# NON-ARCHIMEDEAN TAME TOPOLOGY AND STABLY DOMINATED TYPES

EHUD HRUSHOVSKI AND FRANÇOIS LOESER

Abstract. Let $V$ be a quasi-projective algebraic variety over a non-archimedean valued field. We introduce topological methods into the model theory of valued fields, define an analogue $\widehat{V}$ of the Berkovich analytification $V^{an}$ of $V$, and deduce several new results on Berkovich spaces from it. In particular we show that $V^{an}$ retracts to a finite simplicial complex and is locally contractible, without any smoothness assumption on $V$. When $V$ varies in an algebraic family, we show that the homotopy type of $V^{an}$ takes only a finite number of values. The space $\widehat{V}$ is obtained by defining a topology on the pro-definable set of stably dominated types on $V$. The key result is the construction of a pro-definable strong retraction of $\widehat{V}$ to an o-minimal subspace, the skeleton, definably homeomorphic to a space definable over the value group with its piecewise linear structure.

## 1. Introduction

Model theory rarely deals directly with topology; the great exception is the theory of o-minimal structures, where the topology arises naturally from an ordered structure, especially in the setting of ordered fields. See [29] for a basic introduction. Our goal in this work is to create a framework of this kind for valued fields.

A fundamental tool, imported from stability theory, will be the notion of a definable type; it will play a number of roles, starting from the definition of a point of the fundamental spaces that will concern us. A definable type on a definable set $V$ is a uniform decision, for each definable subset $U$ (possibly defined with parameters from larger base sets), of whether $x \in U$; here $x$ should be viewed as a kind of ideal element of $V$. A good example is given by any semi-algebraic function $f$ from $\mathbb{R}$ to a real variety $V$. Such a function has a unique limiting behavior at $\infty$: for any semi-algebraic subset $U$ of $V$, either $f(t) \in U$ for all large enough $t$, or $f(t) \notin U$ for all large enough $t$. In this way $f$ determines a definable type.

One of the roles of definable types will be to be a substitute for the classical notion of a sequence, especially in situations where one is willing to refine to a subsequence. The classical notion of the limit of a sequence makes little sense in a saturated setting. In o-minimal situations it can often be replaced by the limit of a definable curve; notions such as definable compactness are defined using continuous definable maps from the field $R$ into a variety $V$. Now to discuss the limiting behavior of $f$ at $\infty$ (and thus to define notions such as compactness), we really require only the answer to this dichotomy - is $f(t) \in U$ for large $t$ ? - uniformly, for all $U$; i.e. knowledge of the definable type associated with $f$. For the spaces we consider, curves will not always be sufficiently plentiful to define compactness, but definable types will be, and our main notions will all be defined in these terms.

A different example of a definable type is the generic type of the valuation ring $\mathcal{O}$, or of a closed ball $B$ of $K$, or of $V(\mathcal{O})$ where $V$ is a smooth scheme over $\mathcal{O}$. Here again, for any definable subset $U$ of $\mathbb{A}^1$, we have $v \in U$ for all sufficiently generic $v \in V$, or else $v \notin U$ for all sufficiently generic $v \in V$; where "sufficiently generic" means "having residue outside $Z_U$" for a certain proper Zariski closed subset $Z_U$ of $V(\mathrm{k})$, depending only on $U$. Here k is the residue field. Note that the generic type of $\mathcal{O}$ is invariant under multiplication by $\mathcal{O}^*$ and addition by $\mathcal{O}$, and hence induces a definable type on any closed ball. Such definable types are *stably dominated*, being determined by a function into objects over the residue field, in this case the residue map into $V(\mathrm{k})$. They can also be characterized as *generically stable*. Their basic properties were developed in [14]; some results are now seen more easily using the general theory of NIP, [18].

Let $V$ be an algebraic variety over a field $K$. A valuation or ordering on $K$ induces a topology on $K$, hence on $K^n$, and finally on $V(K)$. We view this topology as an object of the definable world; for any model $M$, we obtain a topological space whose set of points is $V(M)$. In this sense, the topology is on $V$.

In the valuative case however, it has been recognized since the early days of the theory that this topology is inadequate for geometry. The valuation topology is totally disconnected, and does not afford a useful globalization of local questions. Various remedies have been proposed, by Krasner, Tate, Raynaud and Berkovich. Our approach can be viewed as a lifting of Berkovich's to the definable category. We will mention below a number of applications to classical Berkovich spaces, that indeed motivated the direction of our work.

The fundamental topological spaces we will consider will not live on algebraic varieties. Consider instead the set of semi-lattices in $K^n$. These are $\mathcal{O}^n$-submodules of $K^n$ isomorphic to $\mathcal{O}^k \oplus K^{n-k}$ for some $k$. Intuitively, a sequence $\Lambda_n$ of semi-lattices approaches a semi-lattice $\Lambda$ if for any $a$, if $a \in \Lambda_n$ for infinitely many $n$ then $a \in \Lambda$; and if $a \notin \mathcal{M}\Lambda_n$ for infinitely many $n$, then $a \notin \mathcal{M}\Lambda$. The actual definition is the same, but using definable types. A definable set of semi-lattices is *closed* if it is closed under limits of definable types. The set of closed balls in the affine line $\mathbb{A}^1$ can be viewed as a closed subset of the set of semi-lattices in $K^2$. In this case the limit of decreasing sequence of balls is the intersection of these balls; the limit of the generic type of the valuation ring $\mathcal{O}$

(or of small closed balls around generic points of $\mathcal{O}$) is the closed ball $\mathcal{O}$. We also consider subspaces of these spaces of semi-lattices. They tend to be definably connected and compact, as tested by definable types. For instance the set of all semi-lattices in $K^n$ cannot be split into two disjoint closed definable subsets.

To each algebraic variety $V$ over a valued field $K$ we will associate in a canonical way a projective limit $\widehat{V}$ of spaces of the type described above. A point of $\widehat{V}$ does not correspond to a point of $V$, but rather to a stably dominated definable type on $V$. For instance when $V = \mathbb{A}^1$, $\widehat{V}$ is the set of closed balls of $V$; the stably dominated type associated to a closed ball is just the generic type of that ball (which may be a point, or larger). In this case, and in general for curves, $\widehat{V}$ is definable (more precisely, a definable set of some imaginary sort), and no projective limit is needed.

While $V$ admits no definable functions of interest from the value group $\Gamma$, there do exist definable functions from $\Gamma$ to $\widehat{\mathbb{A}^1}$: for any point $a$ of $\mathbb{A}^1$, one can consider the closed ball $B(a;\alpha) = \{x : \mathrm{val}(a-x) \geq \alpha\}$ as a definable function of $\alpha \in \Gamma$. These functions will serve to connect the space $\widehat{\mathbb{A}^1}$. In [13] the imaginary sorts were classified, and moreover the definable functions from $\Gamma$ into them were classified; in the case of $\widehat{\mathbb{A}^1}$, essentially the only definable functions are the ones mentioned above. It is this kind of fact that is the basis of the geometry of imaginary sorts that we study here.

At present we remain in a purely algebraic setting. The applications to Berkovich spaces are thus only to Berkovich spaces of algebraic varieties. This limitation has the merit of showing that Berkovich spaces can be developed purely algebrically; historically, Krasner and Tate introduce analytic functions immediately even when interested in algebraic varieties, so that the name of the subject is rigid *analytic* geometry, but this is not necessary, a rigid algebraic geometry exists as well.

While we discussed o-minimality as an analogy, our real goal is a *reduction* of questions over valued fields to the o-minimal setting. The value group $\Gamma$ of a valued field is o-minimal of a simple kind, where all definable objects are piecewise $\mathbb{Q}$-linear. Our main result is that any variety $V$ over $K$ admits a definable deformation retraction to a subset $S$, a "skeleton", which is definably homeomorphic to a space defined over $\Gamma$. At this point, o-minimal results such as triangulation can be quoted. As a corollary we obtain an equivalence of categories between the category of algebraic varieties over $K$, with homotopy classes of definable continuous maps $\widehat{U} \to \widehat{V}$ as morphisms $U \to V$, and a category of definable spaces over the o-minimal $\Gamma$.

In case the value group is $\mathbb{R}$, our results specialize to similar tameness theorems for Berkovich spaces. In particular we obtain local contractibility for Berkovich spaces associated to algebraic varieties, a result which was proved by Berkovich under smoothness assumptions [4], [5]. We also show that for projective varieties, the corresponding Berkovich space is homeomorphic to a projective limit of finite dimensional simplicial complexes that are deformation retracts of itself. We further obtain finiteness statements that were not known classically; we refer to § 13 for these applications.

We now present the contents of the sections and a sketch of the proof of the main theorem. Section 2 includes some background material on definable sets, definable types, orthogonality and domination, especially in the valued field context.

In § 3 we introduce the space $\widehat{V}$ of stably dominated types on a definable set $V$. We show that $\widehat{V}$ is pro-definable; this is in fact true in any NIP theory, and not just in ACVF. We further show that $\widehat{V}$ is strict pro-definable, i.e. the image of $\widehat{V}$ under any projection to a definable set is definable. This uses metastability, and also a classical definability property of irreducibility in algebraically closed fields. In the case of curves, we note later that $\widehat{V}$ is in fact definable; for many purposes strict pro-definable sets behave in the same way. Still in § 3, we define a topology on $\widehat{V}$, and study the connection between this topology and $V$. Roughly speaking, the topology on $\widehat{V}$ is generated by $\widehat{U}$, where $U$ is a definable set cut out by *strict* valuation inequalities. The space $V$ is a dense subset of $\widehat{V}$, so a continuous map $\widehat{V} \to \widehat{U}$ is determined by the restriction to $V$. Conversely, given a definable map $V \to \widehat{U}$, we explain the conditions for extending it to $\widehat{V}$. This uses the interpretation of $\widehat{V}$ as a set of definable types. We determine the Grothendieck topology on $V$ itself induced from the topology on $\widehat{V}$; the closure or continuity of definable subsets or of functions on $V$ can be described in terms of this Grothendieck topology without reference to $\widehat{V}$, but we will see that this viewpoint is more limited.

In the last subsection of §2 (to step back a little) we present the main result of [14] with a new insight regarding one point, that will be used in several critical points later in the paper. We know that every nonempty definable set over an algebraically closed substructure of a model of ACVF extends to a definable type. A definable type $p$ can be decomposed into a definable type $q$ on $\Gamma^n$, and a map from this type to stably dominated definable types. In previous definitions of metastability, this decomposition involved an uncontrolled base change that prevented any canonicity. We note here that the $q$-germ of $f$ is defined with no additional parameters, and that it is this germ that really determines $p$. Thus a general definable type is a function from a definable type on $\Gamma^n$ to stably dominated definable types.

In § 4 we define the central notion of definable compactness; we give a general definition that may be useful whenever one has definable topologies with enough definable types. The o-minimal formulation regarding limits of curves is replaced by limits of definable types. We relate definable compactness to being closed and bounded. We show the expected properties hold, in particular the image of a definably compact set under a continuous definable map is definably compact.

The definition of $\widehat{V}$ is a little abstract. In §5 we give a concrete representation of $\widehat{\mathbb{A}^n}$ in terms of spaces of semi-lattices. This was already alluded to in the first paragraphs of the introduction.

A major issue in this paper is the frontier between the definable and the topological categories. In o-minimality automatic continuity theorems play a role. Here we did not find such results very useful. At all events in §2.7 we characterize topologically those subspaces of $\widehat{V}$ that can be definably parameterized by $\Gamma^n$. They turn out to be o-minimal in the topological sense too. We use here in an essential way the construction of $\widehat{V}$ in terms of spaces of semi-lattices, and the characterization in [13] of definable maps from $\Gamma$ into such spaces.

§7 is concerned with the case of curves. We show that $\widehat{C}$ is definable (and not just pro-definable) when $C$ is a curve. The case of $\mathbb{P}^1$ is elementary, and in equal characteristic zero it is possible to reduce everything to this case. But in general we use model-theoretic methods. We find a definable deformation retraction from $\widehat{C}$ into a $\Gamma$-internal subset, the kind of subset whose topology was characterized in §2.7. We consider relative curves too, i.e. varieties $V$ with maps $f : V \to U$, whose fibers are of dimension one. In this case we find a deformation retraction of all fibers that is globally continuous and takes $\widehat{C}$ into $\Gamma$-internal subset for almost all fibers $C$, i.e. all outside a proper subvariety of $U$. On curves lying over this variety, the motions on nearby curves do not converge to any continuous motion.

§8 contains some algebraic criteria for the verification of continuity. For the Zariski topology on algebraic varieties, the valuative criterion is useful: a constructible set is closed if it is invariant under specializations. Here we are led to doubly valued fields. These can be obtained from valued fields *either* by adding a valued field structure to the residue field, *or* by enriching the value group with a new convex subgroup. The functor $\widehat{X}$ is meaningful for definable sets of this theory as well, and interacts well with the various specializations. These criteria are used in §9 to verify the continuity of the relative homotopies of §7.

§9 includes some additional easy results on homotopies. In particular, for a smooth variety $V$, there exists an "inflation" homotopy, taking a simple point to the generic type of a small neighborhood of that point. This homotopy has an image that is properly a subset of $\widehat{V}$, and cannot be understood directly in terms of definable subsets of $V$. The image of this homotopy retraction has the merit of being contained in $\widehat{U}$ for any Zariski open subset $U$ of $V$.

§10 contains the statement and proof of the main theorem. For any algebraic variety $V$, we find a definable homotopy retraction from $\widehat{V}$ to an o-minimal subspace of the type described in §2.7. After some modifications we fiber $V$ over a variety $U$ of lower dimension. The fibers are curves. On each fiber, a homotopy retraction can be described with o-minimal image, as in §7; above a certain Zariski open subset $U_1$ of $U$, these homotopies can be viewed as the fibers of a single homotopy $h_1$. The homotopy $h_1$ does not extend to the complement of $U_1$;

but in the smooth case, one can first apply an inflation homotopy whose image lies in $\widehat{V_1}$, where $V_1$ is the pullback of $U_1$. If $V$ has singular points, a more delicate preparation is necessary. Let $S_1$ be the image of the homotopy $h_1$. Now a relative version of the results of §2.7 applies (Proposition 6.3.9); after pulling back the situation to a finite covering of $U$, we show that $S_1$ embeds topologically into $U' \times \Gamma_\infty^N$. Now any homotopy retraction of $\widehat{U}$, fixing $U'$ and certain functions into $\Gamma^m$, can be extended to a homotopy retraction of $S_1$ (Lemma 6.3.13). Using induction on dimension, we apply this to a homotopy retraction taking $U$ to an o-minimal set; we obtain a retraction of $V$ to a subset $S_2$ of $S_1$ lying over an o-minimal set, hence itself o-minimal. At this point o-minimal topology as in [7] applies to $S_2$, and hence to the homotopy type of $\widehat{V}$.

In §10.7 we give a uniform version of Theorem 10.1.1 with respect to parameters. Sections 11 and 12 are devoted to some further results related to Theorem 10.1.1.

Section 13 contains various applications to classical Berkovich spaces. Let $V$ be a quasi-projective variety over a field $F$ endowed with a non-archimedean norm and let $V^{\mathrm{an}}$ be the corresponding Berkovich space. We deduce from our main theorem several new results on the topology of $V^{\mathrm{an}}$ which were not known previously in such a level of generality. In particular we show that $V^{\mathrm{an}}$ admits a strong homotopy retraction to a subspace homeomorphic to a finite simplicial complex and that $V^{\mathrm{an}}$ is locally contractible. We prove a finiteness statement for the homotopy type of fibers in families. We also show that if $V$ is projective, $V^{\mathrm{an}}$ is homeomorphic to a projective limit of finite dimensional simplicial complexes that are deformation retracts of $V^{\mathrm{an}}$.

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## 2. Preliminaries

We will rapidly recall the basic model theoretic notions of which we make use, but we recommend to the non-model theoretic reader an introduction such as [24].

2.1. **Definable sets.** Let us fix a first order language $\mathcal{L}$ and a complete theory $T$ over $\mathcal{L}$. The language $\mathcal{L}$ may be multisorted. If $\mathcal{S}$ is a sort, and $A$ is an $\mathcal{L}$-structure, we denote by $\mathcal{S}(A)$, the part of $A$ belonging to the sort $\mathcal{S}$. For $C$ a set of parameters in a model of $T$ and $x$ any set of variables, we denote by $S_x(C)$ the set of types over $C$ in the variables $x$. Thus, $S_x(C)$ is the Stone space of the Boolean algebra of formulas with free variables contained in $x$ up to equivalence over $T$.

We shall work in a large saturated model $\mathbb{U}$ (a universal domain for $T$). More precisely, we shall fix some an uncountable cardinal $\kappa$ larger than any cardinality of interest, and consider a model $\mathbb{U}$ of cardinality $\kappa$ such that for every $A \subset U$ of cardinality $< \kappa$, every $p$ in $S_x(A)$ is realized in $\mathbb{U}$, for $x$ any finite set of variables. (Such a $\mathbb{U}$ is unique up to isomorphism. Set theoretic issues involved in the choice of $\kappa$ turn out to be unimportant and resolvable in numerous ways; cf. [6] or [15], Appendix A.)

All sets of parameters $A$ we shall consider will be small subsets of $\mathbb{U}$, that is of cardinality $< \kappa$, and all models $M$ of $T$ we shall consider will be elementary substructures of $\mathbb{U}$ with cardinality $< \kappa$. By a substructure of $\mathbb{U}$ we shall generally mean a small definably closed subset of $\mathbb{U}$.

If $\varphi$ is a formula in $\mathcal{L}_C$, involving some sorts $\mathcal{S}_i$ with arity $n_i$, for every small model $M$ containing $C$, one can consider the set $Z_\varphi(M)$ of uplets $a$ in the cartesian product of the $\mathcal{S}_i(M)^{n_i}$ such that $M \models \varphi(a)$. One can view $Z_\varphi$ as a functor from the category of models and elementary embeddings, to the category of sets. Such functors will be called definable sets over $C$. Note that a definable $X$ is completely determined by the (large) set $X(\mathbb{U})$, so we may identify definable sets with subsets of cartesian products of sets $\mathcal{S}_i(\mathbb{U})^{n_i}$. Definable sets over $C$ form a category $\mathrm{Def}_C$ in a natural way. Under the previous identification a definable morphism between definable sets $X_1(\mathbb{U})$ and $X_2(\mathbb{U})$ is a function $X_1(\mathbb{U}) \to X_2(\mathbb{U})$ whose graph is definable.

By a definable set, we mean definable over some $C$. When $C$ is empty one says $\emptyset$-definable or 0-definable. A subset of a given definable set $X$ which is an intersection of $< \kappa$ definable subsets of $X$ is said to be $\infty$-definable.

Sets of $\mathbb{U}$-points of definable sets satisfy the following form of compactness: if $X$ is a definable set such that $X(\mathbb{U}) = \cup_{i \in I} X_i(\mathbb{U})$, with $(X_i)_{i \in I}$ a small family of definable sets, then $X = \cup_{i \in A} X_i$ with $A$ a finite subset of $I$.

Recall that if $C$ is a subset of a model $M$ of $T$, by the algebraic closure of $C$, denoted by $\mathrm{acl}(C)$, one denotes the subset of those elements $c$ of $M$, such that, for some formula $\varphi$ over $C$ with one free variable, $Z_\varphi(M)$ is finite and contains $c$. The definable closure $\mathrm{dcl}(C)$ of $C$ is the the subset of those elements $c$ of $M$, such that, for some formula $\varphi$ over $C$ with one free variable, $Z_\varphi(M) = \{c\}$.

If $X$ is a definable set over $C$ and $C \subseteq B$, we write $X(B)$ for $X(\mathbb{U}) \cap \mathrm{dcl}(B)$.

2.2. **Pro-definable and ind-definable sets.** We define the category $\mathrm{ProDef}_C$ of pro-definable sets over $C$ as the category of pro-objects in the category $\mathrm{Def}_C$ indexed by a small directed partially ordered set. Thus, if $X = (X_i)_{i \in I}$ and $Y = (Y_j)_{i \in J}$ are two objects in $\mathrm{ProDef}_C$,

$$\mathrm{Hom}_{\mathrm{ProDef}_C}(X, Y) = \varprojlim_j \varinjlim_i \mathrm{Hom}_{\mathrm{Def}_C}(X_i, Y_j).$$

Elements of $\mathrm{Hom}_{\mathrm{ProDef}_C}(X, Y)$ will be called $C$-pro-definable morphisms (or $C$-definable for short) between $X$ and $Y$.

By a result of Kamensky [20], the functor of "taking $\mathbb{U}$-points" induces an equivalence of categories between the category $\mathrm{ProDef}_C$ and the sub-category of the category of sets whose objects and morphisms are inverse limits of $\mathbb{U}$-points of definable sets indexed by a small directed partially ordered set (here the word "co-filtering" is also used, synonymously with "directed"). By pro-definable, we mean pro-definable over some $C$. Pro-definable is thus the same as $*$-definable in the sense of Shelah, that is, a small projective limit of definable subsets. One defines similarly the category $\mathrm{IndDef}_C$ of ind-definable sets over $C$ for which a similar equivalence holds.

Let $X$ be a pro-definable set. We shall say it is *strict* pro-definable if it may be represented as a pro-object $(X_i)_{i \in I}$, with surjective transition morphisms $X_j(\mathbb{U}) \to X_i(\mathbb{U})$. Equivalently, it is a $*$-definable set, such that the projection to any finite number of coordinates is definable.

Dual definitions apply to ind-definable sets; thus "strict" means that the maps are injective: in a $\mathbb{U}$, a small union of definable sets is a strict ind-definable set.

By a morphism from an ind-definable set $X = \varinjlim_i X_i$ to a pro-definable one $Y = \varprojlim_j Y_j$, we mean a compatible family of morphisms $X_i \to Y_j$. A morphism

$Y \to X$ is defined dually; it is always represented by a morphism $Y_j \to X_i$, for some $j, i$.

**Remark 2.2.1.** A strict ind-definable set $X$ with a definable point always admits a bijective morphism to a strict pro-definable set. On the other hand, if $Y$ is pro-definable and $X$ is ind-definable, a morphism $Y \to X$ always has definable image.

