(a)/v(b)$ is a well-defined function on $\overline{\Omega}(a) \cap \overline{\Omega}(b)$, though $v(a)$ and $v(b)$ themselves are not well-defined.

Also, any compact subset of $\overline{\Omega}$ is contained in a finite union of sets $\overline{\Omega}(a)$.

In moving from (3) to (2), we will have $F_h(a)$ equal the total mass of μ_a, where h is the height term $\max(0, x)$.

Proof. From (1) to (2): given an admissible measure μ, let $F_t(a_1, \dots, a_n) := \int t(v(a_1), \dots, v(a_n)) d\mu(v)$. As t is bounded by a linear function, $t(x_1, \dots, x_n) \leq m \max |x_i|$, and each $v \mapsto v(a_i)$ is integrable, so is $v \mapsto t(v(a_1), \dots, v(a_n))$. As t is homogeneous, the integral depends on μ only up to renormalization.

From (2) to (3): We will define a finite measure on the compact space $\overline{\Omega}(a)$, identified with $\Omega_1(a) = \{v : v(a) = 1\}$. By the Riesz representation theorem, it suffices to define a positive linear functional I on a uniformly dense space of continuous functions on $\Omega_1(a)$. It suffices to consider functions $G(v)$ of the form $g(a_0(v), a_1(v), \dots, a_n(v))$ where c is a continuous function on $\mathbb{R}^{n+1}$, and where we may choose $a_0 = a$. Let $|\cdot|$ denote the max norm on $\mathbb{R}^{n+1}$; we may change c so that $c = 0$ on $\{y : |y| \leq 1/2\}$. Letting $g(t, y_1, \dots, y_n, t) = tc(y_1/t, \dots, y_n/t)$, we may write $G(v) = g(1, a_1(v), \dots, a_n(v)) = g(a_0(v), \dots, a_n(v))$ where g is a continuous, 1-homogeneous function on $\mathbb{R}^{n+1}$. We may assume, by Lemma 4.3, that $g(x_0, x_1, \ldots, x_n) = t(x_0, x_1, \ldots, x_n)$ for some (necessarily unique) $t \in T_{n+1}$. In this case, define $I(g) = F_t(a_0, a_1, \ldots, a_n)$. Note that if $g \geq 0$ then $t \geq 0$ on all valuations, hence $F_t \geq 0$.

From : (3) to (1): Let $(a_i : i < \kappa)$ be an enumeration of the nonzero elements of F. By transfinite induction we define a Borel section $s_i : \overline{\Omega}(a_i) \to \Omega_F$ such that for $i < j$ we have $s_i|\overline{\Omega}(a_i) \cap \overline{\Omega}(a_j) = s_j|\overline{\Omega}(a_i) \cap \overline{\Omega}(a_j)$ up to measure zero (for either μ_{a_i} or μ_{a_j}.) Assume (a_i, s_i) have been defined for $i < \alpha$. Let $a = a_\alpha$. Let C be a countable set of indices $j < i$ such that $\cup_{j \in C} \overline{\Omega}(a_j) \cap \overline{\Omega}(a)$ has maximal μ_a-measure. One may easily construct a Borel section $s : \overline{\Omega}(a) \to \Omega_F$ such that $s|\overline{\Omega}(a_j) \cap \overline{\Omega}(a)$ agrees with s_j a.e. Let $s_\alpha = s$.

Define a measure ν_i on by $\nu_i = g_i \cdot s_{i*}\mu_{a_i}$, where g_i is the function $v \mapsto s_i(\overline{v})(a_i)$. Then ν_i, ν_j are compatible.

Any compact $C \subset \overline{\Omega}$ is contained in a finite union of sets $\overline{\Omega}(a_i)$, so we obtain a Baire measure, easily checked to be admissible.

□

Remark 4.14. The direction from (1) to (2) does not use the condition on the support of an admissible measure; it suffices to have a a regular Borel measure μ on Ω_F such that for any $a \in F$, the function $v \mapsto v(a)$ is integrable. Thus going from (1) to (2) and back to (1) provides an admissible measure equivalent to μ (i.e. giving the same MVF structure.)

Remark 4.15. The equivalence of (2,3) can also be seen directly; the map from (2) or (3) to (1) is brusquer, making use of the axiom of choice.

Remark 4.16. Let W be a closed set of valuations. For instance, if V is a complete variety over F_0 and $F = F_0(V)$, the set of all divisorial valuations of V, along with the trivial valuation. Assume given an MVF structure such that $F_t(a_1, \ldots, a_n) \geq 0$ as soon as $w(a_1, \ldots, a_n) \geq 0$ for all $w \in W$. Then the proof of ((2) to (3)) shows that it is given by a globalizing measure concentrating on W. This can also be seen by finding continuous functions whose support is a given compact set disjoint from W.

Corollary 4.17. *There is a 1-1 correspondence between globally valued field structures on F, and globalizing admissible measures on Ω_F, up to renormalization.*

Remark 4.18. Let K be a countable field, and μ an admissible measure on VAL_K. Then $\{v : v(a) = \pm\infty\}$ has measure zero; so μ concentrates on elements of VAL_K that nowhere take the values $\pm\infty$ (i.e. on no nonzero element). These can be:

(1) Valuations $v : K \to \mathbb{R}$ in the usual, non-archimdean sense.
(2) Functions of the form $x \mapsto -\log|\sigma(x)|$, where $\sigma : K \to \mathbb{C}$ is an embedding of K in $\mathbb{C}$. We observe that σ is determined precisely up to complex conjugation.

Let F be a field. Let Val_F be the space of all valuations v on F (avoiding infinite values on $F \setminus (0)$.) For $R \subset F$, let $Term_R$ be the $\mathbb{Q}$-vector space of *term functions* $v \mapsto t(v(a_1), \ldots, v(a_n))$ where $a_1, \ldots, a_n \in R$ are nonzero elements, t is a $\mathbb{Q}$-tropical polynomial; here v ranges over valuations $F \to \mathbb{R}$ (not taking the values $\pm\infty$.) Let $Pr_F = \{(f) : f \in F \setminus (0)\}$ be the subspace of *principal* functions, $(f)(v) := v(f)$. Let $\mathbf{V}_F = Term_F / Pr_F$. Let C_F be the cone of non-negative functions, and $\mathbf{C}_F$ the image of C_F in $\mathbf{V}_F$.

4.19. **Proper subvarieties.** Let F be a countable field, $K = F(V)$ for a variety V over F, and let μ be a globalizing measure on Ω_K. For $u \in Val_F$, let μ_u be a measure on the part of Ω_K over u, so that $\mu = \int_u \mu_u$ [1]

By the remark made earlier, as a consequence of integrability, μ gives measure zero to all valuations of K taking an infinite value at any nonzero element of K; i.e. above almost every $u \in Val_F$ (for the pushforward measure), the set of all valuations on any given proper F-subvarieties of V has measure zero.

Restating this in the non-archimedean case, let $B_{F_u}(V_u)$ be the Berkovich space i.e. the space of types in V over F_u. Then μ_u can be viewed as a measure on $B_{F_u}(V)$. let $B^-_{F_u}(V)$ consist of all Berkovich points of dimension $< \dim(V)$, in particular including all classical points (simple points) if $\dim(V) > 0$. Then $\mu_u(B^-_{F_u}(V)) = 0$

Assume now that u is archimedean; then u corresponds to an embedding $i_u : F \to \mathbb{C}$. Let V' be the base-change of V from F to $\mathbb{C}$, via i. Then μ_u can be as a measure on $V'(\mathbb{C})$ (conjugation-invariant in case $i_u(F) \leq \mathbb{R}$), and again the measure of any proper F-subvariety of V' is zero.

Remark 4.20. In the archimedean case for instance, one can also ask about the measure of proper $\mathbb{C}$-subvarieties W. If such a subvariety has positive measure, then the qf type associated to the measure is not strictly transcendental (Definition 8.7) and in fact the canonical base of the variety may be a hyperimaginary definable over K and bounded over F.

4.21. **Extending valuations on subrings.**

Exercise 4.22. Show that GVF structures on F correspond 1-1 to positive linear functionals on $(\mathbf{V}_F, \mathbf{C}_F)$.

The next lemma will be applied specifically with F the compositum field of K_1, K_2, and R the ring generated by $K_1 \cup K_2$; Z is the set of valuations that are nontrivial on K_1 or on K_2.

[1] This presentation of relativization of measures is technically incorrect but is valid for any given countable subalgebra of the measure algebra, in particular for the term functions, which suffice for us. Any assertion about μ_u is understood to be valid for almost all u, with respect to the measure induced by μ on Val_F.

Definition 4.23. Let $Z \subset Val_F$ be a subset, and ν a measure on Z. We say that (F, ν) is *globalizable* if there exists a measure on Val_F extending ν, making F into a GVF.

This will be applied specifically in this situation: K_1, K_2 are field extensions of a GVF F_0; $K_1 \otimes_F K_2$ a domain with field of fractions F; and Z the set of valuations of F that are nontrivial on K_1 or on K_2.

improve.

Lemma 4.24. *Let $Z \subset Val_F$ be a subset including all archimedean valuations; let $R = \{r : v \notin Z \implies v(r) \geq 0\}$. Assume the field of fractions of R is F. Let ν be a measure on Z, such that for any nonzero $r \in R$,*

$$\int v(r) d\nu(r) \leq 0$$

Then there exists a GVF structure on F, such that for any $\phi \in Term_F$ supported on Z we have: $\int \phi = \int \phi(v) d\nu(v)$.

Proof. Let V be the space of term functions on the valuations of F (avoiding infinite values.) Among these, we have the functions (f) for $f \in F \setminus (0)$, defined by: $f(v) = v(f)$ Let $P = \{(f) : f \in F\}$. Let U be the subspace of functions supported on Z. Let $\mathbf{V} = V/P$, $\mathbf{U}$ the image of U in $\mathbf{V}$. For $\alpha \in U$, $\phi(\alpha) = \int \alpha(v) d\nu(v)$. If $\alpha \in P \cap U$, say $\alpha = (f)$; then $(f^{-1}) = -\alpha$ so $f \in R^*$, and by assumption, $\phi((f)) \leq 0, -\phi((f)) = \phi((f^{-1})) \leq 0$, so $\phi((f)) = 0$. Thus we may view ϕ as defined on $\mathbf{U}$.

We define a cone on V: Pos is the set of non-negative functions in V; $\mathbf{Pos}$ the image of Pos in $\mathbf{V}$. To show that ϕ is positive on $\mathbf{V}$, let $y \in \mathbf{Pos} \cap U$; so y is the image of an element of U of the form $\alpha - (f)$, $a \geq 0$. Thus if $v(f) < 0$, $(f)(v) < 0$ so $(\alpha - (f))(v) > 0$; as $\alpha - (f)$ is supported on Z, it follows that $v \notin Z$. So $f \in R$ by definition of R, and hence by assumption, $\int v(f) d\nu(v) \leq 0$, and hence $\int v(\alpha - (f)) d\nu(v) \geq 0$.

