Model Theory of Berkovich Spaces.

The class will develop the model theory of valued fields, including a definable topology generalizing the Berkovich topology. We aim to reach recent results by Loeser and myself, including applications to classical Berkovich spaces of algebraic varieties such as the existence of homotopy retractions to a finite dimensional piecewise linear variety, finite number of homotopy types in a family. No prior knowledge of Berkovich spaces will be assumed.

Prerequisites: a first semester of model theory, algebraic geometry, and any class in algebra covering valued fields, should amply suffice. The material needed from model theory will be quickly reviewed, with an indication of the proof. From algebraic geometry, only a basic familiarity with the language of affine and projective varieties over fields will be used regularly. The language of instruction will be English.

We will begin with a quick introduction to o-minimality, introducing the basic language of model theory at the same time. The basic theory here is the theory RCF of real closed fields. The linear part of this theory, without multiplication, i.e. the theory of ordered $\mathbb{Q}$-vector spaces, will be important to us later; many expansions by analytic functions are o-minimal but will not show up in this class.

We will prove the curve selection theorem, and discuss equivalent definitions of *definable compactness*. In the o-minimal setting one can define compactness using curves, or using *definable types*. The latter notion is more flexible, and will be our central tool when studying valued fields.

1. Basic model theory and o-minimality.

Languages and theories. Examples: RCF (real closed fields). DOAG (divisible ordered Abelian groups.)

Structures. Constants. Definable sets.

Properties:

o-minimality.

Algebraic and definable closure. Let A be a substructure of a model M, and let $a \in M$. Then a is definable (algebraic) over A if there exists a formula $\phi(x)$ over A with one solution (finitely many solutions), such that $M \models \phi(a)$.

Algebraic boundedness. A theory extending the theory of fields is *algebraically bounded* if for pair $A \leq B$ of substructures of a model of t, if A is relatively algebraically closed in B as a field, then it is relatively algebraically closed as a substructure.

Skolemization (definable choice functions). T is Skolemized if for any formula $\phi(x, y)$ in $m+n$ variables, there exists a definable function $f(x)$ such that for any $a \in M \models T$, if $(\exists y)\phi(a, y)$ then $\phi(a, f(a))$.

Saturated models. o-minimal case = Hausdorff's eta-sets. Proof that saturation in one variable implies saturation. Homogeneity.

Proposition 1.1. *Let T be an o-minimal theory containing $Th(\mathbb{Q}, +, <, 0, 1)$, but possibly in a richer language. Then T is Skolemized.*

Proof: reduce to one variable.

The order topology. Induced topology on n-space and on algebraic varieties.

Corollary 1.2. *(T as in the proposition.) Let X be a definable set and suppose a lies in the closure of X. Then there exists a definable function $f : (0, a) \to X$ such that $x = \lim_{t\to 0} f(t)$.*

Lemma 1.3. *Let T be o-minimal, algebraically bounded. Then for any definable function $f : (0, a) \to X$ there exists $0 < b < a$ and an algebraic plane curve C such that $(x, f(x)) \in C$ for $0 < x < b$.*

Together, the corollary and the lemma are the very useful *curve selection principle.*

Theorems stated as background without proof: 1) Automatic piecewise continuity. Cell decomposition. 2) Definable triangulation. When R is an o-minimal field, definable subsets of R^n are definably homeomorphic to a rational polytope.

Corollary 1.4. *Let $E(n, p, q)$ be the collection of subsets of $\mathbb{R}^n$ that can be defined with p inequalities involving polynomials of degree at most q. Then $E(n, p, q)$ contains only finitely many homeomorphism types.*

1.5. **References:** Basic model theory: Lecture notes on Model Theory by Pillay, http://www.maths.leeds.ac.uk/ pillay

o-minimality: Coste, o-minimal geometry, in: perso.univ-rennes1.fr/michel.coste/polyens/OMIN