*Proof.* For the first statement, fix a definable point $p$. If $f : X_i \to X_j$ is injective, define $g : X_j \to X_i$ by setting it equal to $f^{-1}$ on $\mathrm{Im}(f)$, constant equal to $p$ outside $\mathrm{Im}(f)$. The second statement is clear. $\square$

Let $Y = \varprojlim Y_i$ be pro-definable, and let $X \subseteq Y$. The inclusion $X \to Y$ yields maps $X \to Y_i$, with image $X_i$; for any morphism $i \to i'$, we have maps $X_i \to X_{i'}$, forming a commutative diagram. We shall say $X$ is iso-$\infty$-definable if for some $i_0$, for all $i$ and $i'$ mapping to $i_0$ (i.e. above $i_0$ in the partial ordering), all maps $X_i \to X_{i'}$ are bijections. If, in addition, each $X_i$ is definable one calls $X$ *iso-definable*. Thus a set is iso-definable if and only if it strict pro-definable and iso-$\infty$-definable.

**Remark 2.2.2.** If above, all maps $X_i \to X_{i'}$ are surjections for $i \geq i' \geq i_0$, we call $X$ *definably parameterized. We do not know if definably parameterized subsets of the spaces $\widehat{V}$ that we will consider are iso-definable.* A number of proofs would be considerably simplified if this were true; see Question 7.2.1 for a special case. We mention two conditions under which definably parameterized sets are iso-definable.

**Lemma 2.2.3.** *Let $W$ be a definable set, $Y$ a pro-definable set, and let $f : W \to Y$ be a pro-definable map. Then the image of $W$ in $Y$ is strict pro-definable. If $f$ is injective, or more generally if the equivalence relation $f(y) = f(y')$ is definable, then $f(W)$ is iso-definable.*

*Proof.* Say $Y = \varprojlim Y_i$. Let $f_i$ be the composition $W \to Y \to Y_i$. Then $f_i$ is a function whose graph is $\infty$-definable. By compactness there exists a definable function $F : W \to Y_i$ whose graph contains $f_i$; but then clearly $F = f_i$ and so the image $X_i = f_i(W)$ and $f_i$ itself are definable. Now $f(Y)$ is the projective limit of the system $(X_i)$, with maps induced from $(Y_i)$; the maps $X_i \to X_j$ are surjective for $i > j$, since $W \to X_j$ is surjective. Now assume there exists a definable equivalence relation $E$ on $Y$ such that $f(y) = f(y')$ if and only if $(y, y') \in E$. If $(w, w') \in W^2 \smallsetminus E$ then $w$ and $w'$ have distinct images in some $X_i$. By compactness, for some $i_0$, if $(w, w') \in W^2 \smallsetminus E$ then $f_{i_0}(w) \neq f_{i_0}(w')$. So for any $i$ mapping to $i_0$ the map $X_i \to X_{i_0}$ is injective. $\square$

**Corollary 2.2.4.** *$X \subseteq Y$ is iso-definable if and only if $X$ is in (pro-definable) bijection with a definable set.* $\square$

**Lemma 2.2.5.** *Let $Y$ be pro-definable, $X$ an iso-definable subset. Let $G$ be a finite group acting on $Y$, and leaving $X$ invariant. Let $f : Y \to Y'$ be a map of pro-definable sets, whose fibers are exactly the orbits of $G$. Then $f(X)$ is iso-definable.*

*Proof.* Let $U$ be a definable set, and $h : U \to X$ a pro-definable bijection. Define $g(u) = u'$ if $gh(u) = h(u')$. This induces a definable action of $G$ on $U$. We have $f(h(u)) = f(h(u'))$ iff there exists $g$ such that $gu = u'$. Thus the equivalence relation $f(h(u)) = f(h(u'))$ is definable; by Lemma 2.2.3, the image is iso-definable. □

We shall call a subset $X$ of a pro-definable $Y$ *relatively definable* in $Y$ if $X$ is cut out from $Y$ by a single formula. More precisely, if $Y = \varprojlim Y_i$ is pro-definable, $X$ will be prodefinable in $Y$ if there exists some index $i_0$ and and a definable subset $Z$ of $Y_{i_0}$, such that, denoting by $X_i$ the inverse image of $Z$ in $Y_i$ for $i \geq i_0$, $X = \varprojlim_{i \geq i_0} X_i$.

Iso-definability and relative definability are related somewhat as finite dimension is related to finite codimension; so they rarely hold together. In this terminology, a semi-algebraic subset of $\widehat{V}$, that is, a subset of the form $\widehat{X}$, where $X$ is a definable subset of $V$, will be relatively definable, but most often not iso-definable.

**Lemma 2.2.6.** *(1) Let $X$ be pro-definable, and assume that the equality relation $\Delta_X$ is a relatively definable subset of $X^2$. Then $X$ is iso-$\infty$-definable. (2) A pro-definable subset of an iso-$\infty$-definable set is iso-$\infty$-definable.*

*Proof.* (1) $X$ is the projective limit of an inverse system $\{X_i\}$, with maps $f_i : X \to X_{\alpha(i)}$. We have $(x, y) \in \Delta_X$ if and only if $f_i(x) = f_i(y)$ for each $i$. It follows that for some $i$, $(x, y) \in \Delta_X$ if and only if $f_i(x) = f_i(y)$. For otherwise, for any finite set $I_0$ of indices, we may find $(x, y) \notin \Delta_X$ with $f_i(x) = f_i(y)$ for every $i \in I_0$. But then by compactness, and using the relative definability of (the complement of) $\Delta_X$, there exist $(x, y) \in X^2 \smallsetminus \Delta_X$ with $f_i(x) = f_i(y)$ for all $i$, a contradiction. Thus the map $f_i$ is injective. (2) follows from (1), or can be proved directly. □

**Lemma 2.2.7.** *Let $f : X \to Y$ be a morphism between strict pro-definable sets. Then* $\mathrm{Im} f$ *is strict pro-definable, as is the graph of $f$.*

*Proof.* We can represent $X$ and $Y$ as respectively projective limit of definable sets $X_i$ and $Y_j$, and $f$ by $f_j : X_{c(j)} \to Y_j$, for some function $c$ between the index sets. The projection of $X$ to $Y_j$ is the same as the image of $f_j$, using the surjectivity of the maps between the sets $X_j$ and $f_j(Y_j)$ is definable. The graph of $f$ is the image of $\mathrm{Id} \times f : X \to (X \times Y)$. □

**Remark 2.2.8** (on terminology)**.** We often have a set $D(A)$ depending functorially on a structure $A$. We say that $D$ is pro-definable if there exists a pro-definable set $D'$ such that $D'(A)$ and $D(A)$ are in canonical bijection; in other words $D$ and $D'$ are isomorphic functors.

In practice we have in mind a choice of $D'$ arising naturally from the definition of $D$; usually various interpretations are possible, but all are isomorphic as pro-definable sets.

Once $D'$ is specified, so is, for any pro-definable $W$ and any $A$, the set of $A$-definable maps $W \to D'$. If worried about the identity of $D'$, it suffices to specify what we mean by an $A$-definable map $W \to D$. Then Yoneda ensures the uniqueness of a pro-definable set $D'$ compatible with this notion.

The same applies for ind. For instance, let $\mathrm{Fn}(V, V')(A)$ be the set of $A$-definable functions between two given 0-definable sets $V$ and $V'$. Then $\mathrm{Fn}(V, V')$ is an ind-definable set. The representing ind-definable set is clearly determined by the description. To avoid all doubts, we specify that $\mathrm{Fn}(U, \mathrm{Fn}(V, V')) = \mathrm{Fn}(U \times V, V')$.

2.3. **Definable types.** Let $\mathcal{L}_{x,y}$ be the set of formulas in variables $x, y$, up to equivalence in the theory $T$. A type $p(x)$ in variables $x = (x_1, \ldots, x_n)$ can be viewed as a Boolean homomorphism from $\mathcal{L}_x$ to the 2-element Boolean algebra.

A definable type $p(x)$ is defined to be a Boolean retraction $d_p x : \mathcal{L}_{x,y_1,\ldots,} \to \mathcal{L}_{y_1,\ldots,}$. Here the $y_i$ are variables running through all finite products of sorts. Equivalently, for a 0-definable set $V$, let $L_V$ denote the Boolean algebra of 0-definable subset of $V$. Then a type on $V$ is a compatible family of elements of $\mathrm{Hom}(L_V, 2)$; a definable type on $V$ is a compatible family of elements of $\mathrm{Hom}_W(L_{V\times W}, L_W)$, where $\mathrm{Hom}_W$ denotes the set of Boolean homomorphisms $h$ such that $h(V \times X) = X$ for $X \subseteq W$.

Given such a homomorphism, and given any model $M$ of $T$, one obtains a type over $M$, namely $p|M := \{\varphi(x, b_1, \ldots, b_n) : M \models ((d_p x)\varphi)(b_1, \ldots, b_n)\}$.

The type $p|\mathbb{U}$ is $\mathrm{Aut}(\mathbb{U})$-invariant, and determines $p$; we will often identify them. The image of $\phi(x, y)$ under $p$ is called the $\phi$-definition of $p$.

Similarly, for any $C \subset \mathbb{U}$, replacing $\mathcal{L}$ by $\mathcal{L}_C$ one gets the notion of $C$-definable type. If $p$ is $C$-definable, then the type $p|\mathbb{U}$ is $\mathrm{Aut}(\mathbb{U}/C)$-invariant. The map $M \mapsto p|M$, or even one of its values, determines the definable type $p$.

If $p$ is a definable type and $X$ is a definable set, one says $p$ is on $X$ if all realizations of $p|\mathbb{U}$ lie in $X$. One denotes by $S_{def}(X)$ the set of definable types on $X$. Let $f : X \to Z$ be a definable map between definable sets. For $p$ in $S_{def}(X)$ one denotes by $f_*(p)$ the definable type defined by $d_{f_*(p)}(\varphi(z, y)) = d_p(\varphi(f(x), y))$. This gives rise to a mapping $f_* : S_{def}(X) \to S_{def}(Z)$.

Let $U$ be a pro-definable set. By a definable function $U \to S_{def}(V)$, we mean a compatible family of Boolean homomorphisms $L_{V\times W\times U} \to L_{W\times U}$, with $h(V \times X) = X$ for $X \subseteq W \times U$. Any element $u \in U$ gives a Boolean retraction $L_{W\times U} \to L_W(u)$ by $Z \mapsto Z(u) = \{z : (z, u) \in Z\}$. So a definable function $U \to S_{def}(V)$ gives indeed a $U$-parametrized family of definable types on $V$.

Let us say $p$ is *definably generated* over $A$ if it is generated by a partial type of the form $\cup_{(\phi,\theta)\in S} P(\phi,\theta)$, where $S$ is a set of pairs of formulas $(\phi(x,y),\theta(y))$ over $A$, and $P(\phi,\theta)=\{\phi(x,b):\theta(b)\}$.

**Lemma 2.3.1.** *Let $p$ be a type over $\mathbb{U}$. If $p$ is definably generated over $A$, then $p$ is $A$-definable.*

*Proof.* This follows from Beth's theorem: if one adds a predicate for the $p$-definitions of all formulas $\phi(x,y)$, with the obvious axioms, there is a unique interpretation of these predicates in $\mathbb{U}$, hence they must be definable.

Alternatively, let $\phi(x,y)$ be any formula. From the fact that $p$ is definably generated it follows easily that $\{b:\phi(x,b)\in p\}$ is an ind-definable set over $A$. Indeed, $\phi(x,b)\in p$ if and only if for some $(\phi_1,\theta_1),\dots,(\phi_m,\theta_m)\in S$, $(\exists c_1,\cdots,c_m)(\theta_i(c_i)\wedge(\forall x)(\bigwedge_i \phi_i(x,c) \implies \phi(x,b))$. Applying this to $\neg\phi$, we see that the complement of $\{b:\phi(x,b)\in p\}$ is also ind-definable. Hence $\{b:\phi(x,b)\in p\}$ is $A$-definable. $\square$

**Lemma 2.3.2.** *Assume the theory $T$ has elimination of imaginaries. Let $f: X\to Y$ be a $C$-definable mapping between $C$-definable sets. Assume $f$ has finite fibers, say of cardinality bounded by some integer $m$. Let $p$ be a $C$-definable type on $Y$. Then, any global type $q$ on $X$ such that $f_*(q)=p|\mathbb{U}$ is* $\mathrm{acl}(C)$*-definable.*

*Proof.* The partial type $p|\mathbb{U}(f(x))$ admits at most $m$ distinct extensions $q_1$, ..., $q_\ell$ to a complete type. Choose $C'\supset C$ such that all $q_i|C'$ are distinct. Certainly the $q_i$'s are $C'$-invariant. It is enough to prove they are $C'$-definable, since then, for every formula $\varphi$, the $\mathrm{Aut}(\mathbb{U}/C)$-orbit of $d_{q_i}(\varphi)$ is finite, hence $d_{q_i}(\varphi)$ is equivalent to a formula in $L(\mathrm{acl}(C))$. To prove $q_i$ is $C'$-definable note that

$$p(f(x))\cup q_i|C'(x)\vdash q_i(x).$$

Thus, there is a set $A$ of formulas $\varphi(x,y)$ in $L$, a mapping $\varphi(x,y)\to\vartheta_\varphi(y)$ assigning to formulas in $A$ formulas in $L(C')$ such that $q_i$ is generated by $\{\varphi(x,b): \mathbb{U}\models\vartheta_\varphi(b)\}$. It then follows from Lemma 2.3.1 that $q_i$ is indeed $C'$-definable. $\square$

2.4. **Orthogonality to a definable set.** Let $Q$ be a fixed 0-definable set. We give definitions of orthogonality to $Q$ that are convenient for our purposes, and are equivalent to the usual ones when $Q$ is stably embedded and admits elimination of imaginaries; this is the only case we will need.

Let $A$ be a substructure of $\mathbb{U}$. A type $p=\mathrm{tp}(c/A)$ is said to be almost orthogonal to $Q$ if $Q(A(c))=Q(A)$. Here $A(c)$ is the substructure generated by $c$ over $A$, and $Q(A)=Q\cap\mathrm{dcl}(A)$ is the set of points of $Q$ definable over $A$.

An $A$-definable type $p$ is said to be orthogonal to $Q$, and one writes $p\perp Q$, if $p|B$ is almost orthogonal to $Q$ for any substructure $B$ containing $A$. Equivalently, for any $B$ and any $B$-definable function $f$ into $Q$ the pushforward $f_*p$ is a type concentrating on one point, i.e. including a formula of the form $y=\gamma$.

Let us recall that for $F$ a structure containing $C$, $\mathrm{Fn}(W,Q)(F)$ denotes the family of $F$-definable functions $W \to Q$ and that $\mathrm{Fn}(W,Q) = \mathrm{Fn}(W,Q)(\mathbb{U})$ is an ind-definable set.

Let $V$ be a $C$-definable set. Let $p$ be a definable type on $V$, orthogonal to $Q$. Any $\mathbb{U}$-definable function $f : V \to Q$ is generically constant on $p$. Equivalently, any $C$-definable function $f : V \times W \to Q$ (where $W$ is some $C$-definable set) depends only on the $W$-argument, when the $V$-argument is a generic realization of $p$. More precisely, we have a mapping

$$p_*^W : \mathrm{Fn}(V \times W, Q) \longrightarrow \mathrm{Fn}(W,Q)$$

(denoted by $p_*$ when there is no possibility of confusion) given by $p_*(f)(w) = \gamma$ if $(d_p v)(f(v,w) = \gamma)$ holds in $\mathbb{U}$.

Uniqueness of $\gamma$ is clear for any definable type. Orthogonality to $Q$ is precisely the statement that for any $f$, $p_*(f)$ is a function on $W$, i.e. for any $w$, such an element $\gamma$ exists. The advantage of the presentation $f \mapsto p_*(f)$, rather than the two-valued $\phi \mapsto p_*(\phi)$, is that it makes orthogonality to $Q$ evident from the very data.

Let $S^Q_{def,V}(A)$ denote the set of $A$-definable types on $V$ orthogonal to $Q$. It will be useful to note the (straightforward) conditions for pro-definability of $S^Q_{def,V}$. Given a function $g : S \times W \to Q$, we let $g_s(w) = g(s,w)$, thus viewing it as a family of functions $g_s : W \to Q$.

**Lemma 2.4.1.** *Assume the theory $T$ eliminates imaginaries, and that for any formula $\phi(v,w)$ without parameters, there exists a formula $\theta(w,s)$ without parameters such that for any $p \in S^Q_{def,V}$, for some $e$,*

$$\phi(v,c) \in p \iff \theta(c,e).$$

*Then $S^Q_{def,V}$ is pro-definable, i.e. there exists a canonical pro-definable $Z$ and a canonical bijection $Z(A) = S^Q_{def,V}(A)$ for every $A$.*

*Proof.* We first extend the hypothesis a little. Let $f : V \times W \to Q$ be 0-definable. Then there exists a 0-definable $g : S \times W \to Q$ such that for any $p \in S^Q_{def,V}$, for some $s \in S$, $p_*(f) = g_s$. Indeed, let $\phi(v,w,q)$ be the formula $f(v,w) = q$ and let $\theta(w,q,s)$ the corresponding formula provided by the hypothesis of the lemma. Let $S$ be the set of all $s$ such that for any $w \in W$ there exists a unique $q \in Q$ with $\theta(w,q,s)$. Now, by setting $g(s,w) = q$ if and only if $\theta(w,q,s)$ holds, one gets the more general statement.

Let $f_i : V \times W_i \to Q$ be an enumeration of all 0-definable functions $f : V \times W \to Q$, with $i$ running over some index set $I$. Let $g_i : S_i \times W_i \to Q$ be the corresponding functions provided by the previous paragraph. Elimination of imaginaries allows us to assume that $s$ is a canonical parameter for the function $g_{i,s}(w) = g_i(s,w)$, i.e. for no other $s'$ do we have $g_{i,s} = g_{i,s'}$. We then have a natural map $\pi_i : S^Q_{def,V} \to S_i$, namely $\pi_i(p) = s$ if $p_*(f_i) = g_{i,s}$. Let $\pi = \Pi_i \pi_i : S^Q_{def,V} \to \Pi_i S_i$ be the product map. Now $\Pi_i S_i$ is canonically a pro-definable set, and the map $\pi$ is injective. So it suffices to show that the image is $\infty$-definable in $\Pi S_i$. Indeed, $s = (s_i)_i$ lies in the image if and only if for each finite tuple of indices $i_1, \dots, i_n \in I$ (allowing repetitions), $(\forall w_1 \in W_1) \cdots (\forall w_n \in W_n)(\exists v \in V) \bigwedge_{i=1}^n f_i(v, w_i) = g_i(s_i, w_i)$. For given this consistency condition, there exists $a \in V(\mathbb{U}')$ for some $\mathbb{U} \prec \mathbb{U}'$ such that $f_i(a, w) = g_i(s, w)$ for all $w \in W_i$ and all $i$. It follows immediately that $p = \mathrm{tp}(a/\mathbb{U})$ is definable and orthogonal to $\Gamma$, and $\pi(p) = s$. Conversely if $p \in S^Q_{def,V}(\mathbb{U})$ and $a \models p|\mathbb{U}$, for any $w_1 \in W_1(\mathbb{U}), \dots, w_n \in W_n(\mathbb{U})$, the element $a$ witnesses the existence of $v$ as required. So the image is cut out by a set of formulas concerning the $s_i$. □

If $Q$ is a two-element set, any definable type is orthogonal to $Q$, and $\mathrm{Fn}(V, Q)$ can be identified with the algebra of formulas on $V$, via characteristic functions. The presentation of definable types as a Boolean retraction from formulas on $V \times W$ to formulas on $W$ can be generalized to definable types orthogonal to $Q$. An element $p$ of $S^Q_{def,V}(A)$ yields a compatible systems of retractions $p^W_* : \mathrm{Fn}(V \times W, Q) \longrightarrow \mathrm{Fn}(W, Q)$. These retractions are also compatible with definable functions $g : Q^m \to Q$, namely $p_*(g \circ (f_1, \dots, f_m)) = g \circ (p_* f_1, \dots, p_* f_m)$. One can restrict attention to 0-definable functions $Q^m \to Q$ along with compositions of the following form: given $F : V \times W \times Q \to Q$ and $f : V \times W \to Q$, let $F \circ' f(v, w) = F(v, w, f(v, w))$. Then $p_*(F \circ' f) = p_*(F) \circ' p_*(f)$. It can be shown that any compatible system of retractions compatible with these compositions arises from a unique element $p$ of $S^Q_{def,V}(A)$. This can be shown by the usual two way translation between sets and functions: a set can be coded by a function into a two-element set (in case two constants are not available, one can add variables $x$, $y$, and consider functions whose values are among the variables). On the other hand a function can be coded by a set, namely its graph. This characterization will not be used, and we will leave the details to the reader. It does give a slightly different way to see the $\infty$-definability of the image in Lemma 2.4.1.

2.5. **Stable domination.** We shall assume from now on that the theory $T$ has elimination of imaginaries.

**Definition 2.5.1.** A $C$-definable set $D$ in $\mathbb{U}$ is said to be stably embedded if, for every definable set $E$ and $r > 0$, $E \cap D^r$ is definable over $C \cup D$. A $C$-definable set $D$ in $\mathbb{U}$ is said to be stable if the structure with domain $D$, when equipped with all the $C$-definable relations, is stable.

One considers the multisorted structure $\mathrm{St}_C$ whose sorts $D_i$ are the $C$-definable, stable and stably embedded subsets of $\mathbb{U}$. For each finite set of sorts $D_i$, all the $C$-definable relations on their union are considered as $\emptyset$-definable relations $R_j$. The structure $\mathrm{St}_C$ is stable by Lemma 3.2 of [14].

For any $A \subset \mathbb{U}$, one sets $\mathrm{St}_C(A) = \mathrm{St}_C \cap \mathrm{dcl}(CA)$.

**Definition 2.5.2.** A type $\mathrm{tp}(A/C)$ is stably dominated if, for any $B$ such that $\mathrm{St}_C(A) \downarrow_{\mathrm{St}_C(C)} \mathrm{St}_C(B)$, $\mathrm{tp}(B/C\mathrm{St}_C(A)) \vdash \mathrm{tp}(B/CA)$.