The boundedness condition of M. Riesz (2.8 (2)): Any element of V is bounded by a finite sum of elements h^+, and $h = f/g$ for some $f, g \in R$. We have

$$(f/g)^+ \leq (f)^+ + (g) - \sim (f)^- + (g)^-$$

and $(f)^- + (g)^-$ is supported on Z.

By Lemma 2.8 (2) and Exercise 4.22, there exists a GVF structure on F, agreeing with ν on terms supported on Z.

□

We have a Galois correspondence between subsets of Val_F and subrings of F, $Z' \mapsto R = \{r : v \in Z' \implies v(r) \geq 0\}$, and in the opposite direction: $R \mapsto Z' = \{v : r \in R \implies v(r) \geq 0\}$. It will be useful to know in some cases that R is closed in the sense of this correspondence.

Exercise 4.25. (1) Let V_1, V_2 be irreducible affine varieties over F, and h a rational function on $V_1 \times V_2$ whose poles on $V_1 \times V_2$ are all vertical or horizontal, i.e. f restricts to a regular function on $V_1' \times V_2'$ for some Zariski open $V_1' \subseteq V_1, V_2' \subseteq V_2$. Then $f = c^{-1}d^{-1}\sum_{i=1}^{n} a_i b_i$ for some regular $a_1, \dots, a_n, c$ on V_1 and $b_1, \dots, b_n, d$ on V_2.

(2) Let a ring K_1K_2 generated by two fields K_1, K_2 linearly disjoint over $F = K_1 \cap K_2$. Let Z^c be the set of valuations that are non-negative on K_1K_2. Let $R = \{r : v \in Z^c \implies v(r) \geq 0\}$. Then $R = K_1K_2$. (This says that K_1K_2 is integrally closed, but a little more since we are using only real-valued valuations.)

We summarize the contents of Lemma 4.24 in the following terms: ν is *globalizable* iff $\int v(r)d\nu(r) \leq 0$ for $r \in R = R(Z), r \neq 0$. See Proposition 8.1.

4.26. **Symmetric extensions.**

Proposition 4.27. *Let F be a field, and L a finite Galois extension. Let μ be a globalizing measure on F. Then μ extends uniquely to an $Aut(L/F)$-invariant measure on Ω_L, and this is a globalizing measure.*

Proof. Let $G = Aut(L/F)$. Then G acts on Ω_K, and the quotient can be identified with Ω_F. The existence and uniqueness of an $Aut(L/F)$-invariant measure on Ω_L is thus a special case of Lemma 2.1; we have to show it is globalizing. Now if e_a is the function $v \mapsto v(a)$, then for $a \in L$ we have

$$Tr(e_a) = \sum_{g \in G} e_{g(a)} = e_{N(a)}$$

where $N(a)$ is the norm of a (product of $Aut(L/F)$-conjugates of a.) Thus (1) reads:

$$\int e_a(y)d\nu(y) = \frac{1}{|G|}\int e_{N(a)}(x)d\mu(x)$$

so the product formula (3) for ν and a will follow from the same for μ and $N(a)$.

But we have not yet proved integrability of e_a. For this we use not only $N(a)$ but also the symmetric functions $S_k(a)$, $k = 0, \dots, n = [L : F]$; we have $\sum_{k<n} S_k(a)a^k = 0$, so in the non-archimedean case $v(a) = \frac{v(S_k(a)) - v(S_l(a))}{k-l}$ for some $k \neq l$, in particular $|v(a)| \leq \sum |v(S_k(a))|$, and a similar inequality in the archimedean case. As $e_{S_k(a)}$ is integrable for each k, so is e_a. □

Corollary 4.28. *Let F be a field, μ a globalizing measure on F, $L = F^a$ the algebraic closure. Then μ extends uniquely to an $Aut(L/F)$-invariant measure on Ω_L, and this is a globalizing measure.*

Proof. Let $G = Aut(L/F)$. Then G acts on Ω_K, and the quotient can be identified with Ω_F. The existence and uniqueness of an $Aut(L/F)$-invariant measure ν_L on Ω_L is thus a special case of Remark 2.2 (2). Also the restriction to a finite Galois extension K of F is the unique $Aut(K/F)$-invariant extension, hence is globalizing by Proposition 4.27. As every element of L lies in some K, it follows that ν_L is globalizing. □

4.29. **Heights.** Let μ be a globalizing measure. For $x \neq 0 \in K$, define

$$h(x) = \int |v(x)| d\mu(v)$$

Exercise 4.30. [height-finite] Let $Y(d, r)$ be the set of algebraic integers of degree $\leq d$ (i.e. $F(x) = 0$ for some monic $F \in \mathbb{Q}[X]$, $\deg(h) \leq d$) and height $\leq r$. Then $Y(d, r)$ is finite.

(Using symmetric functions as above, find a bound on the height of the coefficients of F.)

Exercise 4.31. $h(x^n) = nh(x)$. However, we need not have $h(xy) = h(x) + h(y)$ (try $1 + \sqrt{-1}$).

Exercise 4.32. $h(xy) \leq h(x) + h(y)$, and $h(x+y) \leq h(x) + h(y) + m \log 2$ where m is the measure of the set of archimedean valuations (if any).

Corollary 4.33 (Kronecker)**.** *If $a \in \mathbb{Q}^a$ and $h(a) = 0$, then a is a root of unity.*

Proof. Say $a \in K$, K a finite extension of $\mathbb{Q}$ of degree d. Then all powers a^n lie in K. Also, $v(a^n) = nv(a)$ so $h(a^n) = 0$. By Exercise 4.30, the set of powers of a is finite, so a is a root of 1. □

4.34. **A criterion for globalization.**

4.35. **GVF's as continuous logic structures.** To present GVF's as an elementary class in continuous, real-valued logic, in the sense of [**?**], we need to choose a *gauge*. The gauge here is natural, the height, and the required properties are proved below.

Definition 4.36. Let K be a GVF. Define $h(0) = 0$ (arbitrarily) and $h(x) = \int v(x)^+ = \int v(x)^-$ for $x \neq 0$.

Thus we may only quantify over height-bounded sets.

Equivalently, an ultrproduct of GVF's is defined as follows: let K_i be a GVF, and let u be an ultrafilter on i. Let K^* be the usual ultraproduct of fields K_i. Let K be the set of elements of K^* represented by sequences $(a_i \in K_i)_{i \in I}$, with $h(a_i)$ bounded. They form a subfield of K^* by Lemma 4.37. Note that if if $a, b, c, \ldots$ are elements of K, represented by sequences of bounded height $(a_i), (b_i), (c_i), \ldots$, then for any $\mathbb{Q}$-tropical polynomial t, $F_t(a_i, b_i, c_i, \ldots)$ is bounded; thus a unique real number r satisfies, for any $\epsilon > 0$, for u-almost all i, $|F_t(a_i, b_i, c_i, \ldots) - r| < \epsilon$; we let $F_t(a, b, c, \ldots) = r$.

Lemma 4.37. *$h(xy) \leq h(x)+h(y)$; $h(x+y) \leq h(2)+\max(h(x), h(y))$. $h(x^{-1}) = h(x)$ for $x \neq 0$.*

4.38. **Archimedean valuations.** Let A be a finitely generated domain over $\mathbb{Q}$, $X = Spec(A)$. Let $V_{arch}(A)$ be the set of archimedean semi-valuations on A, i.e. archimedean valuations on a quotient field K of A. Topologize it as a subspace of $VAL(A)$.

We can identify $V_{arch}(A)$ with $X(\mathbb{C})$ up to complex conjugation; to see that the topologies coincide, we will use a lemma. (It would also be possible to deduce Lemma 4.42 from compactness considerations, and deduce Lemma 4.39 from the equality of the topologies in Lemma 4.42.)

Lemma 4.39. *Let X be a compact subset of $\mathbb{C}^n$, with $X^\tau = X$. Let $f : X \to \mathbb{C}$ be any continuous function on X, satisfying: $\tau \circ f \circ \tau = f$. Then f can be approximated in the uniform norm by a polynomial with rational coefficients. Hence if $g : X/\tau \to \mathbb{R}$ is continuous, then g can be uniformly approximated by a function $-\log|f|$ with $f \in \mathbb{Q}[X]$ (so f has no zeroes or poles on X.)*

Proof. Let $C(X)$ be the space of continuous complex-valued functions on X, and let P be the image in $C(X)$ of the polynomials $\mathbb{C}[X]$ in n variables. By Stone-Weierstrass, P is uniformly dense in $C(X)$. We have $T : C(X) \to C(X)$ defined by: $T(f) = \tau \circ f \circ \tau$. Since $T^2 = Id$, $C(X) = Ker(T-1) \oplus Ker(T+1)$. As P is T-invariant, we have similarly $P = Ker(T-1|P) \oplus Ker(T+1|P)$, and it follows (using continuity of T in the operator norm) that $Ker(T-1|P)$ is uniformly dense in $Ker(T-1)$. Now $Ker(T-1|P)$ is the image in $C(X)$ of $\mathbb{R}[X]$. Thus the image of $\mathbb{R}[X]$ in $C(X)$ is uniformly dense; as any polynomial in $\mathbb{R}[X]$ can be uniformly approximated on X by a polynomial in $\mathbb{Q}[X]$, it follows that the image of $\mathbb{Q}[X]$ is uniformly dense too.

Let $g : X/\tau \to \mathbb{R}$ be continuous. Let $G(x) = exp(-g(x/\tau))$; so $G : X \to \mathbb{R}$ and $G(x) > 0$ for all $x \in X$; by compactness of X, $G : X \to [1/r, r]$ for some $r \in \mathbb{R}, r > 1$. Note that $T(G) = G$. Hence there exists $f \in \mathbb{Q}[X]$ arbitrarily close to G in the uniform norm on X; it follows that $-\log|f|$ is as close as required to g.

□

We pause to record a fact about the uniform measure on annuli, that will be needed when we consider archimedean 1-types. Write $v(z)$ for $-\log(|z|)$. Let μ_r be the rotation-invariant probability measure on the complex circle $\{z : v(z) = r\}$.

Exercise 4.40 (Jensen's formula.)**.** For $a \in \mathbb{C}$, $\int v(z-a)d\mu_r(z) = \min(v(a), r)$.

If K is non-archimedean, nontrivially valued field, we can define a valuation p_r on $K(x)$ such that $v(x-a) = \min(v(a), r)$ for all $a \in K^a$. Let μ_r be the probability measure concentrating on the single point p_r. Then clearly we have, here too: $\int v(z-a)d\mu_r(z) = \min(v(a), r)$.