**Remark 2.5.3.** The type $\mathrm{tp}(A/C)$ is stably dominated if and only if, for any $B$ such that $\mathrm{St}_C(A) \downarrow_{\mathrm{St}_C(C)} \mathrm{St}_C(B)$, $\mathrm{tp}(A/\mathrm{St}_C(A))$ has a unique extension over $C\mathrm{St}_C(A)B$.

By [14] 3.13, if $\mathrm{tp}(a/C)$ is stably dominated, then it is has an an $\mathrm{acl}(C)$-definable extension $p$ to $\mathbb{U}$; this definable type will also be referred to as stably dominated; we will sometimes denote it by $\mathrm{tp}(a/\mathrm{acl}(C))|\mathbb{U}$, and for any $B$ with $\mathrm{acl}(C) \leq B \leq \mathbb{U}$, write $p|B = \mathrm{tp}(a/\mathrm{acl}(C))|B$. For any $|C|^+$-saturated extension $N$ of $C$, $p|N$ is the unique $\mathrm{Aut}(N/\mathrm{acl}(C))$-invariant extension of $\mathrm{tp}(a/\mathrm{acl}(C))$. We will need a slight extension of this:

**Lemma 2.5.4.** *Let $p$ be a stably dominated $C$-definable type, $C = \mathrm{acl}(C)$. Let $C \subseteq B = \mathrm{dcl}(B)$, and assume $p|B$ is $\mathrm{Aut}(B/C)$-invariant. Assume: for any $b \in St_C(B) \setminus \mathrm{dcl}(C)$, there exists $b' \in B$, $b' \neq b$, with $\mathrm{tp}(b/C) = \mathrm{tp}(b'/C)$. Then $p|N$ is the unique $\mathrm{Aut}(N/C)$-invariant extension of $\mathrm{tp}(a/C)$.*

*Proof.* By hypothesis, $p$ is stably dominated via some $C$-definable function $h$ into $\mathrm{St}_C$. Let $q$ be an $\mathrm{Aut}(N/C)$-invariant extension of $\mathrm{tp}(a/C)$. Let $h_*q$ be the pushforward. Then $h_*q$ is $\mathrm{Aut}(\mathrm{St}_C(B)/C)$ invariant, so the canonical base of $h_*q|St_C(B)$ must be contained in $\mathrm{acl}(C) = C$; hence $h_*q$ is a non-forking extension of $h_*p|C$, so $h_*q = h_*p$. By definition of stable domination, it follows that $q = p$. □

**Proposition 2.5.5** ([14], Proposition 6.11)**.** *Assume $\mathrm{tp}(a/C)$ and $\mathrm{tp}(b/aC)$ are stably dominated, then $\mathrm{tp}(ab/C)$ is stably dominated.*

**Remark 2.5.6.** It is easy to see that transitivity holds for the class of symmetric invariant types. Hence Proposition 2.5.5 can be deduced from the characterization of stably dominated types as symmetric invariant types.

A formula $\varphi(x, y)$ is said to *shatter* a subset $W$ of a model of $T$ if for any two finite disjoint subsets $U, U'$ of $W$ there exists $b$ with $\phi(a, b)$ for $a \in U$, and $\neg\phi(a', b)$ for $a \in U'$. Shelah says that a formula $\varphi(x, y)$ *has the independence property* if it shatters arbitrarily large finite sets; otherwise, it has NIP. Finally, $T$ has NIP if every formula has NIP. Stable and o-minimal theories are NIP, as is ACVF.

If $\varphi(x, y)$ is NIP then for some $k$, for any indiscernible sequence $(a_1, \ldots, a_n)$ and any $b$ in a model of $T$, $\{i : \phi(a_i, b)\}$ is the union of $\leq k$ convex segments. If $\{a_1, \ldots, a_n\}$ is an indiscernible set, i.e. the type of $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ does not depend on $\sigma \in \mathrm{Sym}(n)$, it follows that $\{i : \phi(a_i, b)\}$ has size $\leq k$, or else the complement has size $\leq k$.

**Definition 2.5.7.** If $T$ is a NIP-theory, and $p$ is an $\mathrm{Aut}(\mathbb{U}/C)$-invariant type over $\mathbb{U}$, one says that $p$ is generically stable over $C$ if it is $C$-definable and finitely satisfiable in any model containing $C$ (that is, for any formula $\varphi(x)$ in $p$ and any model $M$ containing $C$, there exists $c$ in $M$ such that $\mathbb{U} \models \varphi(c)$).

In general, when $p(x), q(y)$ are $\mathrm{Aut}(\mathbb{U}/C)$-invariant types, there exists a unique $\mathrm{Aut}(\mathbb{U}/C)$-invariant type $r(x, y)$, such that for any $C' \supseteq C$, $(a, b) \models p \otimes q$ if and only if $a \models p|C$ and $b \models q|C(a)$. This type is denoted $p(x) \otimes q(y)$. In general $\otimes$ is associative but not necessarily symmetric. We define $p^n$ by $p^{n+1} = p^n \otimes p$.

The following characterization of generically stable types in NIP theories from will be useful:

**Lemma 2.5.8** ([18] Proposition 3.2)**.** *Assume $T$ has* NIP*. An* $\mathrm{Aut}(\mathbb{U}/C)$*-invariant type $p(x)$ is generically stable over $C$ if and only if $p^n$ is symmetric with respect to permutations of the variables $x_1, \dots, x_n$.*

*For any formula $\varphi(x, y)$, there exists a natural number $n$ such that whenever $p$ is generically stable and $(a_1, \dots, a_N) \models p^N|C$ with $N > 2n$, for every $c$ in $\mathbb{U}$, $\varphi(x, c) \in p$ if and only if $\mathbb{U} \models \bigvee_{i_0 < \dots < i_n} \varphi(a_{i_0}, c) \wedge \dots \wedge \varphi(a_{i_n}, c)$.*

The second part of the lemma is an easy consequence of the definition of a NIP formula, or rather the remark on indiscernible sets just below the definition.

We remark that Proposition 2.5.5 also follows from the characterization of generically stable definable types in NIP theories as those with symmetric tensor powers in Lemma 2.5.8, cf. [18].

2.6. **Review of** ACVF**.** A valued field consists of field $K$ together with a homorphism $v$ from the multiplicative group to an ordered abelian group $\Gamma$, such that $v(x + y) \geq \min v(x), v(y)$. In this paper we shall take write the law on $\Gamma$ additively. We shall write $\Gamma_\infty$ for $\Gamma$ with an element $\infty$ added with usual conventions. In particular we extend $v$ to $K \to \Gamma_\infty$ by setting $v(0) = \infty$. We denote by $R$ the valuation ring, by $\mathcal{M}$ the maximal ideal and by $k$ the residue field.

Now assume $K$ is algebraically closed and $v$ is surjective. The value group $\Gamma$ is then divisible and the residue field $k$ is algebraically closed. By a classical result of A. Robinson, the theory of non trivially valued algebraically closed fields of given characteristic and residue characteristic is complete. Several quantifier elimination results hold for the theory ACVF of algebraically closed valued fields with non-trivial valuation. In particular ACVF admits quantifier elimination in the 3-sorted language $\mathcal{L}_{k\Gamma}$, with sorts VF, $\Gamma$ and k for the valued field, value group and residue field sorts, with respectively the ring, ordered abelian group and ring language, and additional symbols for the valuation $v$ and the map $\mathrm{Res} : \mathrm{VF}^2 \to k$ sending $(x, y)$ the residue of $xy^{-1}$ if $v(x) \geq v(y)$ and $y \neq 0$ and to 0 otherwise, (cf. [13] Theorem 2.1.1). Sometimes we shall also write val instead of $v$ for the valuation. In this paper we shall use the extension $\mathcal{L}_\mathcal{G}$ of $\mathcal{L}_{k\Gamma}$ considered in [13] for which elimination of imaginaries holds. In addition to sorts VF, $\Gamma$ and $k$

there are sorts $S_n$ and $T_n$, $n \geq 1$. The sort $S_n$ is the collection of all codes for free rank $n$ $R$-submodules of $K^n$. For $s \in S_n$, we denote by $\mathrm{red}(s)$ the reduction modulo the maximal ideal of the lattice $\Lambda(s)$ coded by $s$. This has $\emptyset$-definably the structure of a rank $n$ $k$-vector space. We denote by $T_n$ the set of codes for elements in $\cup\{\mathrm{red}(s)\}$. Thus an element of $T_n$ is a code for the coset of some element of $\Lambda(s)$ modulo $\mathcal{M}\Lambda(s)$. For each $n \geq 1$, we have symbols $\tau_n$ for the functions $\tau_n : T_n \to S_n$ defined by $\tau_n(t) = s$ if $t$ codes an element of $\mathrm{red}(s)$. We shall set $\mathcal{S} = \cup_{n\geq 1} S_n$ and $\mathcal{T} = \cup_{n\geq 1} T_n$. The main result of [13] is that ACVF admits elimination of imaginaries in $\mathcal{L}_{\mathcal{G}}$.

With our conventions, if $C \subset \mathbb{U}$, we write $\Gamma(C)$ for $\mathrm{dcl}(C) \cap \Gamma$ and $k(C)$ for $\mathrm{dcl}(C) \cap k$. If $K$ is a subfield of $\mathbb{U}$, one denotes by $\Gamma_K$ the value group, thus $\Gamma(K) = \mathbb{Q} \otimes \Gamma_K$. If the valuation induced on $K$ is non-trivial, then the model theoretic algebraic closure $\mathrm{acl}(K)$ is a model of ACVF. In particular the structure $\Gamma(K)$ is Skolemized.

We shall denote in the same way a finite cartesian product of sorts and the corresponding definable set. For instance, we shall denote by $\Gamma$ the definable set which to any model $K$ of ACVF assigns $\Gamma(K)$ and by k the definable set which to $K$ assigns its residue field. We shall also sometimes write $K$ for the sort VF.

The following follows from the different versions of quantifier elimination (cf. [13] Proposition 2.1.3):

**Proposition 2.6.1.**
(1) *The definable set $\Gamma$ is o-minimal in the sense that every definable subset of $\Gamma$ is a finite union of intervals.*
(2) *Any $K$-definable subset of k is finite or cofinite (uniformly in the parameters), i.e. k is strongly minimal.*
(3) *The definable set $\Gamma$ is stably embedded.*
(4) *If $A \subseteq K$, then $\mathrm{acl}(A) \cap K$ is equal to the field algebraic closure of $A$ in $K$.*
(5) *If $S \subseteq k$ and $\alpha \in k$ belongs to $\mathrm{acl}(S)$ in the $K^{\mathrm{eq}}$ sense, then $\alpha$ belongs to the field algebraic closure of $S$.*
(6) *The definable set k is stably embedded.*

**Lemma 2.6.2** ([13] Lemma 2.17)**.** *Let $C$ be an algebraically closed valued field, and let $s \in S_n(C)$, with $\Lambda = \Lambda_s \subseteq K^n$ the corresponding lattice. Then $\Lambda$ is $C$-definably isomorphic to $R^n$, and the torsor $\mathrm{red}(s)$ is $C$-definably isomorphic to $k^n$.*

A $C$-definable set $D$ is $k$-internal if there exists a finite $F \subset \mathbb{U}$ such that $D \subset \mathrm{dcl}(k \cup F)$.

By Lemma 2.6.2 of [13], we have the following characterisations of $k$-internal sets:

**Lemma 2.6.3** ([13] Lemma 2.6.2)**.** *Let $D$ be a $C$-definable set. Then the following conditions are equivalent:*

1. *$D$ is $k$-internal.*
2. *For any $m \geq 1$, there is no surjective definable map from $D^m$ to an infinite interval in $\Gamma$.*
3. *$D$ is finite or, up to permutation of coordinates, is contained in a finite union of sets of the form $\mathrm{red}(s_1) \times \cdots \times \mathrm{red}(s_m) \times F$, where $s_1, \ldots, s_m$ are $\mathrm{acl}(C)$-definable elements of $\mathcal{S}$ and $F$ is a $C$-definable finite set of tuples from $\mathcal{G}$.*

For any parameter set $C$, let $\mathrm{VC}_{k,C}$ be the many-sorted structure whose sorts are $k$-vector spaces $\mathrm{red}(s)$ with $s$ in $\mathrm{dcl}(C) \cap \mathcal{S}$. Each sort $\mathrm{red}(s)$ is endowed with $k$-vector space structure. In addition, as its $\emptyset$-definable relations, $\mathrm{VC}_{k,C}$ has all $C$-definable relations on products of sorts.

By Proposition 3.4.11 of [13], we have:

**Lemma 2.6.4** ([13] Proposition 3.4.11)**.** *Let $D$ be a $K^{\mathrm{eq}}$-definable set. Then the following conditions are equivalent:*

1. *$D$ is $k$-internal.*
2. *$D$ is stable and stably embedded.*
3. *$D$ is contained in $\mathrm{dcl}(C \cup \mathrm{VC}_{k,C})$.*

By combining Proposition 2.6.1, Lemma 2.6.2, Lemma 2.6.4 and Remark 2.5.3, one sees that (over a model) the $\phi$-definition of a stably dominated type factors through some function into a finite dimensional vector space over the residue field.

**Corollary 2.6.5.** *Let $C$ be a model of* ACVF*, let $V$ be a $C$-definable set and let $a \in V$. Assume $p = \mathrm{tp}(a/C)$ is a stably dominated type. Let $\phi(x, y)$ be a formula over $C$. Then there exists a definable map $g : V \to k^n$ and a formula $\theta$ over $C$ such that, if $g(a) \downarrow_{k(C)} \mathrm{St}_C(b)$, then $\phi(a, b)$ holds if and only if $\theta(g(a), b)$.*

2.7. **$\Gamma$-internal sets.** Let $Q$ be an $F$-definable set. An $F$-definable set $X$ is *$Q$-internal* if there exists $F' \supset F$, and an $F'$-definable surjection $h : Y \to X$, with $Y$ an $F'$-definable subset of $Q^n$ for some $n$. When $Q$ is stably embedded and eliminates imaginaries, as is the case of $\Gamma$ in ACVF, we can take $h$ to be a bijection, by factoring out the kernel. If one can take $F' = F$ we say that $X$ is directly $Q$-internal.

In the case of $Q = \Gamma$ in ACVF, we mention some equivalent conditions.

**Lemma 2.7.1.** *The following conditions are equivalent:*

1. *$X$ is $\Gamma$-internal.*
2. *$X$ is internal to some o-minimal definable linearly ordered set.*
3. *$X$ admits a definable linear ordering.*

(4) *Every stably dominated type on* $X$ *(over any base set) is constant (i.e. contains a formula* $x = a$*).*
(5) *There exists an* $\mathrm{acl}(F)$*-definable injective* $h : Y \to \Gamma^*$.

*Proof.* The fact that (2) implies (3) follows easily from elimination of imaginaries in ACVF: any o-minimal definable linearly ordered set is directly internal to $\Gamma$. Condition (3) clearly implies (4) by the symmetry property of generically stable types $p$: $p(x) \otimes p(y)$ has $x \leq y$ if and only if $y \leq x$, hence $x = y$. The implication (4) $\to$ (5) again uses elimination of imaginaries in ACVF, and inspection of the geometric sorts. Namely, let $A = \mathrm{acl}(F)$ and let $c \in Y$. Assuming (4), we have to show that $c \in \mathrm{dcl}(\Gamma^*)$. This reduces to the case that $\mathrm{tp}(c/A)$ is unary; for if $c = (c_1, c_2)$ and the implication holds for $\mathrm{tp}(c_2/A)$ and for $\mathrm{tp}(c_1/A(c_2)$ we obtain $c_2 \in \mathrm{acl}(A, \Gamma, c_1)$; it follows that (4) holds for $\mathrm{tp}(c_1/A)$, so $c_1 \in \mathrm{dcl}(A, \gamma)$ and the result follows since $\mathrm{acl}(A, \gamma) = \mathrm{dcl}(A, \gamma)$ for $\gamma \in \Gamma^m$. So assume $\mathrm{tp}(c/A)$ is unary, i.e. it is the type of a sub-ball $b$ of a free $\mathcal{O}$-module $M$. The radius $\gamma$ of $b$ is well-defined. Now $\mathrm{tp}(c/A, \gamma(b))$ is a type of balls of constant radius; if $c \notin \mathrm{acl}(A, \gamma(b))$ then there are infinitely many balls realizing this type, and their union fills out a set containing a larger closed sub-ball. In this case the generic type of the closed sub-ball induces a stably dominated type on a subset of $\mathrm{tp}(c/A, \gamma(b))$, contradicting (4). Thus $c \in \mathrm{acl}(A, \gamma(b)) = \mathrm{dcl}(A, \gamma(b))$.

The remaining implications (1) $\to$ (2) and (5) $\to$ (1) are obvious. □

Let $U$ and $V$ be definable sets. A definable map $f : U \to V$ with all fibers $\Gamma$-internal is called a $\Gamma$-internal cover. If $f : U \to V$ is an $F$-definable map, such that for every $v \in V$ the fiber is $F(v)$-definably isomorphic to a definable set in $\Gamma^n$, then by compactness and stable embeddedness of $\Gamma$, $U$ is isomorphic over $V$ to a fiber product $V \times_{g,h} Z$, where $g : V \to Y \subseteq \Gamma^m$, and $Z \subseteq \Gamma^n$, and $h : Z \to Y$. We call such a cover *directly* $\Gamma$*-internal.*

Any finite cover of $V$ is $\Gamma$-internal, and so is any directly $\Gamma$-internal cover.

**Lemma 2.7.2.** *Let* $V$ *be a definable set in* $\mathrm{ACVF}_F$*. Then any* $\Gamma$*-internal cover* $f : U \to V$ *is isomorphic (over* $V$*) to a fiber product over* $V$ *of a finite cover and a directly* $\Gamma$*-internal cover.*

*Proof.* It suffices to prove this at a complete type $p = \mathrm{tp}(c/F)$ of $U$, since the statement will then be true (using compactness) above a (relatively) definable neighborhood of $f_*(p)$, and so (again by compactness, on $V$) everywhere. Let $F' = F(f(c))$. By assumption, $f^{-1}(f(c))$ is $\Gamma$-internal. So over $F'$ there exists a finite definable set $H$, for $t \in H$ an $F'(t)$-definable bijection $h_t : W_t \to U$, with $W_t \subseteq \Gamma^n$, and $c \in \mathrm{Im}(h_t)$. We can assume $H$ is an orbit of $G = \mathrm{Aut}(\mathrm{acl}(F)/F)$. In this case, since $\Gamma$ is linearly ordered, $W_t$ cannot depend on $t$, so $W_t = W$. Similarly let $G_c = \mathrm{Aut}(\mathrm{acl}(F)(c)/F(c)) \leq G$. Then $h_t^{-1}(c) \in W$ depends only on the $G_c$-orbit of $h_t$. Let $H_c$ be such an orbit (defined over $F(c)$), and set $h^{-1}(c) = h_t^{-1}(c)$ for $t$ in this orbit and some $h \in H_c$. Then $H_c$ has a canonical code $g_1(c)$, and we have $g_1(c) \in \mathrm{acl}(F(f(c))$, and $c \in \mathrm{dcl}(F(f(c), g_1(c), h^{-1}(c)))$. Let $g(c) = (f(c), g_1(c))$. Then $\mathrm{tp}(g(c)/F)$ is naturally a finite cover of $\mathrm{tp}(f(c)/F)$, and $\mathrm{tp}(f(c), h^{-1}(c)/F)$ is a directly $\Gamma$-internal cover. □

**Lemma 2.7.3.** *Let $F$ be a definably closed substructure of $\mathrm{VF}^* \times \Gamma^*$, let $B \subseteq \mathrm{VF}^m$ be $\mathrm{ACVF}_F$-definable, and let $B'$ be a definable set in any sorts (including possibly imaginaries). Let $g : B' \to B$ be a definable map with finite fibers. Then there exists a definable $B'' \subseteq \mathrm{VF}^{m+\ell}$ and a definable bijection $B' \to B''$ over $B$.*

*Proof.* By compactness, working over $F(b)$ for $b \in B$, this reduces to the case that $B$ is a point. So $B'$ is a finite $\mathrm{ACVF}_F$-definable set, and we must show that $B'$ is definably isomorphic to a subset of $\mathrm{VF}^\ell$. Now we can write $F = F_0(\gamma)$ for some $\gamma \in \Gamma^*$ with $F_0 = F \cap \mathrm{VF}$. By Lemma 3.4.12 of [13], $\mathrm{acl}(F) = \mathrm{acl}(F_0)(\gamma)$. So $B' = \{f(\gamma) : f \in B''\}$ where $B''$ is some finite $F_0$-definable set of functions on $\Gamma$. Replacing $F$ by $F_0$ and $B'$ by $B''$, we may assume $F$ is a field. Now $\mathrm{acl}(F) = \mathrm{dcl}(F^{alg})$. Indeed, this is clear if $F$ is not trivially valued since then $F^{alg}$ is an elementary substructure of $\mathbb{U}$. In general there exists $\mathrm{acl}(F) \leq \cap_\tau \mathrm{dcl}(F(\tau)^{alg}) = \mathrm{dcl}(F^{alg})$. Now we have $B' \subseteq \mathrm{acl}(F) = \mathrm{dcl}(F^{alg})$. Using induction on $|B'|$ we may assume $B'$ is irreducible, and also admits no nonconstant $\mathrm{ACVF}_F$-definable map to a smaller definable set. If $B'$ admits a nonconstant definable map into VF then it must be 1-1 and we are done. Let $b \in B'$ and let $F' = \mathrm{Fix}(\mathrm{Aut}(F^{alg}/F(b)))$. Then $F'$ is a field, and if $d \in F' \smallsetminus F$, then $d = h(b)$ for some definable map $h$, which must be nonconstant since $d \notin F$. If $F' = F$ then by Galois theory, $b \in \mathrm{dcl}(F)$, so again the statement is clear. □

**Corollary 2.7.4.** *The composition of two definable maps with $\Gamma$-internal fibers also has $\Gamma$-internal fibers. In particular if $f$ has finite fibers and $g$ has $\Gamma$-internal fibers then $g \circ f$ and $f \circ g$ have $\Gamma$-internal fibers.*

*Proof.* Here we may work over a model $A$. By Lemma 2.7.2 and the definition, the class of $\Gamma$-internal covers is the same as compositions $g \circ f$ of definable maps $f$ with finite fibers, and $g$ with directly $\Gamma$-internal covers. Hence to show that this class is closed under composition it suffices to show that if $f$ has finite fibers and $g$ has directly $\Gamma$-internal covers, then $f \circ g$ has $\Gamma$-internal fibers; in other words that if $b \in \mathrm{acl}(A(\gamma))$ with $\gamma$ a tuple from $\Gamma$, then $(a, b) \in \mathrm{dcl}(A \cup \Gamma)$. But this follows from Lemma 3.4.12 quoted above. □

Warning: the corollary applies to definable maps between definable sets, hence also to iso-definable sets. However if $f : X \to Y$ is map between pro-definable sets and $U$ is a $\Gamma$-internal, iso-definable subset of $Y$, we do not know if $f^{-1}(U)$ must be $\Gamma$-internal, even if $f$ is $\leq$ 2-to-one.

**Remark 2.7.5.** Let $\Gamma$ be a Skolemized o-minimal structure, $a \in \Gamma^n$. Let $D$ be a definable subset of $\Gamma^n$ such that $a$ belongs to the topological closure $cl(D)$ of $D$. Then there exists a definable type $p$ on $D$ with limit $a$.

*Proof.* Consider the family $F$ of all rectangles (products of intervals) whose interior contains $a$. This is a definable family, directed downwards under containment. By Lemma 2.19 of [16] there exists a definable type $q$ on $F$ concentrating, for each $b \in F$, on $\{b' \in F : b' \subseteq b\}$. Since $a \in cl(D)$, there exists a definable (Skolem) function $g$ such that for $u \in F$, $g(u) \in u \cap D$. To conclude it is enough to set $p = g_*(q)$. □

An alternative proof is provided, in our case, by Lemma 4.2.12.

It follows that if the limit of any definable type on $D$ exists and lies in $D$, then $D$ is closed. Conversely, if $D$ is bounded, any definable type on $D$ will have a limit, and if $D$ is closed then this limit is necessarily in $D$.

2.8. **Orthogonality to $\Gamma$.** Let $A$ be a substructure of $\mathbb{U}$.

**Proposition 2.8.1.** (a) *Let $p$ be an $A$-definable type. The following conditions are equivalent:*

(1) *$p$ is stably dominated.*
(2) *$p$ is orthogonal to $\Gamma$.*
(3) *$p$ is generically stable.*

(b) *A type $p$ over $A$ extends to at most one generically stable $A$-definable type.*

*Proof.* The equivalence of (1) and (2) follows from [14] 10.7 and 10.8. Using Proposition 10.16 in [14], and [18], Proposition 3.2(v), we see that (2) implies (3). (In fact (1) implies (3) is easily seen to be true in any theory, in a similar way.) To see that (3) implies (2) (again in any theory), note that if $p$ is generically stably and $f$ is a definable function, then $f_*p$ is generically stable (by any of the criteria of [18] 3.2, say the symmetry of indiscernibles). Now a generically stable definable type on a linearly ordered set must concentrate on a single point: a 2-element Morley sequence $(a_1, a_2)$ based on $p$ will otherwise consist of distinct elements, so either $a_1 < a_2$ or $a_1 > a_2$, neither of which can be an indiscernible set. The statement on unique extensions follows from [18], Proposition 3.2(v). □

We shall use the following statement, Theorem 12.18 from [14]:

**Theorem 2.8.2.** (1) *Suppose that $C \leq L$ are valued fields with $C$ maximally complete, $k(L)$ is a regular extension of $k(C)$ and $\Gamma_L/\Gamma_C$ is torsion free. Let $a$ be a sequence in $\mathbb{U}$, $a \in \mathrm{dcl}(L)$. Then $\mathrm{tp}(a/C \cup \Gamma(Ca))$ is stably dominated.*

(2) *Let $C$ be a maximally complete algebraically closed valued field, and $a$ be a sequence in $\mathbb{U}$. Then $\mathrm{tp}(\mathrm{acl}(Ca)/C \cup \Gamma(Ca))$ is stably dominated.*

2.9. $\widehat{V}$ **for stable definable** $V$**.** We end with a description of the set $\widehat{V}$ of definable types concentrating on a stable definable $V$, as an ind-definable set. The notation $\widehat{V}$ is compatible with the one that will be introduced in greater generality in §3.1. Such a representation will not be possible for algebraic varieties

$V$ in ACVF and so the picture here is not at all suggestive of the case that will mainly interest us, but it is simpler and will be lightly used at one point.

A family $X_a$ of definable sets is said to be *uniformly definable* in the parameter $a$ if there exists a definable $X$ such that $X_a = \{x : (a, x) \in X\}$. An ind-definable set $X_a$ depending on a parameter $a$ is said to be uniformly definable in $a$ if it can be presented as the direct limit of a system $X_{a,i}$, with each $X_{a,i}$ and the morphisms $X_{a,i} \to X_{a,j}$ definable uniformly in $a$. If $U$ is a definable set, and $X_u = \lim_i X_{u;i}$ is (strict) ind-definable uniformly in $u$, then the disjoint union of the $X_u$ is clearly (strict) ind-definable too.

Recall $\mathrm{k}$ denotes the residue field sort. Given a Zariski closed $W \subseteq \mathrm{k}^n$, define $\deg(W)$ to be the degree of the Zariski closure of $W$ in projective $n$-space. Let $\mathrm{ZC}_d(\mathrm{k}^n)$ be the family of Zariski closed subset of degree $\leq d$ and let $\mathrm{IZC}_d(\mathrm{k}^n)$ be the sub-family of absolutely irreducible varieties. It is well known that $\mathrm{IZC}_d(\mathrm{k}^n)$ is definable (cf., for instance, §17 of [11]). These families are invariant under $\mathrm{GL}_n(\mathrm{k})$, hence for any definable $\mathrm{k}$-vector space $V$ of dimension $n$, we may consider their pullbacks $\mathrm{ZC}_d(V)$ and $\mathrm{IZC}_d(V)$ to families of subsets of $V$, under a $\mathrm{k}$-linear isomorphism $V \to \mathrm{k}^n$. Then $\mathrm{ZC}_d(V)$ and $\mathrm{IZC}_d(V)$ are definable, uniformly in any definition of $V$.

**Lemma 2.9.1.** *If $V$ is a finite-dimensional $\mathrm{k}$-space, then $\widehat{V}$ is strict ind-definable.*

*The disjoint union $D_{st}$ of the $\widehat{V_\Lambda}$ with $V_\Lambda = \Lambda/\mathcal{M}\Lambda$ and where $\Lambda$ ranges over the definable family $S_n$ of lattices in $K^n$ is also strict ind-definable.*

*Proof.* Since $\widehat{V}$ can be identified with the limit over all $d$ of $\mathrm{IZC}_d(V)$, it is strict ind-definable uniformly in $V$. The family of lattices $\Lambda$ in $K^n$ is a definable family, so the disjoint union of $\widehat{V_\Lambda}$ over all such $\Lambda$ is strict ind-definable. $\square$

If $K$ is a valued field, one sets $\mathrm{RV} = K^\times/1 + \mathcal{M}$. So we have an exact sequence of abelian groups $0 \to k^\times \to \mathrm{RV} \to \Gamma \to 0$. For $\gamma \in \Gamma$, denote by $V_\gamma$ the preimage of $\gamma$ in $\mathrm{RV}$.

**Lemma 2.9.2.** *For $m \geq 0$, $\widehat{\mathrm{RV}^m}$ is strict ind-definable. The function* dim *is constructible (i.e. has definable fibers on each definable piece of $\widehat{\mathrm{RV}^m}$).*

*Proof.* Note that $\mathrm{RV}$ is the union over $\gamma \in \Gamma$ of the $\mathrm{k}$-vector spaces $V_\gamma$. For $\overline{\gamma} = (\gamma_1, \ldots, \gamma_n) \in \Gamma^n$, let $V_{\overline{\gamma}} = \Pi_{i=1}^n V_{\gamma_i}$. Since the image of a stably dominated type on $\mathrm{RV}^m$ under the morphism $\mathrm{RV}^m \to \Gamma^m$ is constant, any stably dominated type must concentrate on a finite product $V_{\overline{\gamma}}$. Thus it suffices to show, uniformly in $\overline{\gamma} \in \Gamma^n$, that $\widehat{V_{\overline{\gamma}}}$ is strict ind-definable. Indeed $\widehat{V_{\overline{\gamma}}}$ can be identified with the limit over all $d$ of $\mathrm{IZC}_d(V_{\overline{\gamma}})$. $\square$

2.10. **Decomposition of definable types.** We seek to understand a definable type in terms of a definable type $q$ on $\Gamma^n$, and the germ of a definable map from $q$ to stably dominated types.

Let $p$ be an $A$-definable type. Define $\mathrm{rk}_\Gamma(p) = \mathrm{rk}_{\mathbb{Q}}\Gamma(M(c))/\Gamma(M)$, where $A \leq M \models \mathrm{ACVF}$ and $c \models p|M$. Since $p$ is definable, this rank does not depend on the choice of $M$, but for the present discussion it suffices to take $M$ somewhat saturated, to make it easy to see that $\mathrm{rk}_\Gamma(p)$ is well-defined.

If $p$ has rank $r$, then there exists a definable function to $\Gamma^r$ whose image is not contained in a smaller dimensional set. We show first that at least the germ of such a function can be chosen $A$-definable.

**Lemma 2.10.1.** *Let $p$ be an $A$-definable type and set $r = \mathrm{rk}_\Gamma(p)$. Then there exists a nonempty $A$-definable set $Q''$ and for $b \in Q''$ a $b$-definable function $\gamma_b = ((\gamma_b)_1, \ldots, (\gamma_b)_r)$ into $\Gamma^r$, such that*

1. *If $b \in Q''$ and $c \models p|A(b)$ then the image of $\gamma_b(c)$ in $\Gamma(A(b,c))/\Gamma(A(b))$ is a $\mathbb{Q}$-linearly-independent $r$-tuple.*
2. *If $b, b' \in Q''$ and $c \models p|A(b,b')$ then $\gamma_b(c) = \gamma_{b'}(c)$.*

*Proof.* Pick $M$ as above, and an $M$-definable function $\gamma = (\gamma_1, \ldots, \gamma_r)$ into $\Gamma^r$, such that if $c \models p|M$ then $\gamma_1(c), \ldots, \gamma_r(c)$ have $\mathbb{Q}$-linearly-independent images in $\Gamma(M(c))/\Gamma(M)$. Say $\gamma = \gamma_a$ and let $Q = \mathrm{tp}(a/A)$. If $b \in Q$ there exist a unique $N = N(a,b) \in \mathrm{GL}_r(\mathbb{Q})$ and $\gamma' = \gamma'(a,b) \in \Gamma^r$ such that for $c \models p|M(b)$, $\gamma_b(c) = N\gamma_a(c) + \gamma'$. By compactness, as $b$ varies the matrices $N(a,b)$ vary among a finite number of possibilities $N_1, \ldots, N_k$; moreover there exists an $A$-definable set $Q'$ such that for $a', b \in Q'$ we have $(\exists t' \in \Gamma^r)(d_p u) \bigvee_i (\gamma_b(u) = N_i \gamma_{a'}(u) + t')$. In other words the definable set $Q'$ has the same properties as $Q$.

Define an equivalence relation on $Q'$: $b'Eb$ if $(d_p x)(\gamma_{b'}(x) = \gamma_b(x))$. Then by the above discussion, $Q'/E \subseteq \mathrm{dcl}(A(a), \Gamma)$ (in particular $Q'/E$ is $\Gamma$-internal, cf. 2.7). By Lemma 2.7 it follows that $Q'/E \subseteq \mathrm{acl}(A, \Gamma)$, and there exists a definable map $g : Q'/E \to \Gamma^\ell$ with finite fibers.

We can consider the following partial orderings on $Q'$: $b' \leq_i b$ if and only if $(d_p x)((\gamma_{b'})_i(x) \leq (\gamma_b)_i(x))$. These induce partial orderings on $Q'/E$, such that if $x \neq y$ then $x <_i y$ for some $i$. This permits a choice of an element from any given finite subset of $Q'/E$; thus the map $g$ admits a definable section.

It follows in particular there exists a non empty $A$-definable subset $Y \subseteq \Gamma$ and for $y \in Y$ an element $e(y) \in Q'/E$. If $Y$ has an $A$-definable element then there exists an $A$-definable $E$-class in $Q'/E$; let $Q''$ be this class. This is always the case unless $\Gamma(A) = (0)$, $0 \notin Y$, and $Y = (0, \infty)$ or $Y = (-\infty, 0)$; but we give another argument that works in general.

For $y \in Y$ we have a $p$-germ of a function $\gamma[y]$ into $\Gamma^r$, and the germs of $y, y' \in Y$ differ by an element $(M(y,y'), d(y,y'))$ of $\mathrm{GL}_r(\mathbb{Q}) \ltimes \Gamma^r$. It is easy to cut down $Y$ so that $M(y,y') = 1$ for all $y, y'$. Indeed, let $q$ be any definable type on $Y$; then for some $M_0 \in \mathrm{GL}_r(\mathbb{Q})$, for $y \models q$ and $y' \models q|y$ we have $M(y,y') = M_0$; it follows that $M_0^2 = M_0$ so that $M_0 = 1$; replace $Y$ by $(d_q y')M(y,y') = 1$. Now we have $d(y,y') = d_1(y) - d_2(y')$ for some definable (linear) maps into $\Gamma^r$. Since $d(y,y'') = d(y,y') + d(y',y'')$ we have $d_1 = d_2$. Replace each germ

$\gamma[y]$ by $\gamma[y] - d_1(y)$. The result is another family of germs with $M(y, y') = 1$ and $d(y, y') = 0$. This means that the germ does not depend on the choice of $y \in Y$. □

**Lemma 2.10.2.** *Let $p$ be an $A$-definable type on some $A$-definable set $V$ and set $r = \mathrm{rk}_\Gamma(p)$. There exists an $A$-definable germ of maps $\delta : p \to \Gamma^r$ of maximal rank. Furthermore for any such $\delta$ the definable type $\delta_*(p)$ is $A$-definable.*

*Proof.* The existence of the germ $\delta$ follows from Lemma 2.10.1. It is clear that any two such germs differ by composition with an element of $\mathrm{GL}_r(\mathbb{Q}) \ltimes \Gamma(A)^r$. So, if one fixes such a germ, it is represented by any element of the $A$-definable family $(\gamma_a : (a \in Q''))$ in Lemma 2.10.1. The definable type $\delta_*(p)$ on $\Gamma^r$ does not depend on the choice of $\delta$ within this family, hence $\delta_*(p)$ is an $A$-definable type. □

Let $q$ be a definable type over $A$. Two pro-definable maps $h = (h_1, h_2, \ldots)$ and $g = (g_1, g_2, \ldots)$ over $B \supseteq A$ are said to have the same $q$-germ if $h(e) = g(e)$ when $e \models q|B$. The $q$-germ of $h$ is the equivalence class of $h$. So $h, g$ have the same $q$-germ if and only if the definable approximation $(h_1, \ldots, h_n), (g_1, \ldots, g_n)$ have the same $q$-germ for each $n$; and the $q$-germ of $h$ is determined by the sequence of $q$-germs of the $h_n$.

In the remainder of this section, we will use the notation $\widehat{V}$ for the space of stably dominated types on $V$, for $V$ an $A$-definable set, introduced in §3.1. In Theorem 3.1.1 we prove that $\widehat{V}$ can be canonicaly identified with a strict pro-definable set.

**Definition 2.10.3.** If $q$ is an $A$-definable type on some $A$-definable set $V$ and $h : V \to \widehat{W}$ is an $A$-definable map, there exists a unique $A$-definable type $r$ on $W$ such that for any model $M$ containing $A$, if $e \models q|M$ and $b \models h(e)|Me$ then $b \models r|M$. We refer to $r$ as the integral $\int_q h$ of $h$ along $q$. As by definition $r$ depends only of the $q$-germ $\underline{h}$ of $h$, we set $\int_q \underline{h} := \int_q h$.

Note that that for $h$ as above, if the $q$-germ $\underline{h}$ is $A$-definable (equivalently $\mathrm{Aut}(\mathbb{U}/A)$-invariant), then so is $r$; again the definition of $r$ depends on on $\underline{h}$ hence if $\underline{h}$ is $\mathrm{Aut}(\mathbb{U}/A)$ then so is $r$ (even if $h$ is not).

**Remark 2.10.4.** The notion of stably dominated type making sense for $*$-types, one can consider the space $\widehat{\widehat{V}}$ of stably dominated types on the strict pro-definable set $\widehat{V}$, for $V$ a definable set. There is a canonical map $h : \widehat{\widehat{V}} \to \widehat{V}$ sending a stably dominated type $q$ on $\widehat{V}$ to $h(q) = \int_q \mathrm{id}_{\widehat{V}}$. So $h(q)$ is a definable type, and by Proposition 2.5.5 it is stably dominated.

The following Proposition states that any definable type may be viewed as an integral of stably dominated types along some definable type on $\Gamma^r$. The proposition states the existence of certain $A$-definable germs of functions; there may be no $A$-definable function with this germ. For the notion of $A$-definable germ, see Definition 6.1 in [14].

**Proposition 2.10.5.** *Let $p$ be an $A$-definable type on some $A$-definable set $V$ and let $\delta : p \to \Gamma^r$ be as in Lemma 2.10.2. There exists an $A$-definable germ of definable function $h$ at $\delta_*(p)$ into $\widehat{V}$ such that $p = \int_{\delta_*(p)} h$.*

*Proof.* Let $M$ be a maximally complete model, and let $c \models p|M$, $t = \delta(c)$. Then $G(M(c))$ is generated over $\Gamma(M)$ by $\delta(c)$. By [13], Corollary 3.4.3 and Theorem 3.4.4, $M(t) := \mathrm{dcl}(M \cup \{t\})$ is algebraically closed. By Theorem 2.8.2 $\mathrm{tp}(c/M(t))$ is stably dominated, hence extends to a unique element $f(t, M)$ of $\widehat{V}(M(t))$.

Let $M \leq N \models \mathrm{ACVF}$, with $N$ large and saturated, and $c \models p|N$. Note that $s = \mathrm{tp}(t/N)$ is $M$-definable. We will show that the homogeneity hypotheses of Lemma 2.5.4 hold. Consider an element $b$ of $N(t) \setminus M(t)$; it has the form $h(e, t)$ with $e \in N$. Let $\bar{e}$ be the class of $e$ modulo the definable equivalence relation: $x \sim x'$ if $(d_s t)(h(x, t) = h(x', t))$. Since $b$ is not $M(t)$-definable, $\bar{e} \notin M$. Hence there exists $e' \in N$ with $\mathrm{tp}(e'/M) = \mathrm{tp}(e/M)$, but $e' \not\sim e$. So $b' = h(e', t) \neq b$, and $\mathrm{tp}(b'/M(t)) = \mathrm{tp}(b/M(t))$. Since $\mathrm{tp}(c/N(t))$ is $\mathrm{Aut}(N(t)/M(t))$-invariant, by Lemma 2.5.4, $\mathrm{tp}(c/N(t)) = f(t, M)|N(t)$.

Given two maximally complete fields $M$ and $M'$ we see by choosing $N$ containing both that $f(t, M) = f(t, M')$, so we can denote this by $f(t)$. We obtain a definable function $f : P \to \widehat{V}$, where $P = \mathrm{tp}(t)$. The $\delta_*(p)$-germ of this function $f$ does not depend on the choice of $\delta$. It follows that the germ is $\mathrm{Aut}(\mathbb{U}/A)$-invariant, hence $A$-definable; and by construction we have $p = \int_{\delta_*(p)} h$. □

2.11. **Pseudo-Galois morphisms.** We finally recall a a notion of Galois cover at the level of points; it is essentially the notion of a Galois cover in the category of varieties in which radicial morphisms (EGA I, (3.5.4)) are viewed as invertible.

[pseudogalois]

Following [30] p. 52, we call a finite surjective morphism $Y \to X$ of integral noetherian schemes a pseudo-Galois covering if the field extension $F(Y)/F(X)$ is normal and the canonical group homomorphism $\mathrm{Aut}_X(Y) \to \mathrm{Gal}(F(Y), F(X))$ is an isomorphism, where by definition $\mathrm{Gal}(F(Y), F(X))$ means $\mathrm{Aut}_{F(X)}(F(Y))$. Injectivity follows from the irreducibility of $Y$.

If $V$ is a normal irreducible variety over a field $F$ and $K'$ is a finite, normal field extension of $F(V)$, the normalization $V'$ of $V$ in $K'$ is a pseudo-Galois covering since the canonical morphism $\mathrm{Aut}_V(V') \to G = \mathrm{Gal}(K', F(V))$ is an isomorphism. This is a special case of the functoriality in $K'$ of the map taking $K'$ to the normalization of $V$ in $K'$. The action of $g \in G$ on $V'$ may be described as follows. To $g$ corresponds to a rational map $V' \to V'$; let $W_g$ be the graph of this map, a closed subvariety of $V' \times V'$. Each of the projections $W_g \to V'$ is birational, and finite. Since $V'$ is normal, these projections are isomorphisms, so $g$ is the graph of an isomorphism $V' \to V'$.

As observed in loc. cit., p. 53, if $Y \to X$ is a pseudo-Galois covering and $X$ is normal, for any morphism $X' \to X$ with $X'$ an integral noetherian scheme, the Galois group $G = \mathrm{Gal}(F(Y), F(X))$ acts transitively on the components of $X' \times_X Y$. Here is a brief argument: as $Y/G \to X$ is generically radicial and finite, it must be radicial since $X$ is normal; hence $G$ is transitive on fibers of $Y/X$. So there are no proper $G$-invariant subvarieties of $Y$. It is clear from Galois theory that $G$ acts transitively on the components of $X' \times_X Y$ mapping dominantly to $X'$; it follows that the union of these components is an $\mathrm{Gal}(F(Y), F(X))$-invariant subset mapping onto $X'$, hence is all of $X' \times_X Y$. So there are no other components.