Lemma 4.41. *Let $X \subset \mathbb{C}^n$ be a bounded set. Let μ, μ' be two measures on X, with equal finite total volume. Assume, for every $f \in \mathbb{Z}[X_1, \ldots, X_n]$, that $v(f(x)) = -\log|f(x)|$ is integrable for μ, μ', and that $\int v(f(x))d\mu(x) \geq \int v(f(x))d\mu'(x)$. Then $\mu = \mu'$.*

Proof. As $\int v(mf(x))d\mu(x) = \int v(f(x))d\mu(x) - \mu(X)\log(m)$ and similarly for μ', the inequality continues to hold for $f \in \mathbb{Q}[X_1, \ldots, X_n]$. By Lemma 4.39, $\int g d\mu \geq \int g d\mu'$ for any Borel step function g. So $\mu - \mu'$ is a non-negative measure with total mass 0, hence equals 0. □

Let $X(\mathbb{C})$ be the set of ring homomorphisms $h : A \to \mathbb{C}$. If we choose generators $a_1, \ldots, a_n \in A$, then h is determined by $h(a_1), \ldots, h(a_n)$; so $X(\mathbb{C})$ can be identified with a subvariety of $\mathbb{C}^n$; in particular we obtain an induced topology on $X(\mathbb{C})$, which does not depend on the choice of generators. Let τ denote complex conjugation, and also the equivalence relation on $X(\mathbb{C})$ identifying h with $\tau \circ h$. We give $X(\mathbb{C})/\tau$ the quotient topology.

Lemma 4.42. *There exists a natural bijection $X(\mathbb{C})/\tau \to V_{arch}(A)$. Moreover it is a homeomorphism.*

Proof. An element of $X(\mathbb{C})$, i.e. a homomorphism $h : A \to \mathbb{C}$, induces a semi-norm on A, namely $a \mapsto |h(a)|$. It is clearly conjugation invariant, and we leave it as an excercise to show that it is bijective and continuous. To show that the map is open, pick a point $x \in X(\mathbb{C})$, and let U be a conjugation-invariant, neighborhood of x. We have to find a smaller open neighborhood of x, cut out by equations $-log|a| < \alpha$ or $-log|a| > \alpha$, where $a \in A$, i.e. a is a rational polynomial in the coordinates $z_1, \ldots, z_n$. First let $C(r)$ be a set of the form: $|z_1| \leq r, \cdots, |z_n| < r$; take r large enough to contain a neighborhood of x. By Lemma 4.39, find a rational polynomial h such that $-\log|h| = 0$ on x, but $-\log|h| \geq 1$ on $C(r) \setminus U$. Thus $C(r)$ along with $-\log|h| < 1/2$ cut out a neighborhood of x contained in U. □

Let K be the field of functions of an irreducible variety V over $\mathbb{Q}$. A complex embedding of K is the same as a point of $V(\mathbb{C})$ which does not belong to any proper $\mathbb{Q}$-subvariety.

Thus μ induces a measure μ' on $V(\mathbb{C})/\tau$ where τ is complex conjugation. There exists a unique conjugation-invariant measure μ_{arch} on $V(\mathbb{C})$ whose pushforward to $V(\mathbb{C})/\tau$ is μ'. (Namely, on $V(\mathbb{R})$, $\mu' = \mu_{arch}$; on each connected component C of $V(\mathbb{C}) \setminus V(\mathbb{R})$, $\mu_{arch} = \frac{1}{2}\mu'$; where we identify $V(\mathbb{R})$ or C with a subset of $V(\mathbb{C})/\tau$.) The measure μ_{arch} restricts to zero on each proper $\mathbb{Q}$-subvariety of V.

Exercise 4.43. Explain the assertions in this subsection. Show that μ_{arch} is a finite measure.

5. Classical GVF structures.

Let K be a GVF. Then either $v(2)$ is not identically zero, or else K contains a constant field C satisfying $h(x) = 0$, and also some element t with $h(t) > 0$. Thus K contains either $\mathbb{Q}$ or a rational function field $k(t)$ as a nontrivial GVF.

Exercise 5.1. Prove the above. Conclude that a globalizing measure is never a finite measure, unless it is zero.

Exercise 5.2. Let r be a non-negative real.

(1) There is a unique GVF structure on $\mathbb{Q}$, with $h(2) = r$. (Keep the usual valuations, and give v_∞ weight $r/log(2)$.)
(2) If k is a field, there is a unique GVF structure on $k(t)$, trivial on k, and with $h(t) = r$.

(Hint: in the case of $\mathbb{Q}$, a GVF structure corresponds to numbers m_p for each prime p including $p = v_\infty$, such that $\sum m_p v_p(x) = 0$ for all $x \in \mathbb{Q} \smallsetminus (0)$. Applying this to a prime number p, since $v_l(p) = 0$ for all $l \neq p, \infty$, and $v_p(p) = 1$, we find that $m_p + m_\infty v_\infty(p) = 0$, i.e. $m_p = -\log(p) m_\infty$.)

We can view $k(t)$ as the function field of the projective line. To extend the analysis to other curves, i.e. to finite extensions of $k(t)$, we will need Riemann-Roch, or rather Riemann's inequality. Let k be a field, that we may take algebraically closed. Let C be a smooth, nonsingular curve over k, and let $t \in k(C)$ be any nonconstant element.

Lemma 5.3. *Let r be a non-negative real. There is a unique GVF structure on $k(C)$, trivial on k, and with $h(t) = r$.*

Proof. The valuations of K/k have the form v_a, $a \in C(k)$, where $v_a(f)$ is the order of zeroes of f at a. Let d be the number of zeroes of f (counted with multiplicites). Let μ be the measure on $C(k)$ giving each point weight r/d. As every $f \in k(C)$ has as many zeroes as poles, this is a globalizing measure on $k(C)$.

To prove uniqueness, let μ be any globalizing measure. Note that $C(k) = Val(K/k)$ has the discrete topology (The open set $v(f) > 0$ picks out a finite set of valuations, corresponding to the zeroes of f.) Thus μ is atomic, giving a weight m_a to v_a, and it suffices to show that all points of $C(k)$ have the same weight. Let a, b be two distinct points of $C(k)$, and let c be another point, $a, b \neq c$. Let n be a large integer. By Riemann's inequality, for some g depending only on C, there exists $f \in k(C)$ with a pole of order at most g at c, a pole of order n at b, a zero of order at least n at a, and no other poles. It follows that $\int v(f) \geq m_a n - m_b n - m_c g$. Thus $m_b \geq m_a + m_c g/n$. Letting $n \to \infty$ we see that $m_b \geq m_a$, so by symmetry $m_b = m_a$ as required. ☐

Lemma 5.4. *Let r be a non-negative real. Let K be a finite extension of $\mathbb{Q}$. There is a unique GVF structure on K with $h(2) = r$.*

We postpone the proof a little, in order to give it in greater generality.

Corollary 5.5. *Let $r \geq 0$. There exists a unique GVF structure on $k(t)^a$, trivial on k, with $h(t) = r$. There exists a unique GVF structure on $\mathbb{Q}^a$ with $h(2) = r$.*

We will refer to the above GVF's as *classical.* Denote the above by $k(t)^a[r]$, $\mathbb{Q}^a[r]$. (The former notation depends on t.)

In addition, ultraproducts of such structures will also be called classical. Notably, let u be an ultrafilter on $\mathbb{N}$, $r = (r_i)$ a divergent sequence of reals, and let $\mathbb{Q}^a_{r;u}$ be the ultraproduct of $\mathbb{Q}^a[r_i^{-1}]$ over this ultrafilter. We obtain a purely non-archimedean GVF $\mathbb{Q}^a_u$; it samples $\mathbb{Q}^a$ at heights r_i. Similarly, $\mathbb{F}^a_{r;u}$ is the ultrapower of $\mathbb{F}_p(t)^a[r_i]$, another characteristic zero, non-archimedean GVF.

Theorem 5.6. *$k(t)^a[r]$ is existentially closed.*

6. Conjectures

Four conjectures will serve as organizing principles for further work.

The various conjectures are tied up with each other. Notably, stability implies a notion of canonical amalgamation, which is clearly relevant to (2). (1) amounts to an open mapping theorem for quantifier-free types, *along with* a finiteness condition for counterexamples to amalgamation.

(4) is a theorem in the case of $k(t)^{alg}$ (with k a constant field), with the standard GVF structure. For the asymptotic versions of $\mathbb{Q}^a$ and $k(t)^a$, it amounts roughly to the statement that there are no further relations among the height functions h_D, $D \in Pic(X)$ beyond the known ones.

(2) is true in the function field case when A has transcendence degree 0 or 1 over the constants, as an immediate corollary of (4).

(1) The theory of GVF's admits a model companion; in other words, the class of existentially closed GVF's is axiomatizable.
(2) The class of GVF's admits amalgamation over structures A that are qf-algebraically closed within some existentially closed extension.
(3) The theory of GVF's is stable.
(4) The classical GVFs are existentially closed.

In (2), the notion of algebraic-closure is the model-theoretic one.

Example 6.1. (over $\mathbb{Q}^a[1](a, b)$). The formula:

$$v(x - a) \geq v(2), v(x - b) \geq v(3), h(x) < (1/2)\log(6)$$

has at most one solution. Indeed if x, x' are two solutions, and $x - x' \neq 0$, then already the height of $x - x'$ for valuations dividing 6 is $\geq \log(6)$, a contradiction. The solution need not lie in $\mathbb{Q}(a, b)^a$: for instance we could first pick c, then pick a close enough to c above 2, and b close to c above 3; so c will be definable over $\mathbb{Q}(a, b)$ but certainly a, b, c will be algebraicallly independent over $\mathbb{Q}$.

The above example is, conjecturally, typical; see the proof of Lemma 10.5 for a variant. We also have ordinary, field-theoretic algebraic closure; GVF structures do not always extend in a Galois-invariant way. This shows that one cannot expect full quantifier-elimination in the model companion; but if (2) holds, only bounded quantifiers (in the sense of model-theoretic, qf algebraic closure) are required.

We can also define the *aq-type* by allowing *algebraicallly bounded quantifiers* $\inf_{f(x_1,\ldots,x_k,y)=0} \phi(x,y)$, where f is a polynomial, monic in the variable y; thus for $L_i = L_i^{alg}$, $a_i \in L_i^m$, $K \leq L_1, L_2$ we have $aqtp(a_1/K) = aqtp(a_2/K)$ iff $(K(a_1)^{alg}, a_1) \cong_K (K(a_2)^{alg}, a_2)$.

7. G_m-fullness and the GVF structure on K^a.

For any GVF K, with admissible globalizing measure μ, let $\underline{\Gamma}(K) = L^1(\Omega_K, \mu)$. If an equivalent admissible μ' is given, there is a canonical isomorphism $L^1(\Omega_K, \mu) \to L^1(\Omega_K, \mu')$; $\underline{\Gamma}(K)$ may be taken to be the limit along these isomorphisms. We have a sequence of group homomorphisms:

$$G_m(K) \to \underline{\Gamma}(K) \to \mathbb{R} \to 0$$

The map $G_m(K) \to \underline{\Gamma}(K)$ is the structure map $a \mapsto [a]$, where $[a](v) = v(a)$. The map $\underline{\Gamma}(K) \to \mathbb{R}$ is the integral, $f \mapsto \int f$. The composition of these two maps is zero by the product formula. We say that K is G_m*-exact* if the sequence is exact; we say it is G_m-full if the image of $G_m(K)$ is dense in the kernel of $\underline{\Gamma}(K) \to \mathbb{R}$, with respect to the L^1-norm. [2]

Say K is G_m-pre-full if the divisible hull (equivalently the $\mathbb{Q}$-span, equivalently the $\mathbb{R}$-span) of the image of $G_m(K)$ is dense in the kernel of $\underline{\Gamma}(K) \to \mathbb{R}$.

Let us analyze the fullness condition from the point of view of definability. An element $f \in L^1(\Omega_K)$ can be approximated in the L^1-norm by another, supported on $\cup_{i=1}^n \Omega_K(a_i)$; where $\Omega_K(a) = \{v : v(a) > 0\}$. On $\Omega_K(a)$, we may find a continuous g with $\int |f-g| < \epsilon/2$, and then uniformly approximate g by a function $F_t(a_1, \ldots, a_n)/v(a)$. Thus f can be approximated by finite linear combinations of such term functions, with finitely many parameters.

check normalization, and add argument for EC fullness.

If K has a full algebraic extension, then K may not be full but it will be pre-full; if $s \in L^1(\Omega_K)$, then s is close to $[f]$ for f in some finite Galois extension L of K; let $r = |G|$; so all Galois conjugates of $[f]$ are close to s, and hence their average is close to s: this average is just $(1/r)[N_{L/K}(s)]$.

In the converse direction, one can say that if K is algebraically closed, or just closed under roots, and is pre-full, then K is full; the presence of n'th roots implies that the image of $G_m(K)$ is a $\mathbb{Q}$-subspace of $\underline{\Gamma}(K)$.

[2]We will later interpret $\underline{\Gamma}(K)$ as $G_m(\mathbb{A})/G_m(\mathbb{O})$, and discuss a similar fullness notion for other groups G.

7.1. **Axiomatizability of G_m-fullness.**

Exercise 7.2. Let K be an ultraproduct of GVF's K_i along an ultrafilter u (in the sense of continuous logic.) Let $f \in L^1(\Omega_K)$. Then there exist $f_i \in L^1(\Omega_{K_i})$ such that if $a, a_1, \dots, a_n \in K$ are represented by $a(i), a_1(i), \dots, a_n(i)$, then for any tropical polynomial t, $\int_{\Omega_{K_i}} |F_t(a_1(i), \dots, a_n(i))/v(a(i)) - f_i(v)|dv$ approaches $int_{\Omega_K} |F_t(a_1, \dots, a_n)/v(a) - f_i(v)|dv$ along the ultrafilter. Moreover for almost all $v \in \Omega_K$ there exists a sequence $v(i) \in \Omega_{K_i}$ such that K_v is the ultrapower of $(K_i)_{v_i}$, and $f_i(v_i) \to_u f(v)$.

check.

Exercise 7.3. The G_m-full GVF's form an elementary class. (Show that G_m-fullness is equivalent to the following 'continuous' version: for all terms $f(v) = F_t(a_1, \dots, a_n)/v(a)$, if $|\int f| < \epsilon/2$, then there exists a with $\int |f(v) - v(a)|dv < \epsilon$.)

Exercise 7.4. Let F_n be GVF's, and let u be a non-principle ultrafilter on $\mathbb{N}$. Assume each F_n is G_m-full. Then the ultrapower F_u is G_m-exact.

Lemma 7.5. *Let K be a G_m-full (or pre-full) GVF. If μ' is a globalizing measure on K, absolutely continuous with respect to μ ($\mu(Z) = 0$ implies $\mu'(Z) = 0$), then μ' is a scalar multiple of μ.*

Proof. It suffices to show that $\mu(Z) = t\mu(Z')$ (with $t \in \mathbb{R}$) implies $\mu'(Z) = t\mu'(Z')$. Let $s \in L^1(\Omega_K)$ be $s = 1_Z - t1_{Z'}$, so $\int s = 0$. By G_m-pre-fullness, s can be approximated in $L^1(\mu)$ by functions $\frac{1}{r}[f]$, $r > 0$, $f \in K$. As $\int [f] d\mu' = 0$, we have $\int s d\mu' = 0$, i.e. $\mu'(Z) = t\mu'(Z')$. □

We will later define and prove:

Proposition 7.6. *K^a is G_m-full iff for any irreducible, algebraic torus T over K, K is T-pre-full.*

Corollary 7.7. *Let K be a GVF. Let μ be a Galois-invariant globalizing measure on K^a, extending the given GVF structure. Assume (K^a, μ) is G_m-full. Then μ is the unique extension of K to a GVF structure on K^a.*

Proof. Let L be a finite Galois extension of K, and let μ be the Galois-invariant extension of μ_K to L. If μ' is any other globalizing measure on L extending μ_K, then μ' must be absolutely continuous with respect to μ. For let Z be a subset of VAL_L, with $\mu(Z) = 0$; then $\mu(GZ) = 0$ where $G = Aut(L/K)$; but GZ is the pullback of a subset of VAL_K, so has measure zero for μ' too. We also saw that L is pre-full; so by Lemma 7.5, $\mu' = \mu$. □

Exercise 7.8. Let C be a curve over a field k; let each valuation of C/k have measure 1. Then $k(C)$ is pre-full. (This was implicitly proved above using Riemann's inequality.)

Exercise 7.9. Let L be a finite extension of $\mathbb{Q}$. Then as a GVF, L is pre-G_m-full. (The proof uses the finiteness of the class number of K, and the Dirichlet unit theorem.)

This now proves Lemma 5.4.

Lemma 7.10. *Let K be an existentially closed GVF. Then K is G_m-full.*

Proof. Let μ be an admissible measure on Ω_K, inducing the given GVF structure. Let $s \in \underline{\Gamma}(K)$. We may take s to be given by a term, with support $\{v : \bigwedge_{i=1}^k v(a_i) > 0\}$. We will find a GVF structure on the rational function field $K(t)$ such that $[t] = s$ (where s is defined by the same term on $K(t)$; as a function, we will see that it is pulled forward to the valuations of $K(t)$ nontrivial on K, and extended by 0 to $\Omega_{K(t)}$.) By existential closure of K, t can be approximated within K.

We may assume $\int s = 0$; we let $r = 0$ in Lemma 9.1. For each $v \in \Omega_K$, let $q(v)$ be the uniform measure on the annulus $v(x) = s(x)$. Let ν be the integral of $q(v)$ over μ. On the non-archimedean part, $q(v)$ is a single valuation above v, and ν is the pushforward of μ via q.

The measure μ_{ad} we constructed has: $0 \geq \int v(f)$, for any $f \in K[t] \smallsetminus K$. Indeed $0 = \int v(t)$ by construction, while $v(t-a) = \min v(a), v(t)$ for $a \neq 0$, so $\int v(t-a) \leq \int v(t) = 0$. Let $r = 0$. By the existence part of Proposition 9.1, there exists a globalizing measure on $\Omega_{K(t)}$ extending μ_{ad}; the construction there shows that $v(t) = 0$ for any $v \in ValK(t)/K$. (Namely: v_∞ has weight $r = 0$; v_t has weight $m_t = 0 - \int v(t) d\mu_{ad}(t) = 0$; and $v_p(t) = 0$ for any other p .) Thus $v(t) = s(v)$ for (almost) any $v \in \Omega_{K(t)}$.

Fill in the details in this proof. In fact for any GVF K there exists an extension L which is G_m-full, and such that any qf type realized in L/K is extendible. (Call a quantifier-free type p over a GVF $F = F$ *extendible* if for any GVF extension $K \geq F$ with F qf-algebraically closed in K, the canonical adelic extension $p \otimes_{adF} K$ is globalizable.) ∎

8. Adelic canonical amalgamation

Assume F is a GVF. In this section, given two GVF extensions K, L of F, we will describe a canonical MVF amalgam $K \otimes_{adF} L$. We will see that if F is existentially algebraically closed in some GVF containing copies of K and of L, then $K \otimes_{adF} L$ is globalizable too. this amalgam extends to a GVF structure.

Let K, L be two MVF extension of F. Assume $K \otimes_F L$ is an integral domain, with field of fractions (KL). We will assume $K = F(a)$ is finitely generated, $p = qftp(a/F)$, $K = F(a)$; when convenient we will think of F, L as finitely generated too. We are interested in MVF structures on the field of fractions (KL); note that GVF structures on (KL) correspond to extensions of p to L.

For the time being, we consider only the *adelic* part of (KL), i.e. the measure on the set $(KL)_{ad}$ of valuations of (KL) that are nontrivial on at least one of K, L.

Four subsets of $(KL)_{ad}$ should be distinguished:

(1) Archimedean valuations. The archimedean part of p gives a conjugation-invariant probability measure $\mu_{F,arch}$ on $U(\mathbb{C})$ (where $F = \mathbb{Q}(U)$). p gives a probability measure $\mu_{p,arch}$ on $V(\mathbb{C})$, projecting to $\mu_{F,arch}$. Similarly L gives $\mu_{L,arch}$ on $W(\mathbb{C})$.
(2) Non-archimedean valuations v of F.
(3) Valuations nontrivial on K but trivial on F.
(4) Valuations nontrivial on L but trivial on F.

Given a valuation of K and a valuation of L of one of the above types, we describe an extension of these to $(KL) = L(a)$, that we call the *canonical extension.*

We describe the measure on valuations of the amalgam above given measures on the nontrivial valuations of K, L. Above a nontrivial valuation v_F of F, this will be done by specifying a probability measure on the set of valuations extending any given pair (v_K, v_L) of valuations of K, L extending v_F, and integrating these measures.

(1) Archimedean case: the fiber product measure. $\mu_{F(a),arch} \times_{\mu_{F,arch}} \mu_L$. In the classical (Kolmogoroff) approach, this is described using the notion of conditional expectation: conditioned on $b \in U(\mathbb{C})$, we take the product of the conditional measures $\mu_{F(a),arch}|b$ and $\mu_{L,arch}|b$.
(2) Above v_0, a non-archimedean, nontrivial valuation of F.

 Consider first two extensions $(K, v_K), (L, v_L)$ of F_{v_0} as valued fields. Consider a valued field (M, v_M) containing as subfields, and such that for c in the ring generated by $K \cup L$ within M, we have

$$v_M(c) = \sup\{\min_i v_K(k_i) v_L(l_i) : c = \sum_{i=1}^{m} k_i l_i, k_i \in K, l_i \in L\}$$

 We say in this case that M is the canonical amalgam. It is clearly uniquely determined, and minimizes the valuation of every element. Write $p \otimes q$ for $qftp((a, b)/F)$ if $p = qftp(a/F), q = qftp(b/F)$ and a, b are taken in M.

 Assume $p * q$ exists for almost all $p \in Val_{K/F_{v_0}}$ and almost all $q \in Val_{L/F_{v_0}}$.