If $Y$ is a finite disjoint union of non empty integral noetherian schemes $Y_i$, we say a finite surjective morphism $Y \to X$ is a pseudo-Galois covering if each restriction $Y_i \to X$ is a pseudo-Galois covering. Also, if $X$ is a finite disjoint union of non empty integral noetherian schemes $X_i$, we shall say $Y \to X$ is a pseudo-Galois covering if its pull-back over each $X_i$ is a pseudo-Galois covering.

## 3. The space of stably dominated types $\widehat{V}$

3.1. **$\widehat{V}$ as a pro-definable set.** We shall now work in a big saturated model $\mathbb{U}$ of ACVF in the language $\mathcal{L}_{\mathcal{G}}$. We fix a substructure $C$ of $\mathbb{U}$. If $X$ is an algebraic variety defined over the valued field part of $C$, we can view $X$ as embedded as a constructible in affine $n$-space, via some affine chart. Alternatively we could make new sorts for $\mathbb{P}^n$, and consider only quasi-projective varieties. At all events we will treat $X$ as we treat the basic sorts. By a "definable set" we mean: a definable subset of some product of sorts (and varieties), unless otherwise specified.

For a $C$-definable set $V$, and any substructure $F$ containing $C$, we denote by $\widehat{V}(F)$ the set of $F$-definable stably dominated types $p$ on $V$ (that is such that $p|F$ contains the formulas defining $V$).

We will now construct the fundamental object of the present work, initially as a pro-definable set. We will later define a topology on $\widehat{V}$.

We show that there exists a canonical pro-definable set $E$ and a canonical identification $\widehat{V}(F) = E(F)$ for any $F$. We will later denote $E$ as $\widehat{V}$.

**Theorem 3.1.1.** *Let $V$ be a $C$-definable set. Then there exists a canonical pro-$C$-definable set $E$ and a canonical identification $\widehat{V}(F) = E(F)$ for any $F$. Moreover, $E$ is strict pro-definable.*

**Remark 3.1.2.** The canonical pro-definable set $E$ described in the proof will be denoted as $\widehat{V}$ throughout the rest of the paper.

If one wishes bringing the choice of $E$ out of the proof and into a formal definition, a Grothendieck-style approach can be adopted. The pro-definable structure of $E$ determines in particular the notion of a pro-definable map $U \to E$, where $U$ is any pro-definable set. We thus have a functor from the category of pro-definable sets to the category of sets, $U \mapsto E(U)$, where $E(U)$ is the set of (pro)-definable maps from $U$ to $\widehat{V}$. This includes the functor $F \mapsto E(F)$ considered above: in case $U$ is a complete type associated with an enumeration of a structure $A$, then $\widehat{V}(U)$ can be identified with $\widehat{V}(A)$. Now instead of describing $E$ we can explicitly describe this functor. Then the representing object $E$ is uniquely determined, by Yoneda, and can be called $\widehat{V}$. Yoneda also automatically yields the functoriality of the map $V \mapsto \widehat{V}$ from the category of $C$-definable sets to the category of $C$-pro-definable sets.

In the present case, any reasonable choice of pro-definable structure satisfying the theorem will be pro-definably isomorphic to the $E$ we chose, so the more category-theoretic approach does not appear to us necessary. As usual in model theory, we will say "$Z$ is pro-definable" to mean: "$Z$ can be canonically identified with a pro-definable $E$", where no ambiguity regarding $E$ is possible.

One more remark before beginning the proof. Suppose $Z$ is a strict ind-definable set of pairs $(x, y)$, and let $\pi(Z)$ be the projection of $Z$ to the $x$-coordinate. If $Z = \cup Z_n$ with each $Z_n$ definable, then $\pi(Z) = \cup \pi(Z_n)$. Hence $\pi(Z)$ is naturally represented as an ind-definable set.

*Proof of Theorem 3.1.1.* A definable type $p$ is stably dominated if and only if it is orthogonal to $\Gamma$. The definition of $\phi(x, c) \in p$ stated in Lemma 2.5.8 clearly runs over a uniformly definable family of formulas. Hence by Lemma 2.4.1, $\widehat{V}$ is pro-definable.

To show strict pro-definability, let $f : V \times W \to \Gamma$ be a definable function. Write $f_w(v) = f(w, v)$, and define $p_*(f) : W \to \Gamma$ by $p_*(f)(w) = p_*(f_w)$. Let $Y_{W,f}$ be the subset of $\mathrm{Fn}(W, \Gamma_\infty)$ consisting of all functions $p_*(f)$, for $p$ varying in $\widehat{V}(\mathbb{U})$. By the proof of Lemma 2.4.1 it is enough to prove that $Y_{W,f}$ is definable Since by pro-definability of $\widehat{V}$, $Y_{W,f}$ is $\infty$-definable, it remains to show that it is ind-definable.

Set $Y = Y_{W,f}$ and consider the set $Z$ of quadruples $(g, h, q, L)$ such that:

(1) $L = k^n$ is a finite dimensional k-vector space
(2) $q \in \widehat{L}$;
(3) $h$ is a definable function $V \to L$ (with parameters);
(4) $g : W \to \Gamma_\infty$ is a function satisfying: $g(w) = \gamma$ if and only if

$$(d_q \bar{v})((\exists v \in V)(h(v) = \bar{v}) \& (\forall v \in V)(h(v) = \bar{v} \implies f(v, w) = \gamma)$$

i.e. for $\bar{v} \models q$, $h^{-1}(\bar{v})$ is nonempty, and for any $v \in h^{-1}(\bar{v})$, $g(w) = f(v, w)$.

Let $Z_1$ be the projection of $Z$ to the first coordinate. Note that $Z$ is strict ind-definable by Lemma 2.9.1 and hence $Z_1$ is also strict ind-definable.

Let us prove $Y \subseteq Z_1$. Take $p$ in $\widehat{V}(\mathbb{U})$, and let $g = p_*(f)$. We have to show that $g \in Z_1$. Say $p \in \widehat{V}(C')$, with $C'$ a model of ACVF and let $a \models p|C$. By Corollary 2.6.5 there exists a a $C'$-definable function $h : V \to L = k^n$ and a formula $\theta$ over $C'$ such that if $C' \subseteq B$ and $b, \gamma \in B$, if $h(a) \downarrow_{k(C')} \mathrm{St}_B$, then $f(a,b) = \gamma$ if and only if $\theta(h(a), b, \gamma)$. Let $q = \mathrm{tp}(h(a)/C')$. Then (1-4) hold and $(g, h, q, L)$ lies in $Z$.

Conversely, let $(g, h, q, L) \in Z$; say they are defined over some base set $M$; we may take $M$ to be a maximally complete model of ACVF. Let $\bar{v} \models q|M$, and pick $v \in V$ with $h(v) = \bar{v}$. Let $\bar{\gamma}$ generate $\Gamma(M(v))$ over $\Gamma(M)$. By Theorem 2.8.2 $\mathrm{tp}(v/M(\bar{\gamma})$ is stably dominated. Let $M' = \mathrm{acl}(M(\bar{\gamma}))$[1]. Let $p$ be the unique element of $\widehat{V'}(M')$ such that $p|M' = \mathrm{tp}(v/M')$. We need not have $p \in \widehat{V}(M)$, i.e. $p$ may not be $M$-definable, but since k and $\Gamma$ are orthogonal, $h_*(p)$ is $M$-definable. Thus $h_*(p)$ is the unique $M$-definable type whose restriction to $M$ is $\mathrm{tp}(\bar{v}/M)$, i.e. $h_*(p) = q$. By definition of $Z$ it follows that $p_*(f) = g$. Thus $Y = Z_1$ and $Y_{W,f}$ is strict ind-definable, hence $C$-definable. □

If $f : V \to W$ is a morphism of definable sets, we shall denote by $\widehat{f} : \widehat{V} \to \widehat{W}$ the corresponding morphism. Sometimes we shall write $f$ instead of $\widehat{f}$.

3.2. **The notion of a definable topological space.** We will consider topologies on definable and pro-definable sets $X$. With the formalism of the universal domain $\mathbb{U}$, we can view these as certain topologies on $X(\mathbb{U})$, in the usual sense. If $M$ is a model, the space $X(M)$ will not be a subspace of $X(\mathbb{U})$. It will be the topological space whose underlying set is $X(M)$, and whose topology is generated by sets $U(M)$ with $U$ an $M$-definable open set. Indeed in the case of an order topology, or any Hausdorff Ziegler topology in the sense defined below, the induced topology on a small set is always discrete.

We will say that a topological space $X$ is *definable in the sense of Ziegler* if the underlying set $X$ is definable, and there exists a definable family $B$ of definable subsets of $X$ forming a neighborhood basis at each point. This allows for a good topological logic, see [32]. But it is too restrictive for our purposes. An algebraic variety with the Zariski topology is not a definable space in this sense; nor is the topology even generated by a definable family.

Let $X$ be an $A$-definable or pro-definable set. Let $\mathcal{T}$ be a topology on $X(\mathbb{U})$, and let $\mathcal{T}_d$ be the intersection of $\mathcal{T}$ with the class of relatively $\mathbb{U}$-definable subsets of $X$. We will say that $\mathcal{T}$ is an *$A$-definable topology* if it is generated by $\mathcal{T}_d$, and for any $A$-definable family $\mathcal{W} = (W_u : u \in U)$ of relatively definable subsets of $X$, $\mathcal{W} \cap \mathcal{T}$ is ind-definable over $A$. The second condition is equivalent to $\{(x, W) : x \in W, W \subseteq X, W \in \mathcal{W} \cap \mathcal{T}\}$ is ind-definable over $A$. An equivalent definition is that the topology is generated by the union of an ind-definable family

[1]Actually $\mathrm{dcl}(M(\bar{(}\gamma)))$ is algebraically closed.

of relatively definable sets over $A$. We will also say that $(X, \mathcal{T})$ is a (pro)-definable space over $A$, or just that $X$ is a (pro)-definable space over $A$ when there is no ambiguity about $\mathcal{T}$. We say $X$ is a (pro)-definable space if it is an (pro)-$A$-definable space for some small $A$. As usual the smallest such $A$ may be recognized Galois theoretically.

If $\mathcal{T}_0$ is any ind-definable family of relatively definable subsets of $X$, the set $\mathcal{T}_1$ of finite intersections of elements of $\mathcal{T}_0$ is also ind-definable. Let $\mathcal{T}$ be the family of subsets of $X(\mathbb{U})$ that are unions of sets $Z(\mathbb{U})$, with $Z \in \mathcal{T}_1$. Then $\mathcal{T}$ is a topology on $X(\mathbb{U})$, generated by the relatively definable sets within it. By compactness, a relatively definable set $Y \subseteq X$ is in $\mathcal{T}$ if and only if for some definable $T' \subseteq \mathcal{T}_1$, $Y$ is a union of sets $Z(\mathbb{U})$ with $Z \in T'$. It follows that the topology $\mathcal{T}$ generated by $\mathcal{T}_0$ is a definable topology. In the above situation, note also that if $Y$ is $A$-relatively definable, then $Y$ is an $A$-definable union of relatively definable open sets from $T'$. Indeed, let $Y' = \{Z \in T' : Z \subseteq Y\}$, then $Y = \cup_{Z \in Y'} Z$. In general $Y$ need not be a union of sets from $\mathcal{T}_1(A)$, for any small $A$.

As is the case with groups, the notion of a pro-definable space is more general than that of pro-(definable spaces). However the spaces we will consider will be pro-(definable spaces).

When $Y$ is a definable topological space, and $A$ a base substructure, the set $Y(A)$ is topologized using the family of $A$-definable open subsets of $Y$. We do not use externally definable open subsets (i.e. $A'$-definable for larger $A$) to define the the topology on $Y(A)$; if we did, we would obtain the discrete topology on $Y(A)$ whenever $Y$ is Hausdorff. The same applies in the pro-definable case; thus in the next subsection we shall topologize $\widehat{X}(K)$ using the $K$-definable open subsets of $\widehat{X}$, restricted to $\widehat{X}(K)$.

When we speak of the topology of $Y$ without mention of $A$, we mean to take $A = \mathbb{U}$, the universal domain; often, any model will also do.

3.3. **$\widehat{V}$ as a topological space.** Assume that $V$ comes with a topology $\mathcal{T}_V$, and a sheaf $\mathcal{O}$ of definable functions into $\Gamma_\infty$. We define a topology on $\widehat{V}$ as follows. A pre-basic open set has the form: $\{p \in \widehat{O} : p_*(\phi) \in U\}$, where $O \in \mathcal{T}_V$, $U \subseteq \Gamma_\infty$ is open for the order topology, and $\phi \in \mathcal{O}(O)$. A basic open set is by definition a finite intersection of pre-basic open sets.

When $V$ is an algebraic variety, we take the topology to be the Zariski topology, and the sheaf to be the sheaf of regular functions composed with val.

When $X$ is a definable subset of a given algebraic variety $V$, we give $\widehat{X}$ the subspace topology.

3.4. **The affine case.** Assume $V$ is a definable subset of some affine variety. Let $\mathrm{Fn}_r(V, \Gamma_\infty)$ denote the functions of the form $\mathrm{val}(F)$, where $F$ is a regular function on the Zariski closure of $V$. By quantifier-elimination any definable function is piecewise a difference of the form $1/nf - 1/mg$ with $f$ and $g$ in $\mathrm{Fn}_r$ and $n$ and

$m$ positive integers. Moreover, by piecewise we mean, sets cut out by Boolean combinations of sets of the form $f \leq g$, where $f, g \in \mathrm{Fn}_r(V, \Gamma_\infty)$. It follows that if $p$ is a definable type and $p_*(f)$ is defined for $f \in \mathrm{Fn}_r(V, \Gamma_\infty)$, then $p$ is stably dominated, and determined by $p_*|\mathrm{Fn}_r(V \times W, \Gamma_\infty)$ for all $W$. A *basic open* set is defined by finitely many strict inequalities $p_*(f) < p_*(g)$, with $f, g \in \mathrm{Fn}_r(V, \Gamma_\infty)$. (In case $f = \mathrm{val}(F)$ and $g = \mathrm{val}(G)$ with $G = 0$, this is the same as $F \neq 0$.) It is easy to verify that the topology generated by these basic open sets coincides with the definition of the topology on $\widehat{V}$ above, for the Zariski topology and the sheaf of functions $\mathrm{val}(f)$, $f$ regular.

Note that if $F_1, \ldots, F_n$ are regular functions on $V$, and each $p_*(f_i)$ is continuous, with $f_i = \mathrm{val}(F_i)$, then $p \mapsto (p_*(f_1(x)), \ldots, p_*(f_n(x)))$ is continuous. Thus the topology on $\widehat{V}$ is the coarsest one such that all $p \mapsto p_*(f)$ are continuous, for $f \in \mathrm{Fn}_r(V, \Gamma_\infty)$. So the basic open sets with $f$ or $g$ constant suffice to generate the topology.

The topology on $\widehat{V}$ is strict pro-definably generated in the following sense: for each definable set $W$, one endows $\mathrm{Fn}(W, \Gamma_\infty)$ with the Tychonoff product topology induced by the order topology on $\Gamma_\infty$. Now for a definable function $f : V \times W \to \Gamma_\infty$ the topology induced on the definable set $Y_{W,f}$ is generated by a definable family of definable subsets of $Y_{W,f}$ (recall that $Y_{W,f}$ is the subset of $\mathrm{Fn}(W, \Gamma_\infty)$ consisting of all functions $p_*(f)$, for $p$ varying in $\widehat{V}(\mathbb{U})$). By definition, the pullbacks to $\widehat{V}$ of the definable open subsets of the $\mathrm{Fn}(W, \Gamma_\infty)$ generate the topology on $\widehat{V}$.

In particular, $\widehat{V}$ is a pro-definable space in the sense of § 3.2.

When $V$ is a definable subset of an algebraic variety over VF, the topology on $\widehat{V}$ can also be defined by glueing the affine pieces. It is easy to check that this is consistent (if $V'$ is an affine open of the affine $V$, obtained say by inverting $g$, then any function $\mathrm{val}(f/g)$ can be written $\mathrm{val}(f) - \mathrm{val}(g)$, hence is continuous on $\widehat{V'}$ in the topology induced from $\widehat{V}$). Moreover, this coincides with the topology defined via the sheaf of regular functions.

**Lemma 3.4.1.** *(1) If $X$ is a definable subset of $\Gamma_\infty^n$ then $X = \widehat{X}$ canonically. More generally if $U$ is a definable subset of $\mathrm{VF}^n$ or a definable subset of an algebraic variety over* VF *and $W$ is a definable subset of $\Gamma_\infty^m$, then the canonical map $\widehat{U} \times W \to \widehat{U \times W}$ is a bijection.*

*(2) Let $h : V \to U$ be a morphism of varieties, and let $X \subset \widehat{V/U}$ be relatively $\Gamma$-internal over $U$. In other words, $X$ is a relatively definable subset of $\widehat{V}$, the projection of $X$ to $\widehat{U}$ consists of simple points, and the fibers $X_u$ of $X \to U$ are $\Gamma$-internal, uniformly in $u \in U$. Then there exists a natural embedding $\theta : \widehat{X} \to \widehat{V}$, over $\widehat{U}$; over a simple point $u \in \widehat{U}$, $\theta$ restricts to the identification of $\widehat{X_u}$ with $X_u$.*

*Proof.* (1) Let $f : U \times W \to U, g : W : U \times W \to W$ be the projections. If $p \in \widehat{U \times W}$ we saw that $g_*(p)$ concentrates on some $a \in W$; so $p = f_*(p) \times g_*(p)$ (i.e. $p(u, w)$ is generated by $f_*(p)(u) \cup g_*(p)(w)$).

(2) Let $h_X : X \to U$ be the natural map. Let $p \in \widehat{X}$; let $A = \mathrm{acl}(A)$ be such that $p$ is $A$-definable; and let $c \models p|A$, $u = h_X(c)$. Since $\mathrm{tp}(c/A(u))$ is $\Gamma$-internal, by Lemma 2.7.1 (5) there exists an $\mathrm{acl}(A(u))$-definable injective map $j$ with $j(c) \in \Gamma^m$. But $\mathrm{acl}(A(c)) \cap \Gamma = \Gamma(A)$. So $j(c) = \alpha \in \Gamma(A)$, and $c = j^{-1}(\alpha) \in \mathrm{acl}(A(u))$. Let $v \models c|\,\mathrm{acl}(A(u))$, and let $\theta(p)$ be the unique stably dominated, $A$-definable type extending $\mathrm{tp}(v/A)$. So $\theta(p) \in \widehat{V}$, and $h_X(p) = h_*\theta(p)$. □

If $U$ is a definable subset of an algebraic variety over VF, we endow $\widehat{U} \times \Gamma_\infty^m \simeq \widehat{U \times \Gamma_\infty^m}$ with the quotient topology for the surjective mapping $\widehat{U \times \mathbb{A}^m} \to \widehat{U \times \Gamma_\infty^m}$ induced by $\mathrm{id} \times \mathrm{val}$.

We will see below (as a special case of Lemma 3.4.3) that the topology on $\Gamma_\infty = \widehat{\Gamma_\infty}$ is the order topology, and the topology on $\widehat{\Gamma_\infty^m} = \Gamma_\infty^m$, is the product topology.

If $b$ is a closed ball in $\mathbb{A}^1$, let $p_b \in \widehat{\mathbb{A}^1}$ be the generic type of $b$: it can be defined by $(p_b)_*(f) = \min\{\mathrm{val} f(x) : x \in b\}$, for any polynomial $f$. This applies even when $b$ has valuative radius $\infty$, i.e. consists of a single point. The generic type of a finite product of balls is defined by exactly the same formula; we have, in the notation of Remark 3.5.3, $p_{b \times b'} = p_b \otimes p_{b'}$.

For $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma_\infty^n$, let $b(\gamma) = \{x = (x_1, \ldots, x_n) \in \mathbb{A}_1^n : \mathrm{val}(x_i) \geq \gamma_i, i = 1, \ldots, n\}$. Let $p_\gamma = p_{b(\gamma)}$.

**Lemma 3.4.2.** *The map $j : \widehat{\mathbb{A}^n} \times \Gamma_\infty \to \widehat{\mathbb{A}^{n+1}}$, $(q, \gamma) \mapsto q \otimes p_\gamma$ is continuous for the product topology of $\widehat{\mathbb{A}^n}$ with the order topology on $\Gamma$.*

*Proof.* We have to show that for each polynomial $f(x_1, \cdots, x_n, y)$ with coefficients in VF, the map $(p, \gamma) \mapsto j(p, \gamma)_* f$ is continuous. The functions min and $+$ extend naturally to continuous functions $\Gamma_\infty^2 \to \Gamma_\infty$. Now if $f(x_1, \cdots, x_n, y)$ is a polynomial with coefficients in VF, there exists a function $P(\gamma_1, \cdots, \gamma_n, \tau)$ obtained by composition of min and $+$, and polynomials $h_i$ such that $\min_{\mathrm{val}(y)=\alpha} \mathrm{val}(f(x_1, \cdots, x_n, y)) = P(\mathrm{val}(h_1(x)), \cdots, \mathrm{val}(h_d(x)), \alpha)$, namely, $\min_{\mathrm{val}(y)=\alpha} \mathrm{val}(\sum h_i(x) y^i) = \min_i(\mathrm{val}(h_i(x)) + i\alpha)$. So $P : \Gamma_\infty^{n+1} \to \Gamma_\infty$ is continuous. Hence $j(p, \gamma)_* f = P(p_*(h_1), \ldots, p_*(h_d), \gamma)$. Continuity follows, by composition. □

**Lemma 3.4.3.** *If $U$ is a definable subset of $\mathbb{A}^n \times \Gamma_\infty^\ell$ and $W$ is a definable subset of $\Gamma_\infty^m$, the induced topology on $\widehat{U \times W} = \widehat{U} \times W$ coincides with the product topology.*

*Proof.* We have seen that the natural map $\widehat{U \times W} \to \widehat{U} \times W$ is bijective; it is clearly continuous, where $\widehat{U} \times W$ is given the product topology. To show that it is closed, it suffices to show that the inverse map is continuous, and we may take $U = \mathbb{A}^n$ and $W = \Gamma_\infty^m$. By factoring $\widehat{U \times G_\infty^m} \to \widehat{U \times \Gamma_\infty^{m-1}} \times \Gamma_\infty \to \widehat{U} \times \Gamma_\infty^{m-1} \times \Gamma_\infty$, we may assume $m = 1$. Having said this, by pulling back to $\mathbb{A}^{n+\ell}$ we may assume $\ell = 0$. The inverse map is equal to the composition of $j$ as in Lemma 3.4.2 with a projection, hence is continuous. □

In the lemma below, $\mathbf{F}$ is not necessarily a field; it could be any structure consisting of field points and $\Gamma$-points.