 The assumption is always correct when $F = F^{alg}$; see § **??**. We also have the dividing property: if $(a_i : i \in \mathbb{N}) \models p^{\mathbb{N}}$, $b \models q$, and $qftp(a_i, b)$ is constant, then $qftp(a_i, b) = p \otimes q$.

 We define the canonical extension above v_0 to be the pushforward of the measures $\mu_K|v_0$ and $\mu_L|v_0$ under $\otimes$.

(3) Let v_K be trivial on F. Assume $v_K \otimes Triv_L$ exists (in this case it is the unique amalgam). Then above $(v_K, triv_L)$ we take $v_K \otimes Triv_L$, with weight 1.
(4) Dually.

From Lemma 4.24 and Exercise 4.25, we obtain:

Proposition 8.1. *Let K_1, K_2 be GVF extensions of F, with $R = K_1 \otimes_F K_2$ a domain, L the field of fractions of R. Then the canonical adelic amalgam of K_1, K_2 over F is globalizable iff it satisfies: $\int v(\sum_{i=1}^n a_i b_i) \leq 0$ for all $a_1, \ldots, a_n \in K_1, b_1, \ldots, b_n \in K_2$.*

Proposition 8.2. *Let L be a GVF extension of F, containing a sequence $(a_i : i \in \mathbb{N})$ of realizations of $qftp(a/F)$. Let $b \in L$. Assume a_i are adelically independent over F, and $qftp(a_i b/F)$ does not depend on i. Then a_i, b are adelically independent over F.*

Proof. (1) The archimedean case follows from Lemma 2.5.
(2) Above v_0, a non-archimedean, nontrivial valuation of F. By the local dividing property (as in Lemma **??**) and again using Lemma 2.5. Continue
(3) For almost all pairs (v_K, v_L) that are trivial on F (but not on K, L respectively), they cannot be in the (essential) image of μ_N. Otherwise, consider a random such pair (v_K, v_L) and the conditional measure ν on valuations of N above it. Being above v_L, it is a probability measure, whereas the v_{K_i} are disjoint, and $[v_{K_i} \wedge v_L]$ has the same weight, so this weight must be zero.

It follows that only $v_K \otimes Triv_L$ lie above v_K, and dually.

□

We did not discuss, so far, the remaining component: valuations of $L(a)$ trivial on L and on K. Thus we cannot expect the amalgam to be a GVF, even when L and K are GVF's. We can ask if it is *globalizable*, with respect to the valuations lying above nontrivial valuations of one of the factors, and the ring $[KL] = L[a]$.

When $K = F(a), L = F(b)$, $p = qftp(a/F), q = qftp(b/F)$,and M is the canonical adelic amalgam, we write $p \underset{adF}{\otimes} q$ for $qftp(ab/F)$, and $p^{(n)}$ for the n-fold amalgam of p with itself. Thus p describes the MVF extension of $F(a)/F$, which we will also write as $F(p)$.

We will now compare an arbitrary amalgam of K, L over F with $K \underset{adF}{\otimes} L$. Let $[KL]$ denote the ring generated by $K \cup L$, and (KL) the field of fractions of $[KL]$.

Recall a local fact, a 'maximum modulus theorem', first in the non-archimedean case. The canonical amalgam $K_v \otimes_{F_u} L_w$ minimizes the valuation of any nonzero $r = \sum a_i b_i \in R = [KL]$, among all possible valued field amalgams.

In the archimedean case, there is no canonical choice of an amalgam of a complex embedding of K, up to conjugation and a complex embedding of L, up to the same. However we will view the archimedean part of the measure on Ω_F,Ω_K, Ω_L is a conjugation-invariant measure on the complex embeddings. Once again there is a canonical extension (indeed a unique extension) of complex embeddings of K, L to one of the field (KL); indeed there is a unique extension.

check: This relegates the issue to one that we will deal with anyway, of considering various measures on VAL_{KL}; in the archimedean case, as in the non-algebraically-closed case, there will not necessarily be a unique measure, but with an additional indiscernibility requirement there will be one.

Thus to prove that a certain MVF extension minimizes $\int v(r)$, it suffices to compare it to extensions that concentrate, above a pair (v, u), only on the single amalgam $K_v \otimes_{F_u} L_w$. More formally, define: $\nu' \leq_{[KL]} \nu''$ if for all nonzero $r \in [KL]$, $\int v(r) d\nu' \leq \int v(r) d\nu''(v)$. Let $\pi : VAL_{(KL)} \to VAL_K \times VAL_F VAL_L$ be the natural projection, and let $s : VAL_K \times_{VAL_F} VAL_L \to VAL_{(KL)}$ be the free amalgam over F. Then:

Lemma 8.3. *For any ν there exists $\nu' \leq_{[KL]} \nu$ such that $\pi_* \nu = \pi_* \nu'$ and $\nu' = s_* \pi_* \nu'$.*

Proof. Let $\nu' = s_* \pi_* \nu$; then show that $\nu' \leq_{[KL]} \nu$ by decomposing the integral over VAL_F, and invoking the local maximum modulus principle □

The problem remains of determining the globalizable measures, among those supported on the locally canonical extensions.

We begin with the part over the trivial valuation of F. We assume the field M is the field of fractions of $K \otimes_F L$. Let Z_K, Z_L be the nontrivial valuations of K (respectively L) that are trivial on F; let $Z_{M|K}$ denote the set of valuations of M lying above Z_K, and $\pi_K : Z_{M|K} \to Z_K$ the natural restriction map; similarly for L; and let $Z_M = Z_{M|K} \cup Z_{M|L}$. Recall that the measure given by $K \underset{adF}{\otimes} L$ on this part is $\mu_Z = \mu_{K,0} \otimes (0)_L \bigsqcup (0)_K \otimes \mu_{L,0}$, where $\mu_{K,0} = \mu_K | Z_K$ is the measure of K restricted to valuations trivial on F, $\otimes(0_L)$ denotes the canonical base change to L at each place, dually for L, and $\bigsqcup$ is the the disjoint union of measures.

Lemma 8.4. *Let ν be any measure on Z_M, such that the restriction to $(\pi_K)_*(\nu | Z_{M,K}) = \mu_{K,0}$, and similarly for L. Then for all nonzero $r \in [KL]$,*

$$\int v(r) d\mu_Z(v) \leq \int v(r) d\nu(v)$$

Proof. By Lemma 8.3, we may assume $\nu = s_* \pi_* \nu$. If $\pi_* \nu$ is supported on $Z_K \otimes (0)_L \cup (0)_K \otimes Z_L$, then $\pi_* \nu$ is uniquely determined, and hence so is ν. We proceed to reduce the general case to this one. Write $\nu = \nu_0 + \nu_1$, where ν_0 concentrates on $Z_{M,K} \cap Z_{M,L}$ and ν_1 on the complement. It suffices to show that $\nu_0 = 0$. Let $\psi = \pi_* \nu_0$; it is a measure on $Z_K \times Z_L$, and $\nu_0 = s_* \psi$. For any $v \in Z_K$ and $w \in Z_L$, consider $\delta_{v \otimes 0} + \delta_{0 \otimes w}$, the sum of two Dirac measures on Z_M. Let $\nu_0' = \int (\delta_{v \otimes 0} + \delta_{0 \otimes w}) d\psi(v, w)$.

Claim 1. For any $(v, w) \in Z_K \times Z_L$, and $0 \neq r \in [KL]$,

$$(v \otimes 0_L)(r) + (0_K \otimes w)(r) \leq (v \otimes w)(r)$$

.

Proof. Write $r = \sum_{i=1}^{n} a_i b_i$, $a_i \in K, b_i \in L$; we may choose $a_1, \ldots, a_n$ to form a valuation basis for the F-space they generate. So $(v \otimes w)(r) = \min_i (v(a_i) + w(b_i))$. On the other hand, $(v \otimes 0_L)(r) = \min_i v(a_i)$ and $(0_K \otimes w)(r) = \min_i v(b_i)$. Clearly $\min_i (v(a_i) + w(b_i)) \geq \min_i v(a_i) + \min_i v(b_i)$, so the displayed formula holds. □

Claim 2. $\nu_0' \leq_{[KL]} \nu_0$.

Proof. This follows from Claim 1 by integrating over ψ, recalling the definition of ν_0' and also that $\nu_0 = s_* \psi$. □

On the other hand, $\pi_K \nu_0 = \pi_K \nu_0'$ and $\pi_L \nu_0 = \pi_L \nu_0'$, so ν_0' has the same projections as ν_0, and thus $\nu_0' + \nu_1$ has the same as $\nu = \nu_0 + \nu_1$. By the first paragraph, as ν' is supported away from $Z_{M,K} \cap Z_{M,L}$, we have $\mu_Z \leq_{[KL]} \nu'$; by Claim 2, $\nu' \leq_{[KL]} \nu$; so $\mu_Z \leq_{[KL]} \nu$. □

Next we consider the F-adelic part. Here we do not work with arbitrary extensions, but only ones that can be self-amalgamated to give an indiscernible sequence. We show that the independent amalgam is best among all such sequences, in the sense of minimizing volumes.

Assume M is the field of fractions of $K_1 \otimes_F K_2 \cdots$ where the K_i are (ACF)-independent isomorphic copies of K over F, by an isomorphism $\alpha_i : K \to K_i$. We let $\alpha_{i,j} = \alpha_j \circ \alpha_i^{-1}$. Let $\Omega(M, ad_F)$ be the space of valuations of M nontrivial on F; and similarly for K_i. Let $\pi_{i_1,\ldots,i_k}$ be the projection to $\Omega((K_{i_1} \cdots K_{i_k}))$. A measure ν on $\Omega(M, ad_F)$ is said to be *indiscernible* if $(\alpha_{i_1,j_1} \otimes \cdots \alpha_{i_k,j_k})_* (\pi_{i_1,\ldots,i_k})_* \nu = \pi_{j_1,\ldots,j_k} \nu$.

Given $r = (r_1, \ldots, r_n) \in K$, linearly independent over F, let $e(r, \nu)$ be ν-expectation of $v(\det((\alpha_i r_j)_{1 \leq i,j \leq n}))$. Write $\nu \prec \nu'$ if for any such r we have $e(r, \nu) \leq e(r, \nu')$.

Lemma 8.5. *Let ϑ be any ACF-independent indiscernible measure on $\Omega(M, ad_F)$. Let $\mu(M, ad_F)$ be the canonical adelic amalgam of $(K_i, (\pi_i)_* \nu)$ over F. Then $\mu \prec \vartheta$.*

Proof. We fix $r = (r_1, \ldots, r_n)$ as above. By relativizing, we may assume ϑ concentrates on valuations lying above a fixed nontrivial valuation v_0 of F. Let B be the Berkovich space of all real-valued extensions of v_0 to K. Given any sequence $(v_i)_{i \in \mathbb{N}} \in B^{\mathbb{N}}$, we have the canonical extension $\otimes_F^{i \in \mathbb{N}} v_i$. (Where in the i'th place we transpose v_i to a valuation on K_i via α_i.) In particular we have the symmetric power $v^\infty = \otimes_F^{i \in \mathbb{N}} v$. By Lemma 8.3 (extended to products of more than two factors), we may assume ϑ concentrates on the image of this map; thus so we may view ϑ as an indiscernible measure on $B^{\mathbb{N}}$. By de Finetti's principle, ϑ can be expressed as an integral of independent measures ν_y^∞ , with respect to a certain auxiliary measure θ on the space $Y = \mathcal{P}(B)$ of measures on B:

$$\vartheta = \int \mu_y^\infty d\theta(y)$$

Now we have a canonical map from B to $\mathbb{S}_n$: first map $v \in B$ to the valued-field type $p = tp(r/F)$; then, $v \mapsto J_1^*(p)$, see § 3.4. Thus $J_1^*(p)$ is the $\mathcal{O}$-module generated by $r, \alpha_1(r), \ldots, \alpha_n(r)$. Thus if $v_i \mapsto L_i$, then the $v_1 \otimes_F \cdots \otimes v_n$-valuation of $\det((\alpha_i r_j)_{1 \le i,j \le n})$ equals $[L_1, \ldots, L_n]$. By convexity (Lemma 3.8), the function $e(\nu) = e(r, \nu^{\otimes n})$ is a convex function of $\nu \in Y$. Hence (as $\mu = \int \mu_y d\theta(y)$),

$$e(\mu) \le \int e(\mu_y) d\theta(y) = \int e(r, \mu_y^\infty) d\theta(y) = e(r, \nu)$$

as required.

□

Example 8.6. To check. If we simply consider amalgams of measures on valuations of K and of $L<$ it is not the case that the canonical F-adelic self-amalgamation assigns least possible valuation to each element of the ring KL (i.e. $\le_{[KL]}$-minimality does not hold in this generality.) Consider an adelic qf type of an element x such that for $v' \neq v_0$, x is generic with respect to $v'(x) = 0$, while for v_0 it is a measure with 2-point support in the space of types over F_{v_0}, each with weight 1/2; such that one of the types gives x value 1, the other 4. Taking adelically independent realizations a, b, we obtain $v(a - b) = (3/4)(-1) + (1/4)4 > 0$. But if we take the measure 1/2 to $(-1, 4)$ and 1/2 to $(4, -1)$, obtain -1.

One can realize this over the non-algebraically closed field $k(t)$, via the roots of the quadratic equation: $tX^2 + X + t^4$. It would be good to extend the example over algebraically closed fields.

Definition 8.7. Say a qf type q over a GVF F is *strictly transcendental* if there exists a GVF extension L of F and realizations $c_1, c_2, \ldots,$ of q such that the fields $F(c_i)$ are linearly disjoint over F.

(Note that a type of elements of F^a is strictly transcendental, of dimension 0.)

Call F *eqf-definably closed* if for some existentially closed extension M of F, for any existential formula over F with a unique solution c in M, we have $c \in F$. Note that if M is a witness to this, then so is any extension M' of M.

Lemma 8.8. *Let F be eqf-definably closed Then every qf type p realized in M/F is strictly transcendental, i.e. in some elementary extension of M there exists an indiscernible sequence (a_i) of realizations of p such that the field extensions are linearly disjoint over F.*

Proof. Let M witness the eqf-definable closedness of F, and let M^* be a somewhat saturated elementary extension of M. Let V be the smallest variety over M^*, such that all realizations of p lie in V. By saturation, for some formula

$\pi \in p$, all realizations of p lie in V. So $V = V_a$ has a canonical parameter a; and $p(M^*)$ is Zariski dense in $V(M^*)$. If $a' \neq a$, then $p(M^*)$ is contained in $V \cap V_{a'}$ which is a proper subvariety of $V_{a'}$, moreover this is uniform in parameters. So a is characterized by the existence of $c_1, \ldots, c_N \in p(M^*) \subset V_a$, where the c_i are sufficiently generic in V_a that they are contained in no other $V_{a'}$. This last condition is a purely field-theoretic statement $\theta(c_1, \ldots, c_N, a)$, and a is defined uniquely by $(\exists x_1, \ldots, x_N)(\theta(x_1, \ldots, x_N, a) \& \bigwedge_i \pi(x_i))$. As F is eqf-definably closed, $a \in F$, so V is F-definable. Now inductively we may find $a_1, a_2, \ldots$ such that $a_k \models p$ and a_k lies in no proper $F(a_1, \ldots, a_{k-1})$-definable subvariety of V. So the fields $F(a_i)$ are linearly disjoint. Using Ramsey's theorem we may obtain a similar indiscernible sequence. □

Parenthetically, we include a generalization of Lemma 8.8:

Lemma 8.9. *Let M be an existentially closed structure, sufficiently saturated for existential types. Let M' be a stable reduct of M with EI,QE. Assume F is algebraically closed in M with respect to existential formulas. Let $a \in M$. Then there exists an indiscernible sequence (a_i) over F that forms a Morley sequence in M'.*

Proof. Find an indiscernible sequence a_i with $etp(a_i/F) = etp(a/F)$, and such that in M', $tp(a_0/F(a_1, a_2, \ldots))$ is minimal in the fundamental order (i.e. no strictly smaller possibility exists. This can be achieved by Zorn's lemma and an ultraproduct argument. Let e be the canonical base of $stp_{M'}(a_0/F(a_1, a_2, \ldots)$. If $e \subset F$ then (a_i) forms a Morley sequence over F in M' and we are done. Suppose for contradiction that $e \notin F$, and (using the fact that F is qf-algebraically closed) let (e_i, b_i) be an infinite indiscernible sequence, $b_i = (b_{ij})$, $etp(ea_1a_2 \cdots) = etp(e_i b_{i,1}, b_{i,2} \cdots)$, e_i distinct, $p_i = stp_{M'}(b_{i,1}/F(e_i))$. We now choose c_i with $qftp(e_i c_i/F) = qftp(e_i b_i/F)$, and such that in M', $c_i \models p_i | F(e_i, e_{<i}, b_{<i})$. This is possible since in M', $(b_{i,j})$ is an infinite Morley sequence over $F(e_i)$. We may further assume that (c_i) is indiscernible. Now computing in M' we see that $tp(c_0/F(c_1, c_2, \ldots))$ is lower in the fundamental order than $tp(a_0/F(a_1, a_2, \ldots))$, a contradiction. □

Proposition 8.10. *Assume $E1_\infty$. Let p, q be strictly transcendental qf types over the GVF F, corresponding to MVF extensions K, L of F. Also assume K, L, as fields, are regular extensions of F (linearly disjoint from F^{alg} over F.) Then $p \underset{adF}{\otimes} q$ is globalizable.*

Proof. Since p is strictly transcendental, p^∞ is globalizable by Lemma 8.5. Similarly q^∞ is globalizable. Consider the MVF structure $p^\infty \underset{adF}{\otimes} q^\infty$; the underlying field M is the compositum of linearly disjoint fields K_i realizing p and L_j realizing q. Suppose $[KL]$, with this structure, is not globalizable; then there exists $a_1, \ldots, a_n \in K$, linearly independent over F, and $b^1, \ldots, b^n \in L$, such that (with

$r_1 = (a_1, \dots, a_n), s^1 = (b^1, \dots, b^n))$ $\int v(r_1 \cdot s^1) dv = \gamma > 0$. Let $r_2, \dots, r_n$ be the conjugates of r in $K_2, \dots, K_n$ respectively, and similarly $s^j \in L_j$. Then $\int v(r_i \cdot s^j) > 0$. Note that $v(r_i \cdot s^j)$ does not depend on i, j since all $K_i \underset{ad_F}{\otimes} L_j$ are isomorphic. Let $\alpha(v) = v(r_i \cdot s^j)$; this is a function in $L^1(Val_M)$, with $\int \alpha > 0$. Using E1, find a symmetric power of r or of s with determinant of positive valuation integral, a contradiction. modify p - over each v of F, keep the same measures space over v, but replace each point of the Berkovich space $V^{an}(F_v)$ by the generic point of the linear type obtained by restriction to linear polynomials (it is the generic point of a certain lattice.)

□

Corollary 8.11. *If $F = F^{alg}$ is eqf-algebraically closed in M, and p, q are realized in M, then the adelic canonical amalgam $p \underset{ad_F}{\otimes} q$ is globalizable.*

Proof. By Lemma 8.8. □

Remark 8.12. It is clear conversely that if qf types over F admit linearly disjoint amalgamation, then $F = F^{alg}$ is eqf-algebraically closed; construct M reflecting this amalgamation, then no existential formula can characterize an element not in F^{alg}.

Remark 8.13. Let the GVF F be *linearly* existentially closed; i.e. any ∞-definable subspace of K^n (over the constants) with a point in an extension, is approximable in F. Then canonical adelic amalgamation is valid over F. For if a, b are not canonically amalgamalizable over F, $a \models p, b \models q$, we may assume (after going to symmetric powers or other elements of the rings generated by a, b) that $v(a \cdot b) > 0$. Then b lies in the linear ∞-definable $\mathcal{O}$-module: $\Omega := \{y : (d_p x) v(x \cdot y) > 0\}$. By linear existential closure there is $b' \in \Omega(F)$; but then p is not consistent.

Remark 8.14. We have defined a canonical amalgam at the adelic level, and shown existence of an extension to the qf level. It appears that among such extensions, one may again be canonical; this seems to be a cohomological issue of a rather different nature. Beyond this, to understand the complete type, one would need to describe the possible extensions to the qf-algebraic closure.

add the dividing lemma for blow-ups.

9. One variable

Let K be a GVF. By an *adelic* or *supported* measure on an extension L of K, we mean an admissible measure on the elements of Ω_L extending an element of Ω_K. Thus an admissible measure μ on L decomposes into an adelic measure μ_{ad} and a measure $\mu_{L/K}$ on $VAL_{L/K}$, the space of valuations of L trivial on K; so $\mu = \mu_{ad} + \mu_{L/K}$. Correspondingly, we can decompose the height $h(x) = h_{ad_K}(x) + h_{L/K}(x)$, where $h_{ad_K}(x) = \int v(x)^- d\mu_{ad}(x)$ and $h_{L/K}(x) = \int v(x)^- d\mu_{L/K}(x)$.

Lemma 9.1. *Let K be a GVF. Let μ_{ad} be a K-adelic measure on $K(t)$. Let r be a real with $r \geq \int v(f)/\deg(f) d\mu_{ad}(v)$, for any (irreducible) $f \in K[t] \setminus K$. Then there exists a unique globalizing measure on $\Omega_{K(t)}$ extending μ_{ad}, and such that $h_{L/K}(t) = r$.*

Proof. We have to specify the measure μ on the K-valuations of $K(t)$; they have the form $v_\infty(f)$ (where $v_\infty(t) = -1$), and v_p for any irreducible polynomial $p \in K[t]$ (where $v_p(p) = 1$.)