**Lemma 3.4.4.** *Let $V$ be a variety over a valued field $F$ and let $U$ be an $F$-definable subset of $V$. Let $\mathbf{F}$ be any structure consisting of field points and $\Gamma$-points, including at least one positive element of $\Gamma$. Let $\mathbf{F} \leq A$. If $\widehat{U}(A)$ is open in $\widehat{V}(A)$, then $\widehat{U}(\mathbf{F})$ is open in $\widehat{V}(\mathbf{F})$.*

*Proof.* Covering $V$ by affines, we may assume $V$ is affine.

Assume first $\mathbf{F} \subseteq \mathrm{dcl}(F)$. In particular, by assumption, $F$ is not trivially valued. Let $p \in \widehat{U}(F)$. There exist regular functions $G_1, \dots, G_n$ and intervals $I_j$ of $\Gamma_\infty$ such that $p \in \cap_j \widehat{g_j}^{-1}(I_j) \subset \widehat{U}$, with $g_j = \mathrm{val}(G_j)$. By definability of $p$, and since $F^{alg}$ is an elementary submodel, we can choose $G_1, \dots, G_n$ to be definef over over $F^{alg}$. So it suffices to show, for each $j$, that the intersection of Galois conjugates of $\widehat{g_j}^{-1}(I_j)$ contains an open neighborhood of $p$ in $\widehat{V}(F)$. Let $G = G_j$, $g = g_j$ and $I = I_j$, and let $G^\nu$ be the Galois conjugates of $G$ over $F$, $g^\nu = \mathrm{val}(G^\nu)$.

Let $b \models p$. Then the $G^\nu$ are Galois conjugate over $F(b)$, $p$ being $F$-definable. The elements $c_\nu = G^\nu(b)$ are Galois conjugate over $F(b)$; they are the roots of a polynomial $H(b, y) = \Pi_\nu(y - G^\nu(b)) = \sum_m h_\mu(b) y^m$. For all $b'$ in some $F$-definable Zariski open set $U'$ containing $b$, the set of roots of $H(b', y)$ is equal to $\{G^\nu(b')\}$. Within $U'$, the set of $b'$ such that, for all $\nu$, $g^\nu(b') \in I$ can therefore be written in terms of the Newton polygon of $H(b', y)$, i.e. in terms of certain inequalities between convex expressions in $\mathrm{val}(h_k(b'))$. This shows that the intersection of Galois conjugates of $\widehat{G}^{-1}(I)$ contains an open neighborhood of $p$.

This argument shows more generally the topology of $\widehat{V}(\mathbf{F})$ is the same as the topology induced from $\widehat{V}(\mathrm{acl}(\mathbf{F}))$. Hence from now on we assume $F$ is algebraically closed.

We now have to deal with the case that $\mathbf{F}$ is bigger than $F$; we may assume $\mathbf{F}$ is generated over $F$ by finitely many elements of $\Gamma$, and indeed, adding one element at a time, that $\mathbf{F} = F(\gamma)$ for some $\gamma \in \Gamma$. Let $c$ be a field element with $\mathrm{val}(c) = \gamma$; it suffices to show that if $U$ is open over $F(c)$, then it is over $F$ too. Let $G(x, c) = \sum G_i(x) c^i$ be a polynomial (where $x = (x_1, \dots, x_n), V \leq \mathbb{A}^n$.) Let $g(p, c)$ be the generic value of $\mathrm{val}(G(x, c))$ at $p$ and $g_i(p)$ the one of $\mathrm{val}(G_i)$. Then $g(p, c) = \min_i g_i(p) + i\gamma$. From this the statement is clear. □

Note that the lemma would not quite be true over a trivially valued field $F$, though it is true over the two-sorted $(F, \mathbb{R})$; the latter will be used in the Berkovich setting.

3.5. **Simple points.** For any definable set $X$, we have an embedding $X \to \widehat{X}$, taking a point $x$ to the definable type concentrating on $x$. The points of the image are said to be *simple*.

**Lemma 3.5.1.** *Let $X$ be a definable subset of* $\mathrm{VF}^n$*.*

(1) *The set of simple points of $\widehat{X}$ (which we identify with $X$) is a relatively definable dense subset of $\widehat{X}$. If $M$ is a model of* ACVF*, then $X(M)$ is dense in $\widehat{X}(M)$.*
(2) *The induced topology on $X$ agrees with the valuation topology on $X$.*

*Proof.* (1) For relative definability, note that a point of $\widehat{X}$ is simple if and only if each of its projections to $\widehat{\mathbb{A}^1}$ is simple and that on $\mathbb{A}^1$, the points are a definable subset of the closed balls (cf. Example 7.1.2). For density, consider (for instance) $p \in \widehat{X}(M)$ with $p_*(f) > \alpha$. Then $\mathrm{val} f(x) > \alpha \wedge x \in X$ is satisfiable in $M$, hence there exists a simple point $q \in \widehat{X}(M)$ with $q_*(f) > \alpha$.

(2) Clear from the definitions. The basic open subsets of the valuation topology are of the form $\mathrm{val} f(x) > \alpha$ or $\mathrm{val} f(x) < \alpha$. □

We write $\mathrm{VF}^*$ for $\mathrm{VF}^n$ when we do not need to specify $n$.

**Lemma 3.5.2.** *Let $f : U \to V$ be a definable map between definable subsets of* $\mathrm{VF}^*$*. If $f$ has finite fibers, then the preimage of a simple point of $\widehat{V}$ under $\widehat{f}$ is simple in $\widehat{U}$.*

*Proof.* It is enough to prove that if $X$ is a finite definable subset of $\mathrm{VF}^n$, then $X = \widehat{X}$, which is clear by (1) of Lemma 3.5.1. □

**Remark 3.5.3.** The natural projection $S_{def}(U \times V) \to S_{def}(U) \times S_{def}(V)$ admits a natural section, namely $\otimes : S_{def}(U) \times S_{def}(V) \to S_{def}(U \times V)$. It restricts to a section of $\widehat{U \times V} \to \widehat{U} \times \widehat{V}$. This map is not continuous in the logic topology, nor is its restriction to $\widehat{U} \times \widehat{V} \to \widehat{U \times V}$ continuous. Indeed when $U = V$ the pullback of the diagonal $\widehat{\Delta_U}$ consists of simple points on the diagonal $\Delta_{\widehat{U}}$. But over a model, the set of simple points is dense, and hence not closed.

3.6. **v-open and g-open subsets, v+g-continuity.**

**Definition 3.6.1.** Let $V$ be a an algebraic variety over a valued field $F$. A definable subset of $V$ is said to be v-open if open for the valuation topology. It is called g-open if it is defined by a positive Boolean combination of Zariski closed and open sets, and sets of the first form above, $\{u : \mathrm{val} f(u) > \mathrm{val} g(u)\}$. More generally, if $V$ is a definable subset of an algebraic variety $W$, a definable subset of $V$ is said to be v-open (resp. g-open) if it is of the form $V \cap O$ with $O$ v-open (resp. g-open) in $W$. A definable subset of $V \times \Gamma_\infty^m$ is called v-or g-open if its pullback to $V \times \mathbb{A}^m$ via $\mathrm{id} \times \mathrm{val}$ is.

**Remark 3.6.2.** If $X$ is $A$-definable, the regular functions $f$ and $g$ in the definition of g-openness are *not* assumed to be $A$-definable; in general when $A$ consists of imaginaries, no such $f, g$ can be found. However when $A = \mathrm{dcl}(F)$ with $F$ a valued field, they may be taken to be $F$-definable, by Lemma 8.1.1. For $A$ a substructure consisting of imaginaries, this is not the case.

**Definition 3.6.3.** Let $V$ be an algebraic variety over a valued field $F$ or a definable subset of such a variety. A definable function $h : V \to \Gamma_\infty$ is called g-continuous if the pullback of any g-open set is g-open. A function $h : V \to \widehat{W}$ with $W$ an affine $F$-variety is called g-continuous if, for any regular function $f : W \to \mathbb{A}^1$, $\mathrm{val} \circ f \circ h$ is g-continuous.

Note that the topology generated by v-open subsets on $\Gamma_\infty$ is discrete on $\Gamma$, while the neighborhoods of $\infty$ in this topology are the same as in the order topology. The topology generated by g-open subsets is the order topology on $\Gamma$, with $\infty$ isolated. We also have the topology on $\Gamma_\infty$ coming from its canonical identification with $\widehat{\Gamma_\infty}$, or the v+g topology; this is the intersection of the two previous topologies, that is, the order topology on $\Gamma_\infty$.

From now on let $V$ be an algebraic variety over a valued field $F$ or a definable subset of such a variety. We say that a definable subset is v+g-open if it is both v-open and g-open. If $W$ has a definable topology, a definable function $V \to W$ is called v+g-continuous if the pullback of a definable open subset of $W$ is both v-and g-open, and similarly for functions to $V$.

Note that v, g and v+g-open sets are *definable* sets. Over any given model is possible to extend v to a topology in the usual sense, the valuation topology, whose restriction to definable sets is the family of v-open sets. But this is not true of g and of v+g; in fact they are not closed under definable unions.

Any $g$-closed subset $W$ of an algebraic variety is defined by a disjunction $\bigvee_{i=1}^m (\neg H_i \wedge \phi_i)$, with $\phi_i$ a finite conjunction of weak valuation inequalities $v(f) \leq v(g)$ and equalities, and $H_i$ defining a Zariski closed subset. If $W$ is also v-closed, $W$ is equal to the union of the $v$-closures of the sets defined by $\neg H_i \wedge \phi_i$, $1 \leq i \leq m$.

**Lemma 3.6.4.** *Let $W$ be a v+g-closed definable subset of the affine space $\mathbb{A}^n$ over a valued field. Then $\widehat{W}$ is closed in $\widehat{\mathbb{A}^n}$. More generally, if $W$ is g-closed then $cl(\widehat{W}) \subseteq \widehat{cl_v(W)}$, with $cl_v$ denoting the v-closure.*

*Proof.* Let $M$ be a model, $p \in \widehat{\mathbb{A}^n}(M)$, with $p \in cl(\widehat{W}(M))$. We will show that $p \in \widehat{cl_v(W)}$. Let $(p_i)$ be a net in $\widehat{W}(M)$ approaching $p$. Let $a_i \models p_i|M$. Let $\mathrm{tp}(a/M)$ be a limit type in the logic topology (so $a$ can be represented by an ultraproduct of the $a_i$). For each $i$ we have $\Gamma(M(a_i)) = \Gamma(M)$, but $\Gamma(M(a))$

may be bigger. Consider the subset $C$ of $\Gamma(M(a))$ consisting of those elements $\gamma$ such that $-\alpha < \gamma < \alpha$ for all $\alpha > 0$ in $\Gamma(M)$. Thus $C$ is a convex subgroup of $\Gamma(M(a))$; let $N$ be the valued field extension of $M$ with the same underlying $M$-algebra structure, obtained by factoring out $C$. Let $\bar{a}$ denote $a$ as an element of $N$. We have $a_i \in W$, so $a \in W$; since $W$ is g-closed it is clear that $\bar{a} \in W$. (This is the easy direction of Lemma 8.1.1.) Let $b \models p|M$. For any regular function $f$ in $M[U]$, with $U$ Zariski open in $\mathbb{A}^n$, we have $\mathrm{val} f(a_i) \to \mathrm{val} f(b)$ in $\Gamma_\infty(M)$. In particular if $\mathrm{val} f(\bar{a}) = \infty$, or just if $\mathrm{val} f(\bar{a}) > \mathrm{val}(M)$, then $f(b) = 0$. Let $R = \{x \in N : (\exists m \in M)(\mathrm{val}(x) \geq \mathrm{val}(m))\}$. Then $R$ is a valuation ring of $N$ over $M$, with residue field isomorphic to $M(b)$, the residue map taking $\bar{a}$ to $b$. Since $\bar{a} \in W$, it follows that $b \in cl_v(W)$ (see § 8.2 for more detail), hence $p \in \widehat{cl_v(W)}$. □

3.7. **Canonical extensions.** Let $V$ be a definable set over some $A$ and let $f : V \to \widehat{W}$ be a $A$-definable map (that is, a morphism in the category of pro-definable sets), where $W$ is an $A$-definable subset of $\mathbb{P}^n \times \Gamma_\infty^m$. We can define a canonical extension to $F : \widehat{V} \to \widehat{W}$, as follows.

If $p \in \widehat{V}(M)$, say $p|M = \mathrm{tp}(c/M)$, let $d \models f(c)|M(c)$. By transitivity of stable domination (Proposition 2.5.5), $\mathrm{tp}(cd/M)$ is stably dominated, and hence so is $\mathrm{tp}(d/M)$. Let $F(c) \in \widehat{W}(M)$ be such that $F(c)|M = \mathrm{tp}(d/M)$; this does not depend on $d$. Moreover $F(c)$ depends only on $\mathrm{tp}(c/M)$, so we can let $F(p) = F(c)$. Note that $F : \widehat{V} \to \widehat{W}$ is a pro-$A$-definable morphism.

**Lemma 3.7.1.** *Let $f : V \to \widehat{W}$ be a definable function, where $V$ is an algebraic variety and $W$ is a definable subset of $\mathbb{P}^n \times \Gamma_\infty^m$. Let $X$ be a definable subset of $V$. Assume $f$ is g-continuous and v-continuous at each point of $X$; i.e. $f^{-1}(G)$ is g-open whenever $G$ is open, and $f^{-1}(G)$ is open at $x$ whenever $G$ is open, for any $x \in X \cap f^{-1}(G)$. Then the canonical extension $F$ is continuous at each point of $\widehat{X}$.*

*Proof.* The topology on $\widehat{\mathbb{P}^n}$ may be described as follows. It is generated by the preimages of open sets of $\Gamma_\infty^N$ under continuous definable functions $\mathbb{P}^n \to \Gamma_\infty^N$ of the form: $(x_0 : \ldots : x_n) \mapsto (\mathrm{val}(x_0^d) : \ldots : \mathrm{val}(x_n^d) : \mathrm{val}(h_1) : \ldots : \mathrm{val}(h_{N-n}))$ for some homogeneous polynomials $h_i(x_0 : \ldots : x_n)$ of degree $d$; where in $\Gamma^N$ we define $(u_0 : \ldots : u_m)$ to be $(u_0 - u_*, \ldots, u_m - u_*)$, with $u_* = \min u_i$. Composing with such functions we reduce to the case of $\Gamma_\infty^m$, and hence to the case of $f : V \to \Gamma_\infty$.

Let $U = f^{-1}(G)$ be the $f$-pullback of a definable open subset $G$ of $\Gamma_\infty$. Then $F^{-1}(G) = \widehat{U}$. Now $U$ is g-open, and v-open at any $x \in X \cap U$. By Lemma 3.6.4 applied to the complement of $U$, it follows that $\widehat{U}$ is open at any $x \in \widehat{X}$. □

**Lemma 3.7.2.** *Let $K$ be a valued field and $V$ be an algebraic variety over $K$. Let $X$ be a $K$-definable subset of $V$ and let $f : X \to \widehat{W}$ be a $K$-definable function,* *with $W$ is a $K$-definable subset of $\mathbb{P}^n \times \Gamma_\infty^m$. Assume $f$ is v+g-continuous. Then $f$ extends uniquely to a continuous pro-$K$-definable morphism $F : \widehat{X} \to \widehat{W}$.*

*Proof.* Existence of a continuous extension follows from Lemma 3.7.1. There is clearly at most one such extension, because of the density in $\widehat{X}$ of the simple points $X(\mathbb{U})$, cf. Lemma 3.5.1. □

**Lemma 3.7.3.** *Let $K$ be a valued field and $V$ be an algebraic variety over $K$. Let $f : I \times V \to \widehat{V}$ be a g-continuous $K$-definable function, where $I = [a, b]$ is a closed interval. Let $i_I$ denote one of $a$ or $b$ and $e_I$ denote the remaining point. Let $X$ be a $K$-definable subset of $V$. Assume $f$ restricts to a definable function $g : I \times X \to \widehat{X}$ and that $f$ is v-continuous at every point of $I \times X$. Then $g$ extends uniquely to a continuous pro-$K$-definable morphism $G : I \times \widehat{X} \to \widehat{X}$. If moreover, for every $v \in X$, $g(i_I, v) = v$ and $g(e_I, v) \in Z$, with $Z$ a $\Gamma$-internal subset, then $G(i_I, x) = x$, and $G(e_I, x) \in Z$.*

*Proof.* Since $\widehat{I \times V} = I \times \widehat{V}$ by Lemma 3.4.1, the first statement follows from Lemma 3.7.1, by considering the pull-back of $I$ in $\mathbb{A}^1$. The equation $G(i_I, x) = x$ extends by continuity from the dense set of simple points to $\widehat{X}$. We have by construction $G(e_I, x) \in Z$, using the fact that any stably dominated type on $Z$ is constant. □

3.8. **Good metrics.** By a definable metric on an algebraic variety $V$ over a valued field $F$, we mean an $F$-definable function $d : V^2 \to \Gamma_\infty$ which is v+g-continuous and such that

(1) $d(x, y) = d(y, x)$; $d(x, x) = \infty$.
(2) $d(x, z) \geq \min(d(x, y), d(y, z))$,
(3) If $d(x, y) = \infty$ then $x = y$.

Note that given a definable metric on $V$, for any $v \in V$, $B(v; d, \gamma) := \{y : d(v, y) \geq \gamma\}$ is a family of g-closed, v-clopen sets whose intersection is $\{v\}$.

We call $d$ a *good metric* if there exists a v+g-continuous definable function $\rho : V \to \Gamma$ (so $\rho(v) < \infty$), such that for any $v \in V$ and any $\alpha > \rho(v)$, $B(v; d, \alpha)$ has a unique generic type; i.e. if there exists a definable type $p$ such that for any Zariski closed $V' \subseteq V$ not containing $B(v; d, \alpha)$ and any regular $f$ on $V \smallsetminus V'$, $p$ concentrates on $B(v; d, \alpha) \smallsetminus V'$, and $p_*(f)$ attains the minimum valuation of $f$ on $B(v; d, \alpha) \smallsetminus V'$. Such a type is orthogonal to $\Gamma$, hence stably dominated.

**Lemma 3.8.1.** (1) *$\mathbb{P}^n$ admits a good definable metric, with $\rho = 0$.*
(2) *Let $F$ be a valued field, $V$ a quasi-projective variety over $F$. Then there exists a definable metric on $V$.*

*Proof.* Consider first the case of $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$. Define $d(x, y) = d(x^{-1}, y^{-1}) = \mathrm{val}(x - y)$ if $x, y \in \mathcal{O}$, $d(x, y) = 0$ if $v(x), v(y)$ have different signs. This is easily checked to be consistent, and to satisfy the conditions (1-3). It is also clearly v+g-continuous. If $F \leq K$ is a valued field extension, $\pi : \Gamma(K) \to \bar{\Gamma}$ a homomorphism of ordered $\mathbb{Q}$-spaces extending $\Gamma(F)$, and $\mathbf{K} = (K, \pi \circ v)$, we have to check (Lemma 8.1.1) that $\pi(d_K(x, y)) = d_{\mathbf{K}}(x, y)$. If $x, y \in \mathcal{O}_K$ then $x, y \in \mathcal{O}_{\mathbf{K}}$ and $\pi d_K(x, y) = \pi v_K(x - y) = d_{\mathbf{K}}(x, y)$. Similarly for $x^{-1}, y^{-1}$. If $v(x) < 0 < v(y)$, then $v(x-y) < 0$ so $\pi(v(x-y)) \leq 0$, hence $d_{\mathbf{K}}(x, y) = 0 = d_K(x, y)$. This proves the g-continuity. It is clear that the metric is good, with $\rho = 0$.

Now consider $\mathbb{P}^n$ with homogeneous coordinates $[X_0 : \cdots : X_n]$. For $0 \leq i \leq n$ denote by $U_i$ the subset $\{x \in \mathbb{P}^n : X_i \neq 0 \wedge \inf \mathrm{val}(X_j/X_i) \geq 0\}$. If $x$ and $y$ belong both to $U_i$ one sets $d(x, y) = \inf \mathrm{val}(X_j/X_i - Y_j/Y_i)$. If $x \in U_i$ and $y \notin U_i$, one sets $d(x, y) = 0$. One checks that this definition in unambigous and reduces to the former one when $n = 1$. The proof it is v+g-continuous is similar to the case $n = 1$ and the fact it is good with $\rho = 0$ is clear. This metric restricts to a metric on any subvariety of $\mathbb{P}^n$. □

A good metric provides, in a uniform way, the kind of descending family of closed balls that we noted for curves; but uniqueness of the germ of this family is special to curves.

3.9. **Zariski topology.** We shall occasionally use the Zariski topology on $\widehat{V}$. If $V$ is an algebraic variety over a valued field, a subset of $\widehat{V}$ of the form $\widehat{F}$ with $F$ Zariski closed, resp. open, in $V$ is said to be Zariski closed, resp. open. Similarly, a subset $E$ of $\widehat{V}$ is said to be Zariski dense in $\widehat{V}$ if $\widehat{V}$ is the only Zariski closed set containing $E$. For $X \subset \widehat{V}$, the Zariski topology on $X$ is the one induced from the Zariski topology on $\widehat{V}$.

## 4. Definable compactness

4.1. **Definition of definable compactness.** Let $X$ be a definable or pro-definable topological space in the sense of § 3.2. Let $p$ be a definable type on $X$.

**Definition 4.1.1.** A point $a \in X$ is a *limit* of $p$ if for any definable neighborhood $U$ of $a$ (defined with parameters), $p$ concentrates on $U$.

When $X$ is Hausdorff, it is clear that a limit point is unique if it exists.

**Definition 4.1.2.** Let $X$ be a definable or pro-definable topological space. One says $X$ is *definably compact* if any definable type $p$ on $X$ has a limit point in $X$.