As $Val(K(t)/K)$ is discrete, we must specify a weight for v_∞ and each v_f. We will have $h(t) = \int [t]^- d\mu(t) = \int v(t)^- d\mu_{ad}(v) + m_\infty$, so $m_\infty = r$ is forced. Now the product formula for an irreducible $p \in K[t]$ reads:

$$0 = \int v(p) d\mu(v) = \int v(p) d\mu_{ad}(v) + m_p - \deg(p) m_\infty$$

Recalling $r = m_\infty$, we set $m_p = r\deg(p) - \int v(p) d\mu_{ad}(v)$; this is non-negative by assumption. We obtain a globalizing measure extending μ_{ad}, which we saw was unique. □

Remark 9.2. Let $K \leq K'$ be GVF's. Let μ_{ad} be an adelic measure on $K(t)$, and μ'_{ad} be an adelic measure on $K'(t)$, and assume $r \geq \int v(f)/\deg(f) d\mu_{ad}(v)$ for $f \in K[t] \setminus K$ and $r \geq \int v(f)/\deg(f) d\mu'_{ad}(v)$ for $f \in K'[t] \setminus K'$. Let μ, μ' be the associated globalizing measures. Then $(K(t), \mu) \leq (K'(t), \mu')$ iff μ is the pusforward to $K(t)$ of the restriction of μ'_{ad} to valuations nontrivial on K.

Definition 9.3. If $p = qftp(a/K)$, write $h_{rel}(p)$ for the relative height, i.e. $h_{rel}(p) = h_{L/K}(a)$ where $L = K(a)$ and $a \models p$.

Remark 9.4. The relative height can only go down, upon field extensions; and it stays constant for the canonical adelic amalgam. Hence by Lemma 9.1, the canonical adelic amalgam extends *uniquely* to a GVF type, if at all.

A second invariant is the *adelic volume* of p, that we define as follows. For $v \in \Omega_K$,

This invariant also stays fixed for the canonical extension. Inasfar as valuations extending valuations of K go it can only go down in any extension, but the relative valuations can (in part) become adelic, so only the *sum* $h_{rel}(p) + vol_{ad}(p)$ can be said to be non-increasing; or better, the lexicographically ordered pair $(h_{rel}(p), vol_{ad}(p))$.

Lemma 9.5. *Let $K \leq L, a \in L, p = qftp(a/K)$. If $vol_{ad}(p) + 2e(p) > 0$, then p is algebraic.*

10. 1-types of height zero

We look here at the simplest possible quantifier-free types: one-variable x, satisfying $h(x) = 0$; equivalently, $\int v(x)^- = \int v(x)^+ = 0$, i.e. $v(x) = 0$ for almost all v.

We will see that this set is qf-stably embedded. The qf-induced structure is field structure in the purely non-archimedean case, and an Abelian group structure in the case including archimedean valuations.

We recover theorems of Bilu and Zhang, (see [5]), in adelic versions due to Antoine Chambert-Loir, A. Thuillier, Favre-J. Rivera-Letelier,Baker-Rumely.

The solution set to $h(x) = 0$ is clearly a subgroup of G_m. Let us denote it G_m^0.

10.1. **Local case.** Let $(K, v) \models ACVF$ or $= \mathbb{C}, \log||$. We discussed in class the equation $v(x) = 0$; in $\mathbb{C}$ it defines the unit circle, in $K \models ACVF$ the invertible elements of the valuation ring. In each case, a unique generically stable, translation-invariant measure μ_{K_v} exists; it can be viewed as a measure on the space of quantifier-free types of RCV_{K_v} or $ACVF_{K_v}$, where K_v denotes an embedding of K into $\mathbb{C}$ in the archimdean case, and a valued field structure on K in the non-archimdean case. In the non-archimedean case it is the Dirac measure on a single complete type.

Let $p^0|K$ denote the unique qf GVF-type over K whose local part over (K_v) is given by μ_{K_v}; it is uniquely determined, using Lemma 9.1.

10.2. **The purely non-archimedean case.** By the product formula, $v(x) = 0$ for almost all v iff $v(x) \geq 0$ for almost all v. In the purely non-archimdean case, the latter defines a subring. As $h(x) = h(x^{-1})$, it is in fact a subfield k, the *field of constants.*

Lemma 10.3. *Other than $qftp(\omega/\mathbb{Q})$ for ω a root of unity, there exists a unique qf 1-type q over $\mathbb{Q}^a[1]$, extending $h(x) = 0$ (namely $p^0|\mathbb{Q}$).*

Proof. Existence is clear by compactness, since there are infinitely many roots of unity, all satsifying $h(x) = 0$. (Excercise: verify it directly.)

The condition $h(x) = 0$ implies that the adelic and relative heights are both zero. Hence, according to Lemma 9.1, it suffices to show that the adelic part of the type is determined. Thus for each prime p we have to determine the measure on valuations with $v(p) > 0$; and also for the archimedean primes. In both cases, the condition we will use is: $\int v(x - \omega) dv \leq 0$, for ω a root of unity.

Let p be a prime number. Let μ_p be the globalizing measure μ, restricted to valuations v with $v(p) > 0$. We have $\int v(x - \omega) d\mu(v) \leq 0$; as $v(x - \omega) \geq 0$ for all v, we have $v(x - \omega) = 0$ for μ-almost all v. We now show that this implies that μ_p concentrates on a single element of $\Omega_{\mathbb{Q}(x)}$.

Let v be a valuation on $\mathbb{Q}^a(x)$ extending the p-adic valuation on $\mathbb{Q}$, and with $v(x) = 0$ and $v(x - \omega) = 0$ for every root of unity ω. As the residue field is locally finite, and any element of the residue field lifts to a root of 1 (of order prime to p), for any element b of $\mathbb{Q}^a$ with $v(b) = 0$ there exists a root of unity ω with $v(b - \omega) > 0$. So $v(x - b) = 0$. Thus $v(x - b) = \min(0, v(b))$ for all $b \in \mathbb{Q}^a$; so v is uniquely determined by the condition and by $v|\mathbb{Q}^a$. So the p-part of q over $\mathbb{Q}^a$ is determined; it follows that the p-part of a non-root-of-unity qf type of height 0 over $\mathbb{Q}$ is also uniquely determined (since we can extend any such type to $\mathbb{Q}(x)^a$, hence to $\mathbb{Q}^a(x)$.)

Next consider the measure μ_∞ on archimedean valuations, equivalently on complex embeddings of $\mathbb{C}(x)$ satisfying $-\log|x| = 0$ or $|x| = 1$, equivalently on the unit circle (identify an embedding with the image of x under it.) Again we have $\int -\log|z-\omega| d\mu_\infty(z) \leq 0$ for all roots of unity ω. By continuity, $\int -\log|z - b| d\mu_\infty(z) \leq 0$ for all b with $|b| = 1$. Hence $\int\int -\log|z-z'| d\mu(z)\delta\mu(z') \leq 0$. This implies that μ is the uniform measure, and equality holds; it is a result of potential theory, proved in books on real analysis such as [7]. (Physically, the double integral represents the potential energy of a ring with a negative charge distribution; the unique minimum is obtained when the distribution is uniform.) □

Corollary 10.4 (Bilu equidistribution)**.** *uppose (a_n) is a sequence in $\mathbb{Q}^a$ such that $h(a_n) \to 0$. For each n let A_n be the Galois orbit of a_n over $\mathbb{Q}$. In addition, for each n let μ_n be the measure on $\mathbb{C}$ defined by*

$$\mu_n = \frac{1}{|A_n|} \sum_{x \in A_n} \delta_x$$

Then the sequence (μ_n) weakly converges to the uniform probability measure on the unit circle.

Proof. First we use ultra-products to find an element in an extension of $\mathbb{Q}^a$ which is the limit of the sequence a_n. Specifically, let K^* be an ultra-power of $\mathbb{Q}^a$. Let a be the ultra-limit of (a_n) in K^*. We note that this is well defined since the sequence (a_n) has bounded height. Furthermore, since $h(a_n) \to 0$, $h(a) = 0$.

This implies that $qf - tp(a/\mathbb{Q}^a)$ is the unique quantifier free type $p_0(x)$ over $\mathbb{Q}^a$ such that $(h(x) = 0) \in p_0(x)$ and p_0 is not realized in $\mathbb{Q}^a$ (Lemma 10.5.) We saw that in this type the measure on the archimedean valuations (which we think of as embeddings $\sigma : \mathbb{C}(x) \to \mathbb{C}$ such that $|\sigma(x)| = 1$) is simply the uniform probability measure μ on the unit circle.

This means that for any polynomial $G(x) \in \mathbb{Q}$ we have

$$\int_{arch} v(G(z)) d_{p_0(z)} = \int_{|z|=1} -log|G(z)| d_\mu$$

Now, let's define $q_n = qf - tp(a_n/\mathbb{Q})$. By the definition of a, we know that

$$\int_{arch} v(G(z)) d_{q_n(z)} \underset{n\to\infty}{\to} \int_{arch} v(G(z)) d_{p_0(z)} = \int_{|z|=1} -log|G(z)| d_\mu$$

As we saw in class, convergence on the rational polynomials is enough to show that the measures given by $q_n(z)$ weakly converge to μ.

Finally, by our classification of finite GVF extensions of $\mathbb{Q}$, if v_∞ is the archimedean valuation on $\mathbb{Q}$ then

$$\int_{arch} v(G(z))d_{q_n(z)} = \frac{1}{|Aut(\mathbb{Q}[a_n]/\mathbb{Q})|}\sum_{v_\infty|w} w(G(z)) = \frac{1}{|A_n|}\sum_{\sigma\in Aut(\mathbb{Q}[a_n]/\mathbb{Q})} -log|(G(\sigma(z)))|$$

Therefore, the measure on the archimedean valuations given by q_n is precisely $\mu_n = \frac{1}{|A_n|}\sum_{x\in A_n}\delta_x$.

This proves the theorem as we've already seen that the measures determined by q_n converge to μ.

□

We will soon see that the qf-type is unique over $\mathbb{Q}^a$ as well, and in fact so are non-algebraic extensions to bigger GVF's. (See also 10.10.) It is reasonable to conjecture here that $p_0|K$ in fact determines a unique *type*, when K is existentially closed; in particular, the GVF structure on the algebraic closure is determined.