For subspaces of $\Gamma^n$ with $\Gamma$ o-minimal, our definition of definable compactness Definition 4.1.2 lies between the definition of [23] in terms of curves, and the property of being closed and bounded; so all three are equivalent. This will be treated in more detail later.

4.2. **Characterization.** A subset of $\mathrm{VF}^n$ is said to be *bounded* if for some $\gamma$ in $\Gamma$ it is contained in $\{(x_1, \ldots, x_n : v(x_i) \geq \gamma, 1 \leq i \leq n\}$. This notion extends to varieties $V$ over a valued field, cf., e.g., [26] p. 81: $X \subseteq V$ is defined to be bounded if there exists an affine covering $V = \cup_{i=1}^m U_i$, and bounded subsets $X_i \subseteq U_i$, with $X \subseteq \cup_{i=1}^m X_i$. Note that projective space $\mathbb{P}^n$ is bounded within itself, and so any subset of a projective variety $V$ is bounded in $V$.

We shall say a subset of $\Gamma_\infty^m$ is *bounded* if it is contained in $[a, \infty]^m$ for some $m$. More generally a subset of $\mathrm{VF}^n \times \Gamma_\infty^m$ is bounded if its pullback to $\mathrm{VF}^{n+m}$ is bounded.

We will use definable types as a replacement for the curve selection lemma, whose purpose is often to use the definable type associated with a curve at a point. Note that the curve selection lemma itself is not true for $\Gamma_\infty$, e.g. in $\{(x, y) \in \Gamma_\infty^2 : y > 0, x < \infty\}$ there is no curve approaching $(\infty, 0)$.

Note that if $V$ is a definable set, the notion of definable type on the strict pro-definable set $\widehat{V}$ makes sense, since the notion of a definable $*$-type, i.e. type in infinitely many variables, or equivalently a definable type on a pro-definable set, is clear.

Let $Y$ be a definable subset of $\Gamma_\infty$. Let $q$ be a definable type on $Y$. Then $\lim q$ be the unique $\alpha \in Y$, if any, such that $q$ concentrates on any neighborhood of $\alpha$. It is easy to see that if $Y$ is bounded then $\alpha$ exists, by considering the $q(x)$-definition of the formula $x > y$; it must have the form $y < \alpha$ or $y \leq \alpha$.

Let $V$ be a definable set and let $q$ be a definable type on $\widehat{V}$. Clearly $\lim q$ exists if there exists $r \in \widehat{V}$ such that for any continuous pro-definable function $f : \widehat{V} \to \Gamma_\infty$, $\lim f_*(q)$ exists and

$$f(r) = \lim f_*(q).$$

If $r$ exists it is clearly unique, and denoted $\lim q$.

**Lemma 4.2.1.** *Let $V$ be an affine variety over a valued field and let $q$ be a definable type on $\widehat{V}$. We have $\lim q = r$ if and only if for any regular $H$ on $V$, setting $h = \mathrm{val} \circ H$,*

$$r_*(h) = \lim h_*(q).$$

*Proof.* One implication is clear, let us prove the reverse one. Indeed, by hypothesis, for any pro-definable neighborhood $W$ of $a$, $p$ implies $x \in W$. In particular, if $U$ is a definable neighborhood of $f(a)$, $p$ implies $x \in f^{-1}(U)$, hence $f_*(p)$ implies $x \in U$. It follows that $\lim f_*(p) = f(a)$. □

**Lemma 4.2.2.** *Let $X$ be a bounded definable subset of an algebraic variety $V$ over a valued field and let $q$ be a definable type on $\widehat{X}$. Then $\lim q$ exists in $\widehat{V}$.*

*Proof.* It is possible to partition $V$ into open affine subsets such that $X$ intersects each affine open in a bounded set. We may thus assume $V$ is affine; and indeed that $X$ is a bounded subset of $\mathbb{A}^n$. For any regular $H$ on $V$, setting $h = \mathrm{val} \circ H$,

$h(X)$ is a bounded subset of $\Gamma_\infty$ and $h_*(q)$ is a definable type on $h(X)$, hence has a limit $\lim h_*(q)$.

Now let $K$ be an algebraically closed valued field containing the base of definition of $V$ and $q$. Fix $d \models q|K$ and $a \models p_d|K(d)$, where $p_d$ is the type coded by the element $d \in \widehat{V}$. Let $B = \Gamma(K)$, $N = K(d,a)$ and $B' = \Gamma(N)$. Hence $B$ is a divisible ordered abelian group. We have $\Gamma(N) = \Gamma(K(d))$ by orthogonality to $\Gamma$ of $p_d$. Since $q$ is definable, for any $e \in B'$, $\mathrm{tp}(e/B)$ is definable; in particular the cut of $e$ over $B$ is definable. Set $B'_0 = \{b' \in B' : (\exists b \in B) b < b'\}$. It follows that if $e \in B'_0$ there exists an element $\pi(e) \in B \cup \{\infty\}$ which is nearest $e$. Note $\pi : B'_0 \to B_\infty$ is an order-preserving retraction and a homomorphism in the obvious sense. The ring $R = \{a \in K(d) : \mathrm{val}(a) \in B'_0\}$ is a valuation ring of $K(d)$, containing $K$. Also $d$ has its coordinates in $R$, because of the boundedness assumption on $X$. Consider the maximal ideal $M = \{a \in K(d) : \mathrm{val}(a) > B\}$ and set $K' = R/M$. We have a canonical homomorphism $R[d] \to K'$; let $d'$ be the image of $d$. We have a valuation on $K'$ extending the one on $K$, namely $\mathrm{val}(x + M) = \pi(\mathrm{val}(x))$. So $K'$ is a valued field extension of $K$, embeddable in some elementary extension. Let $r = \mathrm{tp}(d'/K)$. Then $r$ is definable and stably dominated; the easiest way to see that is to assume $K$ is maximally complete (as we may); in this case stable domination follows from $\Gamma(K(d')) = \Gamma(K)$ by Theorem 2.8.2. The fact that $r_*(h) = \lim h_*(q)$ is a direct consequence from the definitions. $\square$

Let $V$ be a definable set. According to Definition 4.1.2 a pro-definable $X \subseteq \widehat{V}$ is definably compact if for any definable type $q$ on $X$ we have $\lim q \in X$.

**Remark 4.2.3.** Under this definition, any intersection of definably compact sets is definably compact. In particular an interval such as $\cap_n [0, 1/n]$ in $\Gamma$. However we mostly have in mind strict pro-definable sets.

**Lemma 4.2.4.** *Let $V$ be an algebraic variety over a valued field, $Y$ a closed pro-definable subset of $\widehat{V}$. Let $q$ be a definable type on $Y$, and suppose* $\lim q$ *exists. Then* $\lim q \in Y$.

*Hence if $Y$ is bounded (i.e. it is a subset of $\widehat{X}$ for some bounded definable $X \subseteq V$) and closed in $\widehat{V}$, then $Y$ is definably compact.*

*Proof.* The fact that $\lim q \in Y$ when $Y$ is closed follows from the definition of the topology on $\widehat{V}$. The second statement thus follows from Lemma 4.2.2. $\square$

**Definition 4.2.5.** Let $T$ be a theory with universal domain $\mathbb{U}$. Let $\Gamma$ be a stably embedded sort with a $\emptyset$-definable linear ordering. Recall $T$ is said to be *metastable* over $\Gamma$ if for any small $C \subset \mathbb{U}$, the following condition is satisfied:

(MS) For some small $B$ containing $C$, for any $a$ belonging to a finite product of sorts, $\mathrm{tp}(a/B, \Gamma(Ba))$ is stably dominated.

Such a $B$ is called a metastability base. It follows from Theorem 12.18 from [14] that ACVF is metastable.

Let $T$ be any theory, $X$ and $Y$ be pro-definable sets, and $f : X \to Y$ a surjective pro-definable map. The $f$ induces a map $f_{def} : S_{def}(X) \to S_{def}(Y)$ from the set of definable types on $X$ to the set of definable types on $Y$.

**Lemma 4.2.6.** *Let $f : X \to Y$ a surjective pro-definable map between pro-definable sets.*

(1) *Assume $T$ is o-minimal. Then $f_{def}$ is surjective.*
(2) *Assume $T$ is metastable over some o-minimal $\Gamma$. Then $f_{def}$ is surjective, moreover it restricts to a surjective $\widehat{X} \to \widehat{Y}$.*

**Remarks 4.2.7.**
(1) It is not true that either of these maps is surjective over a given base set $F$, nor even that the image of $S_{def}(X)$ contains $\widehat{Y}(F)$ (e.g. take $X$ a finite set, $Y$ a point).
(2) It would also be possible to prove the C-minimal case analogously to the o-minimal one, as below.

*Proof.* First note it is enough to consider the case where $X$ consists of real elements. Indeed if $X$, $Y$ consist of imaginaries, find a set $X'$ of real elements and a surjective map $X' \to X$; then it sufices to show $S_{def}(X') \to S_{def}(Y)$ is surjective.

The lemma reduces to the case that $X \subseteq U \times Y$ is a complete type, $f : X \to Y$ is the projection, and $U$ is one of the basic sorts. Indeed, we can first let $U = X$ and replace $X$ by the graph of $f$. Any given definable type $r(y)$ in $Y$ restricts to some complete type $r_0(y)$, which we can extend to a complete type $r_0(u, y)$ implying $X$. Thus we can take $X \subseteq U \times Y$ to be complete. Now writing an element of $X$ as $a = (b, a_1, a_2, \ldots)$, with $b \in Y$ and $(a_1, a_2, \ldots) \in U$, given the lemma for the case of 1-variable $U$, we can extend $r(y)$ to a definable type one variable at a time. Note that when $X = \varprojlim X_j$, we have $S_{def}(X) = \varprojlim S_{def}(X_j)$ naturally, so at the limit we obtain a definable type on $X$. If $X$ is pro-definable in uncountably many variables, we repeat this transfinitely.

Let us now prove (1). We can take $X, Y$ to be complete types with $X \subseteq \Gamma \times Y$, and $f$ the projection. It follows from completeness that for any $b \in Y$, $f^{-1}(b)$ is convex. Let $r(y)$ be a definable type in $Y$. Let $M$ be a model with $r$ defined over $M$, let $b \models r|M$, and consider $f^{-1}(b)$.

If for any $M$, $x \in X \,\&\, f(x) \models r|M$ is a complete type $p|M$ over $M$, then $x \in X \cup p(f(x))$ already generates a definable type by Lemma 2.3.1 and we are done. So, let us assume for some $M$, and $b \models p|M$, $x \in X \,\&\, f(x) = b$ does not generate a complete type over $M(b)$. Then there exists an $M(b)$-definable set $D$ that splits $f^{-1}(b)$ into two pieces. We can take $D$ to be an interval. Then since $f^{-1}(b)$ is convex, one of the endpoints of $D$ must fall in $f^{-1}(b)$. This endpoint is $M(b)$-definable, and can be written $h(b)$ with $h$ an $M$-definable function. In this case $\mathrm{tp}(b, h(b)/M)$ is $M$-definable, and has a unique extension to an $M$-definable type.

In either case we found $p \in S_{def}(X)$ with $f_*(p) = r$. Note that the proof works when only $X$ is contained in the definable closure of an o-minimal definable set, for any pro-definable $Y$.

For the proof of (2) consider $r \in S_{def}(Y)$. Let $M$ be a metastability base, with $f$, $X$, $Y$, and $r$ defined over $M$. Let $b \models r|M$, and let $c \in f^{-1}(b)$. Let $b_1$ enumerate $\Gamma(M(b))$. Then $\mathrm{tp}(b/M(b_1)) = r'|M(b_1)$ with $r'$ stably dominated, and $\mathrm{tp}(b_1/M) = r_1|M$ with $r_1$ definable. Let $c_1$ enumerate $G(M(c))$; then $\mathrm{tp}(cb/M(c_1)) = q'|M(c_1)$ with $q'$ stably dominated. By (1) it is possible to extend $\mathrm{tp}(c_1b_1/M) \cup r_1$ to a definable type $q_1(x_1, y_1)$. Let $M \prec M'$ with $q_1$ defined over $M'$, with $c_1b_1 \models q_1|M'$, and $cb \models q'|M'(c_1b_1)$. Then $\mathrm{tp}(bc/M')$ is definable, and $\mathrm{tp}(b/M') = r|M'$. Let $p$ be the $M'$-definable type with $p|M' = \mathrm{tp}(bc/M')$. Then $f_*(p) = r$.

The surjectivity on stably dominated types is similar; in this case there is no $b_1$, and $q_1$ can be chosen so that $c_1 \in M'$. Indeed $\mathrm{tp}(c_1/M)$ implies $\mathrm{tp}(c_1/M(b))$ so it suffices to take $M'$ containing $M(c_1)$. □

**Proposition 4.2.8.** *Let $X$ and $Y$ be definable sets and let $f : \widehat{X} \to \widehat{Y}$ be a continuous and surjective morphism. Let $W$ be a definably compact pro-definable subset of $\widehat{X}$. Then $f(W)$ is definably compact.*

*Proof.* Let $q$ be a definable type on $f(W)$. By Lemma 4.2.6 there exists a definable type $r$ on $W$, with $f_*(r) = q$. Since $W$ is definably compact, $\lim r$ exists and belongs to $W$. But then $\lim q = f(\lim r)$ belongs to $f(W)$ (since this holds after composing with any continuous morphism to $\Gamma_\infty$). So $f(W)$ is definably compact. □

**Lemma 4.2.9.** *Let $V$ be an algebraic variety over a valued field, and let $W$ be a definably compact pro-definable subset of $\widehat{V \times \Gamma_\infty^m}$. Then $W$ is contained in $\widehat{X}$ for some bounded definable v+g closed subset $X$ of $V \times \Gamma_\infty^m$.*

*Proof.* By using Proposition 4.2.8 for projections $\widehat{V \times \Gamma_\infty^m} \to \widehat{V}$ and $\widehat{V \times \Gamma_\infty^m} \to \Gamma_\infty$, one may assume $W$ is a pro-definable subset of $\Gamma_\infty$ or $V$. The first case is clear. For the second one, one may assume $V$ is affine contained in $\mathbb{A}^n$ with coordinates $(x_1, \cdots, x_n)$. Consider the function $\min \mathrm{val}(x_i)$ on $V$, extended to $\widehat{V}$; it's a continuous function on $\widehat{V}$. The image of $W$ is a definably compact subset of $\Gamma_\infty$, hence is bounded below, say by $\alpha$. Let $X = \{(x_1, \ldots, x_n) : \mathrm{val}(x_i) \geq \alpha\}$. Then $W \subseteq \widehat{X}$. □

By countably-pro-definable set we mean a pro-definable set isomorphic to one with a countable inverse limit system. Note that $\widehat{V}$ is countably pro-definable.

**Lemma 4.2.10.** *Let $X$ be a strict, countably pro-definable set over a model $M$, $Y$ a relatively definable subset of $X$ over $M$. If $Y \neq \emptyset$ then $Y(M) \neq \emptyset$.*

*Proof.* Write $X = \varprojlim_n X_n$ with transition morphisms $\pi_{m,n} : X_m \to X_n$, and $X_n$ and $\pi_{m,n}$ definable. Let $\pi_n : X \to X_n$ denote the projection. Since $X$ is strict pro-definable, the image of $X$ in $X_n$ is definable; replacing $X_n$ with this image, we may assume $\pi_n$ is surjective. Since $Y$ is relatively definable, it has the form $\pi_n^{-1}(Y_n)$ for some nonempty $Y_n \subseteq X_n$. We have $Y_n \neq \emptyset$, so there exists $a_n \in Y_n(M)$. Define inductively $a_m \in Y_m(M)$ for $m > n$, choosing $a_m \in Y_m(M)$ with $\pi_{m,m-1}(a_m) = a_{m-1}$. For $m < n$ let $a_m = \pi_{n,m}(a_n)$. Then $(a_m)$ is an element of $X(M)$. □

Let $X$ be a pro-definable set with a definable topology (in some theory). Given a model $M$, and an element $a$ of $X$ in some elementary extension of $M$, we say that $\mathrm{tp}(a/M)$ has a limit $b$ if $b \in X(M)$, and for any $M$-definable open neighborhood $U$ of $b$, we have $a \in U$.

**Lemma 4.2.11.** *Let $M$ be an elementary submodel of $\Gamma_\infty^n$, and $p_0 = \mathrm{tp}(a/M)$. Assume $\lim p_0$ exists. Then there exists a (unique) $M$-definable type $p$ extending $p_0$.*

*Proof.* In case $n = 1$, $\mathrm{tp}(a/M)$ is determined by a cut in $\Gamma_\infty(M)$. If this cut is irrational then by definition there can be no limit in $M$. So this case is clear.

We have to show that for any formula $\phi(x, y)$ over $M$, $x = (x_1, \dots, x_n), y = (y_1, \dots, y_m)$, $\{c \in M : \phi(a, c)\}$ is definable. Any formula is a Boolean combination of unary formulas and of formulas of the form: $\sum \alpha_i x_i + \sum \beta_j y_j + \gamma \diamond 0$, where $i, j$ range over some subset of $\{1, \dots, n\}$, $\{1, \dots, m\}$ respectively, $\alpha_i, \beta_j \in \mathbb{Q}, \gamma \in \Gamma(M)$, and $\diamond \in \{=, <\}$. This case follows from the case $n = 1$ already noted, applied to $\mathrm{tp}(\sum \alpha_i a_i/M)$. □

**Proposition 4.2.12.** *Let $X$ be a pro-definable subset of $\widehat{V} \times \Gamma_\infty^m$ with $V$ an algebraic variety over a valued field. Let $a$ belong to the closure of $X$. Then there exists a definable type on $\widehat{V}$ concentrating on $X$, with limit point $a$.*

*Proof.* We may assume $V$ is affine; let $V' = V \times \Gamma_\infty^m$, so $\widehat{V} \times \Gamma_\infty^m = \widehat{V'}$. Since $X$ is a pro-definable subset of $\widehat{V'}$ we may write $X = \cap_{i \in I} X_i$, with $X_i$ a relatively definable subset of $\widehat{V'}$. We may take the family $(X_i)$ to be closed under finite intersections.

Let $M$ be a metastability base model, over which the $X_i$ and $a$ are defined. Let $\mathfrak{U}$ be be the family of $M$-definable open subsets $U$ of $\widehat{V'}$ with $a \in U$. Given any $U \in \mathfrak{U}$ and $i \in I$, choose $b_{U,i} \in (X_i \cap U)(M)$; this is possible by Lemma 4.2.10. Let $p_{U,i} = b_{U,i}|M$. Choose an ultrafilter $\mu$ on $\mathfrak{U} \times I$ such that for any $U_0 \in \mathfrak{U}$ and $i_0 \in I$,

$$\{(U, i) \in \mathfrak{U} \times I : U \subseteq U_0, X_i \subseteq X_{i_0}\} \in \mu$$

By compactness of the type space, there exists a limit point $p_M$ of the points $p_{U,i}$ along $\mu$, in the type space topology. In other words for any $M$-definable set $W$, if $W \in p_M$ then $W \in p_{U,i}$ for $\mu$-almost all $(U, i)$. In particular, $p_M(x)$ implies

$x \in X_i$ for each $i$, so $p_M(x)$ implies $x \in X$. On the other hand $a$ is the limit of the $b_{U,i}$ along $\mu$ in the space $\widehat{V'}(M)$. View $p_M$ as the type of elements of $V'$ over $M$, of $\Gamma$-rank $\rho$ say; let $f = (f_1, \dots, f_\rho)$ be an $M$-definable function $V' \to \Gamma$ witnessing this rank. Since $V$ is affine we can take $f_i$ to have the form $\mathrm{val}(F_i)$, with $F_i$ a polynomial, or coordinate functions on $\Gamma_\infty^m$. Since $a$ is stably dominated, $f_*(a)$ concentrates on a single point $\alpha \in \Gamma^\rho$.