Lemma 10.5. *Over any purely non-archimedean GVF K, there exists a unique non-algebraic qf 1-type over K, extending $h(x) = 0$; namely $p_0|K$. More generally, over any GVF K, there exists a unique non-algebraic qf 1-type over K, extending $h(x) = 0$, and agreeing with $p_0|K$ on the archimedean part.*

Proof. We may assume K is countable. Consider a GVF extension L of K and $a \in L$, with $h(a) = 0$. Assume $qftp(a/K)$ is not algebraic; then we may choose L to contain an infinite number of distinct realizations $a_1, a_2, \ldots$ of $qftp(a/K)$, and we may assume $L = K(a, a_1, \ldots)$. We have to show that $a \models p_0|K$. By Lemma 9.1, the qf type is entirely determined by the adelic part of the type; so it suffice to show that the relativization to each K_v of $qftp(a/K)$ is the same as that of $p_0|K$. For archimedean v this is true by assumption, so assume v is non-archimedean. We may ignore a measure zero set of valuations v. Let $\mu_L|v$ denote the measure μ_L conditioned on v; so μ_L lives on $\{v' \in \Omega_L : v'|K = v\}$, and (symbolically) $\mu = \int_K \mu_{L|v} d\mu_K(v)$. Note that $p_0|K \models v(x-b) = \min(0, v(b))$ for $b \in K$. If $v(b) \neq 0$, then $v(x) = 0$ implies $v(a-b) = \min(0, v(b))$ by the strong triangle inequality. So it suffices to prove that $v(a-b) = 0$ when $b \in K$, $v(b) = 0$. Fix such a $b \in K$. Let $X_i = \{v' : v'(a_i - b) > 0\}$

Claim . If $i \neq j$ then $\mu_L|v(X_i \cap X_j) = 0$.

Proof. Let $X_{i,\epsilon} = \{v' : v'(a_i - b) > \epsilon\}$. If $\mu_L|v(X_i \cap X_j) = 0$, then for some $\epsilon > 0$, $\mu(X_{i,\epsilon} \cap X_{j,\epsilon}) > 0$. For $v' \in (X_{i,\epsilon} \cap X_{j,\epsilon})$ we have $v'(a_i - a_j) > \epsilon$, so $v'(1-c) > \epsilon$, where $c = a_i^{-1}a_j$. Thus $\int \min(v(2)^+, v(c-1)^+) \geq \epsilon\mu(X_{i,\epsilon} \cap X_{j,\epsilon}) > 0$.

On the other hand, we have $h(c) = 0$, and $qftp(c/K)$ is non-algebraic. If $\mathbb{Q}[1] \leq K$, then by Lemma 10.3, $c \models p_{\mathbb{Q}}$. In particular, $\int \min(v(2)^+, v(c-1)^+) = 0$, so $v'(c-1) = 0$ for almost all non-archimedean v'. If $F[0] \leq K$ for some (prime) field F, then more trivially, $v(c-1) \geq 0$ for almost all v so by the product formula, $v'(c-1) = 0$. In any case, $v'(c-1) = 0$ so $v'(a_i - a_j) = 0$ for almost all non-archimedean v'. So we cannot have $v' \in X_i \cap X_j$. Thus $\mu_L | v(X_i \cap X_j) = 0$. ☐

Thus for any $c \in K$, if $W(c) = \{v' : v'(c) > 0, v(2) \geq 0\}$, then $W(c)$ has finite measure, the sets $X_i \cap W(c)$ have measures independent of i, and are pairwise disjoint up to measure 0; so $\mu_L(X_i \cap W(c)) = 0$. Thus for almost all $v \in \Omega_K$, for almost all v' extending v, we have $v'(a-b) = 0$, as required. ☐

Lemma 10.6. *Over any GVF K, there exists a unique non-algebraic qf 1-type over K, extending $h(x) = 0$.*

Proof. Let $a \in L, h(a) = 0$, with a/K not qf-algebraic. Let a_i be distinct (in some GVF extension) with $qftp(a/K) = qftp(a_i/K)$. We may choose a_i with $qftp(a_i/K(a_1, \ldots, a_{i-1}))$ non-algebraic.

Claim . $a_n \models p_0 | F(a_1, \ldots, a_{n-1})$, where F is the prime field.

Proof. In the purely non-archimedean case this is clear, since all a_i lie in the field of constants. Assume therefore that $F = \mathbb{Q}[1]$. Consider any nonzero vector $m = (m_0, \ldots, m_s) \in \mathbb{Z}^{1+s}$. Then $c(m) = a^{m_0} b_1^{m_1} \cdots b_s^{m_s}$ is not a root of unity. By Lemma 10.3, $c(m) \models p_{\mathbb{Q}[1]}$. By Lemma 2.6 the qf-type of $(a, b_1, \ldots, b_s)$ over $v(2) < 0$ is the uniform measure on T^{m+1} where T is the unit circle. By Lemma 10.5, $a \models q_L$. ☐

By Proposition 8.2, $F(a), K$ are adelically independent over F. By Lemma 9.1, this (along with $h(a) = 0$) determines $qftp(a/F)$.

☐

Remark 10.7. Lemma 10.6 could be stated as: p_0 has qf-U-rank one.

Note: The call on Proposition 8.2 could have been avoided, had we had a stronger version of Lemma 2.6.

Describe the algebraic closure within p_0 of a subfield outside it (more or less implicit in the proof.).

We still have to determine the qf-algebraic closure relation within p_0. It reduces to field-theoretic algebraic closure in the purely non-archimedean case, and to multiplicative algebraic closure over $\mathbb{Q}[1]$.

Corollary 10.8. *[zhang] Assume K is a GVF extending $\mathbb{Q}[1]$, $a_1, \ldots, a_n \in K$ satisfy $h(a_i) = 0$, and $a_1, \ldots, a_n$ are algebraically dependent. Then they are multiplicatively dependent.*

Proof. Let T be the unit circle, and let μ_a be the archimedean part of a globalizing measure of K; so μ_a is a measure on T^n, concentrating on a proper Zariski closed subset. In particular μ_a is not the uniform measure, so by Lemma 2.6 there exists a nonzero $m \in \mathbb{Z}^n$ such that $(h_m)_* \mu_a$ is not the uniform measure on the circle. However $h_m(a_1, \ldots, a_n)$ has height zero. By Lemma 10.3 $h_m(a_1, \ldots, a_n)$ must be a root of 1, say of order d. Hence $\prod a_i^{dm_i} = 1$. ☐

Corollary 10.8 can also be expressed in terms of the intersection with subvarieties of the group of n-tuples of elements of height zero. By compactness, it applies to the intersection with subvarieties of degree d of points of small enough height compared to d. This implies Zhang's theorem concerning such intersections, and is equivalent to it assuming $\mathbb{Q}_a[1]$ is existentially closed.

We have almost determined the qf structure on the solution set to $\mu(x) = 0$, but not completely; the potential difference between qf algebraic closure and algebraic dependence in the field-theoretic sense remains to be addressed. This is done in the next lemma.

Lemma 10.9. *Let K be a GVF prime field (i.e. $K = \mathbb{Q}[r]$ with $r > 0$, or K trivially valued.) Let $L' = K(a_1, \ldots, a_n)$, $L = K(b_1, \ldots, b_s)$ be GVF extensions, with $h(a_i) = h(b_j) = 0$, and $qftp(a_i/L(a_1, \ldots, a_{i-1}))$ non-algebraic. Then there is a unique GVF amalgam with L, L' linearly disjoint over K. In case $\mathbb{Q}[r] = K$, $r > 0$, there is a unique GVF amalgam with $a_1, \ldots, b_s$ multiplicatively independent.*

Proof. This reduces to the case $n = 1$, where we have to show that if $a \models p_0|K$ and $a \notin L^{alg}$ (or just: a is multiplicatively independent from $b_1, \ldots, b_s$, in case $K = \mathbb{Q}[1]$) then $a \models p_L$. In the purely non-archimedean case this is clear, since a and all b_i lie in the field of constants. Assume $K = \mathbb{Q}[1]$. Consider any nonzero vector $m = (m_0, \ldots, m_s) \in \mathbb{Z}^{1+s}$. Then $c(m) = a^{m_0} b_1^{m_1} \cdots b_s^{m_s}$ is not a root of unity. By Lemma 10.3, $c(m) \models p_{\mathbb{Q}[1]}$. By Lemma 2.6 the qf-type of $(a, b_1, \ldots, b_s)$ over $v(2) < 0$ is the uniform measure on T^{m+1} where T is the unit circle.

For p-adic valuations v, we must show that $v(\sum_{i=0}^d c_i a^i) = 0$ for any $c_0, \ldots, c_d \in L$ with $\min v(c_i) = 0$. Equivalently, $v(\sum_{i=0}^d c_{i,\nu} b^\nu a^i) = 0$ for any $c_{i,\nu} \in \mathbb{Q}^a$ with $\min v(c_{i,\nu}) = 0$. This is similar to the proof in 10.3. Let $c'_{i,\nu}$ be a root of unity which is v-adically close to $c_{i,\nu}$. We have $v'(\sum_{i=0}^d c'_{i,\nu} b^\nu a^i) \geq 0$ for all v', so by the product formula, $v'(\sum_{i=0}^d c'_{i,\nu} b^\nu a^i) = 0$ for all v' and in particular for $v = v'$. As $v(\sum_{i=0}^d (c_{i,\nu} - c'_{i,\nu}) b^\nu a^i) > 0$, the desired equation follows. So adelically, $a \models p_L$.

It follows that the product formula holds on $L(a)$ even if one restricts to valuations nontrivial on $\mathbb{Q}$. Hence, it also holds for valuations trivial on $\mathbb{Q}$. But for these, the constant set is a field, which includes $a, b_1, \ldots, b_s$ and thus all elements. □

Corollary 10.10 (relative Bilu equidistribution)**.** *$a_n \in \bar{\mathbb{Q}}$, $b_n \in \bar{\mathbb{Q}}^m$, $c_n = (a_n, b_n) \in \bar{\mathbb{Q}}^{m+1}$. Let A_n, B_n, C_n be the Galois orbits of a_n, b_n, c_n respectively. Assume $h(b_n)$ is bounded, $h(a_n) \to 0$, and $[\mathbb{Q}[c_n] : \mathbb{Q}[b_n]] \to \infty$. Assume B_n are equi-distributed along ν. Then C_n are equi-distributed along $\mu \times \nu$ (where μ is the uniform measure on the unit circle.)*

Proof. Take an ultraproduct (K^*, a, b) of $(\bar{\mathbb{Q}}, a_n, b_n)$; apply Lemma 10.6 to $tp(a/\bar{\mathbb{Q}}(b))$. □

Question 10.11. Let $K = K^{alg}$ be an existentially closed GVF. Let $L = K(a)^{alg}$, and endow L with the Galois invariant GVF structure extending that on K, and with $ht(a) = 0$.

(1) If K is G_m-full, is L also G_m-full?
(2) Is there any other GVF structure on L, extending the given one on $K(a)$?

Remark 10.12. A positive answer to (1) would imply one for (2) by Lemma 7.7. This would in turn imply an equidistribution result for curves, which is surely already in the literature: let C be a curve and $f : C \to \mathbb{P}^1$ a rational map. Let $d_n \in C(\mathbb{Q}^a)$ be such that $ht(f(d_n))$ approaches zero. Let D_n be the Galois orbit of d_n. Then the D_n are equi-distributed along a certain measure on $C(\mathbb{C})$, namely, the Haar integral along the unit circle of the counting measure on the fibers of f.

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