By definition of the topology on $\widehat{V'}$, and since $a$ is the $\mu$-limit of the $p_{U,i}$ in $\widehat{V'}$, $\lim_\mu f_*(b_{U,i}) = \alpha$. In particular, for any $M$-definable open neighborhood $W$ of $\alpha$ in $\Gamma^\rho$, $f_*(b_{U,i}) \in W$ for almost all $(U,i)$. So $f_*(p_{U,i})$ concentrates on $W$ for almost all $(U,i)$, and hence so does $f_*(p_M)$. Thus $f_*(p_M)$ has $\alpha$ as a limit. By Lemma 4.2.11, $f_*(p_M)$ is a definable type. By metastability, $p_M$ is a definable type, the restriction to $M$ of an $M$-definable type $p$. To show that the limit of $p$ is $a$, it suffices to consider $M$-definable neighborhoods $U_0$ of $a$ in $\widehat{V'}$; for any such $U_0$, we have $b_{U,i} \in U_0$ for all $U$ with $U \subseteq U_0$, so $a \in U_0$. □

**Corollary 4.2.13.** *Let $X$ be a pro-definable subset of $\widehat{V}$ with $V$ an algebraic variety over a valued field. If $X$ is definably compact, then $X$ is closed in $\widehat{V}$. Moreover $X$ is contained in a bounded subset of $\widehat{V}$. If $X$ is a definably compact pro-definable subset of $\widehat{V} \times \Gamma_\infty^n$, then again $X$ is closed.*

*Proof.* We may embed $V$ in a complete variety $\bar{V}$. The fact that $X$ is closed in $\bar{V}$ is immediate from Proposition 4.2.12 and the definition of definable compactness. Let $Z$ be the complement of $V$ in $\bar{V}$. Then $X$ is disjoint from $\widehat{Z}$. Let $\gamma$ be any continuous function into $\Gamma_\infty$, taking values in $\Gamma$ for arguments outside $Z$, and $\infty$ on $Z$. Then $\gamma(X)$ is a definably compact subset of $\Gamma$, hence bounded above by some $\alpha$. So $X$ is contained in $\{x : \gamma(x) \leq \alpha\}$ which is bounded. □

Even for $\mathrm{Th}(\Gamma)$, definability of a type $\mathrm{tp}(ab/M)$ does not imply that $\mathrm{tp}(a/M(b))$ is definable. For instance $b$ can approach $\infty$, while $a \sim \alpha b$ for some irrational real $\alpha$, i.e. $qb < a < q'b$ if $q, q' \in \mathbb{Q}$, $q < \alpha < q'$. However we do have:

**Lemma 4.2.14.** *Let $p$ be a definable type of $\Gamma$, over $A$. Then up to a definable change of coordinates, $p$ decomposes as the join of two orthogonal definable types $p_f, p_i$, such that $p_f$ has a limit in $\Gamma^m$, and $p_i$ has limit point $\infty^\ell$.*

*Proof.* Let $\alpha_1, \dots, \alpha_k$ be a maximal set of linearly independent vectors in $\mathbb{Q}^n$ such that the image of $p$ under $(x_1, \dots, x_n) \mapsto \sum \alpha_i x_i$ has a limit point in $G$. Let $\beta_1, \dots, \beta_\ell$ be a maximal set of vectors in $\mathbb{Q}^n$ such that for $x \models p|M$, $\alpha_1 x, \dots, \alpha_k x, \beta_1 x, \cdots, \beta_\ell x$ are linearly independent over $M$. If $a \models p|M$, let $a' = (\alpha_1 a, \dots, \alpha_k a)$, $a'' = (\beta_1 a, \dots, \beta_k a)$. For $\alpha \in \mathbb{Q}(\alpha_1, \dots, \alpha_k)$ we have $\alpha a$ is bounded between elements of $M$. On the other hand each $\beta a$, with $\beta \in \mathbb{Q}(\beta_1, \dots, \beta_k)$, satisfies $\beta a > M$ or $\beta a < M$. For if $m \leq \beta a'' \leq m'$ for some $m \in M$, since $\mathrm{tp}(\beta a''/M)$ is definable it must have a finite limit, contradicting the maximality of $k$. It follows that $\mathrm{tp}(\alpha a/M) \cup \mathrm{tp}(\beta a/M)$ extends to a complete 2-type, namely

$\mathrm{tp}((\alpha a, \beta a)/M)$; in particular $\mathrm{tp}(\alpha a + \beta a/M)$ is determined; from this, by quantifier elimination, $\mathrm{tp}(a'/M) \cup \mathrm{tp}(a''/M)$ extends to a unique type in $k+\ell$ variables. So $\mathrm{tp}(a'/M)$ and $\mathrm{tp}(a''/M)$ are orthogonal. After some sign changes in $a''$, so that each coordinate is $> M$, the lemma follows. □

**Remark 4.2.15.** It follows from Lemma 4.2.14 that to check for definable compactness of $X$, it suffices to check definable maps from definable types on $\Gamma^k$ that either have limit 0, or limit $\infty$. From this an alternative proof of the g- and v-criteria of §9 for closure in $\widehat{V}$ can be deduced.

**Lemma 4.2.16.** *Let $S$ be a definably compact definable subset of an o-minimal structure. If $\mathcal{D}$ is a uniformly definable family of nonempty closed definable subsets of $S$, and $\mathcal{D}$ is directed (the intersection of any two elements of $\mathcal{D}$ contains a third), then $\cap\mathcal{D} \neq \emptyset$.*

*Proof.* By Lemma 2.19 of [16] there exists a cofinal definable type $q(y)$ on $\mathcal{D}$; concentrating, for each $U \in \mathcal{D}$, on $\{V \in \mathcal{D} : V \subset U\}$.

Using the lemma on extension of definable types Lemma 4.2.6, let $r(w, y)$ be a definable type extending $q$ and implying $w \in U_y \cap S$. Let $p(w)$ be the projection of $r$ to the $w$-variable. By definable compactness $\lim p = a$ exists. Since $a$ is a limit of points in $D$, we have $a \in D$ for any $D \in \mathcal{D}$. So $a \in \cap\mathcal{D}$. □

Lemma 4.2.16 gives another proof that a definably compact set is closed: let $\mathcal{D} = \{S \smallsetminus U\}$, where $U$ ranges over basic open neighborhoods of a given point $a$ of the closure of $S$.

**Proposition 4.2.17.** *Let $V$ be an algebraic variety over a valued field, and let $W$ be a pro-definable subset of $\widehat{V \times \Gamma_\infty^m}$. Then $W$ is definably compact if and only if it is closed and bounded.*

*Proof.* If $W$ is definably compact it is closed and bounded by Lemma 4.2.13 and 4.2.9. If $W$ is closed and bounded, its preimage $W'$ in $\widehat{V \times \mathbb{A}^m}$ under $\mathrm{id} \times \mathrm{val}$ is also closed and bounded, hence definably compact by Lemma 4.2.4. It follows from Proposition 4.2.8 that $W$ is definably compact. □

**Proposition 4.2.18.** *Let $V$ be a variety over a valued field $F$, and let $W$ be an $F$-definable subset of $V \times \Gamma_\infty^m$. Then $W$ is v+g-closed (resp. v+g-open) if and only if $\widehat{W}$ is closed (resp. open) in $\widehat{V}$.*

*Proof.* A Zariski-locally v-open set is v-open, and similarly for g-open; hence for v+g-open. So we may assume $V = \mathbb{A}^n$ and by pulling back to $V \times \mathbb{A}^m$ that $m = 0$. It enough to prove the statement about closed subsets. Let $V_\alpha = (c\mathcal{O})^n$ be the closed polydisk of valuative radius $\alpha = \mathrm{val}(c)$. Let $W_\alpha = W \cap V_\alpha$, so $\widehat{W_\alpha} = \widehat{W} \cap \widehat{V_\alpha}$. Then $W$ is v-closed if and only if $W_\alpha$ is v-closed for each $\alpha$; by Lemma 8.1.2, the same holds for g-closed; also $\widehat{W}$ is closed if and only if $\widehat{W_\alpha}$ is closed for each $\alpha$. This reduces the question to the case of bounded $W$.

By Lemma 3.6.4, if $W$ is v+g-closed then $\widehat{W}$ is closed.

In the reverse direction, if $\widehat{W}$ is closed it is definably compact. It follows that $W$ is v-closed. For otherwise there exists an accumulation point $w$ of $W$, with $w = (w_1, \dots, w_m) \notin W$. Let $\delta(v) = \min_{i=1}^m \mathrm{val}(v_i - w_i)$. Then $\delta(v) \in \Gamma$ for $v \in W$, i.e. $\delta(v) < \infty$. Hence the induced function $\delta : \widehat{W} \to \Gamma_\infty$ also has image contained in $\Gamma$; and $\delta(\widehat{W})$ is definably compact. It follows that $\delta(\widehat{W})$ has a maximal point $\gamma_0 < \infty$. But then the $\gamma_0$-neighborhood around $w$ contains no point of $W$, a contradiction.

It remains to show that when $\widehat{W}$ is definably compact, $W$ must be g-closed. This follows from Lemma 8.1.3. $\square$

**Corollary 4.2.19.** *Let $V$ be an algebraic variety over a valued field, and let $W$ be a definable subset of $V \times \Gamma_\infty^m$. Then $W$ is bounded and v+g-closed* if and only if *$\widehat{W}$ is definably compact.*

*Proof.* Since $W$ is v+g-closed if and only if $\widehat{W}$ is closed by Lemma 4.2.18, this is a special case of Proposition 4.2.17. $\square$

**Lemma 4.2.20.** *Let $V$ be an algebraic variety over a valued field and let $Y$ be a v+g-closed, bounded subset of $V \times \Gamma_\infty^m$. Let $W$ be a definable subset of $V' \times \Gamma_\infty^m$, with $V'$ another variety, and $f : \widehat{Y} \to \widehat{W}$ be continuous. Then $f$ is a closed map.*

*Proof.* By Propositions 4.2.18 and 4.2.17 $\widehat{Y}$ is definably compact and any closed subset of $\widehat{Y}$ is definably compact, so the result follows from Proposition 4.2.8 and 4.2.13. $\square$

**Lemma 4.2.21.** *Let $X$ and $Y$ be v+g-closed, bounded definable subsets of a product of an algebraic variety over a valued field with some $\Gamma_\infty^m$. Then, the projection $\widehat{X} \times \widehat{Y} \to \widehat{Y}$ is a closed map.*

*Proof.* By Lemma 4.2.20 the mapping $\widehat{X \times Y} \to \widehat{Y}$ is closed. Since this map factorizes as $\widehat{X \times Y} \to \widehat{X} \times \widehat{Y} \to \widehat{Y}$, the mapping on the right, $\widehat{X} \times \widehat{Y} \to \widehat{Y}$, is also closed. $\square$

**Corollary 4.2.22.** *Let $U$ and $V$ be v+g-closed, bounded definable subsets of a product of an algebraic variety over a valued field with some $\Gamma_\infty^m$. If $f : \widehat{U} \to \widehat{V}$ is a pro-definable morphism with closed graph, then $f$ is continuous.*

*Proof.* By Lemma 4.2.21, the projection $\pi_1$ from the graph of $f$ to $U$ is a homeomorphism onto the image. The projection $\pi_2$ is continuous. Hence $f = \pi_2 \pi_1^{-1}$ is continuous. $\square$

**Lemma 4.2.23.** *Let $f : V \to W$ be a proper morphism of algebraic varieties. Then $\widehat{f}$ is a closed map. So is $\widehat{f} \times \mathrm{Id} : \widehat{V} \times \Gamma_\infty^m \to \widehat{W} \times \Gamma_\infty^m$.*

*Proof.* $\widehat{V \times \Gamma_\infty^m}$ can be identified with a subset $S$ of $\widehat{V} \times \mathbb{A}^m$ (projecting on generics of balls around zero in the second coordinate); with this identification, $\widehat{f} \times \mathrm{Id}$ identifies with the restriction of $\widehat{f \times \mathrm{Id}_{\mathbb{A}^m}}$ to $S$. Thus the second statement, for $V \times \Gamma_\infty^m$, reduces to first for the case of the map $f \times \mathrm{Id} : V \times \mathbb{A}^m \to W \times \mathbb{A}^m$.

To prove the statement on $f : V \to W$, let $V', W'$ be complete varieties containing $V, W$, and let $\bar{V}$ be the closure of the graph of $f$ in $V' \times W'$. The map $\mathrm{Id} \times f : V' \times V \to V' \times W$ is closed by properness (universal closedness). So the graph of $f$, a subset of $V \times W$, is closed as a subset of $V' \times W$. Let $\pi : \bar{V} \to W'$ be the projection. Then $\pi^{-1}(W) \subseteq V' \times W$. Since $f$ is closed in $V' \times W$, $f = \pi|(\pi^{-1}(W))$. Now $\widehat{\pi}$ is a closed map by Lemma 4.2.20. So the restriction $f$ is a closed map too. (We could also obtain the result directly from Lemma 4.2.12. □

**Remark 4.2.24.** The previous lemmas apply also to $\infty$-definable sets.

**Lemma 4.2.25.** *Let $X$ be a v+g-closed bounded definable subset of an algebraic variety $V$ over a valued field. Let $f : X \to \Gamma_\infty$ be v+g-continuous. Then the maximum of $f$ is attained on $X$. Similarly if $X$ is a closed bounded pro-definable subset of $\widehat{V}$.*

*Proof.* By Lemma 3.7.2, $f$ extends continuously to $F : \widehat{X} \to \Gamma_\infty$. By Lemma 4.2.18 and Proposition 4.2.17 $\widehat{X}$ is definably compact. It follows from Lemma 4.2.8 that $F(\widehat{X})$ is a definably compact subset of $\Gamma_\infty$ and hence has a maximal point $\gamma$. Take $p$ such that $F(p) = \gamma$, let $c \models p$, then $f(c) = \gamma$. □

For $\Gamma^n$, Proposition 4.2.17 is a special case of [23], Theorem 2.1.

## 5. A closer look at $\widehat{V}$

5.1. $\widehat{\mathbb{A}^n}$ **and spaces of semi-lattices.** Let $K$ be a valued field. Let $H = K^N$ be a vector space of dimension $N$. By a lattice in $H$ we mean a free $\mathcal{O}$-submodule of rank $N$. By a semi-lattice in $H$ we mean an $\mathcal{O}$-submodule $u$ of $H$, such that for some $K$-subspace $U_0$ of $H$ we have $U_0 \subseteq u$ and $u/U_0$ is a lattice in $H/U_0$. Note that every semi-lattice is uniformly definable with parameters and that the set $L(H)$ of semi-lattices in $H$ is definable. Also, a definable $\mathcal{O}$-submodule $u$ of $H$ is a semi-lattice if and only if there is no $0 \neq v \in H$ such that $Kv \cap u = \{0\}$ or $Kv \cap u = \mathcal{M}v$ where $\mathcal{M}$ is the maximal ideal.

We define a topology on $L(H)$: the pre-basic open sets are those of the form: $\{u : h \notin u\}$ and those of the form $\{u : h \in \mathcal{M}u\}$, where $h$ is any element of $H$. We call this family the linear pre-topology on $L(H)$.

Any finitely generated $\mathcal{O}$-submodule of $K^N$ is generated by $\leq N$ elements; hence the intersection of any finite number of open sets of the second type is the intersection of $N$ such open sets. However this is not the case for the first kind, so we do not have a definable topology in the sense of Ziegler.

We say that a definable subset $X$ of $L(H_d)$ is *closed for the linear topology* if for any definable type $q$ on $X$, if $q$ has a limit point $a$ in $L(H_d)$, then $a \in X$. The complements of the pre-basic open sets of the linear pre-topology are clearly closed.

Another description can be given in terms of linear semi-norms. By a *linear semi-norm* on a vector space $V$ over $K$ we mean a definable map $w : V \to \Gamma_\infty$ with $w(x_1+x_2) \geq \min(w(x_1)+w(x_2))$ and $w(cx) = \mathrm{val}(c)+w(x)$. Any linear semi-norm $w$ determines a semi-lattice $\Lambda_w$, namely $\Lambda_w = \{x : w(x) \geq 0\}$. Conversely, any semi-lattice $\Lambda \in LV$ has the form $\Lambda = \Lambda_w$ for a unique $w$. We may thus identify $LV$ with the set of linear semi-norms on $V$. On the set of semi-norms there is a natural topology, with basic open sets of the form $\{w : (w(f_1), \ldots, w(f_k)) \in O\}$, with $f_1, \ldots f_k \in V$ and $O$ an open subset of $\Gamma_\infty^k$. The linear pre-topology on $LV$ coincides with the semi-norm topology.

We say $X$ is *bounded* if the pullback to $\mathrm{End}(H_d)$ is bounded.

**Lemma 5.1.1.** *The space $L(H)$ with the linear pre-topology is Hausdorff. Moreover, any definable type on a bounded subset of $L(H)$ has a (unique) limit point in $L(H)$.*

*Proof.* Let $u' \neq u'' \in L(H)$. One, say $u'$, is not a subset of the other. Let $a \in u', a \notin u''$. Let $I = \{c \in K : ca \in u''\}$. Then $I = \mathcal{O}c_0$ for some $c_0$ with $\mathrm{val}(c_0) > 0$. Let $c_1$ be such that $0 < \mathrm{val}(c_1) < \mathrm{val}(c_0)$ and let $a' = c_1 a$. Then $a' \in \mathcal{M}u'$ but $a' \notin u''$. This shows that $u'$ and $u''$ are separated by the disjoint open sets $\{u : a' \notin u\}$ and $\{u : a' \in \mathcal{M}u\}$.

For the second statement, let $B_\alpha = B(0, \alpha) = \{x : \mathrm{val}(x) \geq \alpha\}$ be the closed ball of valuative radius $\alpha$. Then $\widehat{B_\alpha^m}$ is a closed subset of $\widehat{\mathbb{A}^m}$. Let $Z_\alpha$ be the set of semi-lattices $u \in L(H)$ containing all the linear monomials $cx_i$, $i = 1, \ldots, m$, with $\mathrm{val}(c) \geq -\alpha$. Then $J_d^{-1}(Z_\alpha) = B_\alpha$. Note that $Z_\alpha$ is closed. Any bounded subset of $L(H)$ is contained in $Z_\alpha$ for some $\alpha$, so for the "moreover", it suffices to see that $Z_\alpha$ is definably compact in the linear topology. Let $p$ be a definable type on $Z_\alpha$. Let

$$\Lambda = \{h \in H : (d_p x)(h \in x)\}$$

the "generic intersection" of the semi-lattices on which $p$ concentrates. $\Lambda$ is a submodule of $H$ containing $Z_\alpha$, hence generating $H$ as a vector space. If $h \in \Lambda$, but $Kh$ is not contained in $\Lambda$, then for any $a \in H$ there exists a unique minimal $\gamma \in \Gamma$ with $\gamma = \mathrm{val}(c)$ for some $c$ with $ch \in a$; write $\gamma = \gamma(a)$. Then $\gamma$ is generically constant on $p$, i.e. $\gamma(a) = \gamma_0$ for $a \models p$. If $\mathrm{val}(c) = \gamma_0$ then $\Lambda \cap Kh = \mathcal{O}h$. So $\Lambda$ is a semi-lattice, $\Lambda \in L(H)$. It is easy to see that any pre-basic open set containing $\Lambda$ must also contain a generic point of $p$. □

Let $H_{m;d}$ be the space of polynomials of degree $\leq d$ in $m$ variables. For the rest of this subsection $m$ will be fixed; we will hence suppress the index and write $H_d$.

**Lemma 5.1.2.** *For $p$ in $\widehat{\mathbb{A}^m}$, the set*

$$J_d(p) = \{h \in H_d : p_*(\mathrm{val}(h)) \geq 0\}$$

*belongs to $L(H_d)$.*

*Proof.* Note that $J_d(p)$ is a definable $\mathcal{O}$-submodule of $H_d$. For fixed nonzero $h_0 \in H_d$, it is clear that $J_d(p) \cap K h_0 = \{\alpha \in K : \alpha p_*(\mathrm{val}(h_0)) \geq 0\}$ is either a closed ball in $K$, or all of $K$, hence $J_d(p)$ is a semi-lattice. □

Hence we have a mapping $J_d = J_{d,m} : \widehat{\mathbb{A}^m} \to L(H_d)$ given by $p \mapsto J_d(p)$. It is clearly a continuous map, when $H_d$ is given the linear pre-topology: $f \notin J_d(p)$ if and only if $p_*(f) > 0$, and $f \in \mathcal{M}D_{d,m}(p)$ if and only if $p_*(f) < 0$.

**Lemma 5.1.3.** *The system $(J_d)_{d=1,2,\ldots}$ induces a continuous morphism of pro-definable sets*

$$J : \widehat{\mathbb{A}^m} \longrightarrow \varprojlim L(H_d).$$

*The morphism $J$ is injective and induces a homeomorphism between $\widehat{\mathbb{A}^m}$ and its image.*

*Proof.* Let $f : \mathbb{A}^m \times H_d \to \Gamma_\infty$ given by $(x, h) \mapsto \mathrm{val}(h(x))$. Since $J_d$ factors through $Y_{H_d,f}$, $J$ is a morphism of pro-definable sets.

For injectivity, recall that types on $\mathbb{A}^n$ correspond to equivalence classes of $K$-algebra morphisms $\varphi : K[x_1, \cdots, x_n] \to F$ with $F$ a valued field, with $\varphi$ and $\varphi'$ equivalent if they are restrictions of a same $\varphi''$. In particular, if $\varphi_1$ and $\varphi_2$ correspond to different types, one should have

$$\{f \in K[x_1, \cdots, x_m] : \mathrm{val}(\varphi_1(f)) \geq 0\} \neq \{f \in K[x_1, \cdots, x_m] : \mathrm{val}(\varphi_2(f)) \geq 0\},$$

whence the result.

We noted already continuity. To see that $J$ is an open map onto the image, since bijective maps commute with finite intersections and arbitrary unions, it suffices to see that the image of a generating family of open sets $S$ is open. For this it suffices to see that $J_d(S)$ is open for large enough $d$. The topology on $\widehat{\mathbb{A}^n}$ is generated by sets of the form $\{p : p_*(f) > \gamma\}$ or $\{p : p_*(f) < \gamma\}$, where $f \in H_d$ for some $d$. Replacing $f$ by $cf$ for appropriate $p$, it suffices to consider sets of the form $\{p : p_*(f) > 0\}$ or $\{p : p_*(f) < 0\}$. Now the image of these sets is precisely the intersection with the image of $J$ of the open sets $\Lambda \in L(H_d) : f \notin \Lambda$ or $\{p : p_*(f) \in \mathcal{M}\Lambda\}$. □

The above lemma shows that the linear pre-topology is adequate when one takes all "jets" into account, but does not describe the image of $J$, and gives no information about the individual $J_d$.

Fix a standard (monomial) basis for $H_d$, and let $\Lambda_0$ be the $\mathcal{O}$-module generated by this basis. Given $M \in \mathrm{End}(H_d)$, let $\Lambda(M) = M^{-1}(\Lambda_0)$. We identify $\mathrm{Aut}(\Lambda_0)$ with the group of automorphisms $T$ of $H_d$ with $T(H_0) = H_0$.

**Lemma 5.1.4.** *The mapping* $M \mapsto \Lambda(M)$ *induces a bijection between* $\mathrm{Aut}(\Lambda_0)\backslash\mathrm{End}(H_d)$ *and* $L(H_d)$.

*Proof.* It is clear that $M \mapsto \Lambda(M)$ is a surjective map from $\mathrm{End}(H_d)$ to $L(H_d)$, and also that $\Lambda(N) = \Lambda(TN)$ if $T \in \mathrm{Aut}(\Lambda_0)$. Conversely suppose $\Lambda(M) = \Lambda(N)$. Then $M, N$ have the same kernel $E = \{a : Ka \subseteq M^{-1}(\Lambda_0)\}$. So $NM^{-1}$ is a well-defined homomorphism $MH_d \to NH_d$. Moreover, $MH_d \cap \Lambda_0$ is a free $\mathcal{O}$-submodule of $H_d$, and $(NM^{-1})(MH_d \cap \Lambda_0) = (NH_d \cap \Lambda_0)$. Let $C$ (resp. $C'$) be a free $\mathcal{O}$-submodule of $\Lambda_0$ complementary to $MH_d \cap \Lambda_0$ (resp. $NH_d \cap \Lambda_0$), and let $T_2 : C \to C'$ be an isomorphism. Let $T = (NM^{-1})|(MH_d \cap \Lambda_0) \oplus T_2$. Then $T \in \mathrm{Aut}(\Lambda_0)$, and $NM^{-1}\Lambda_0 = T^{-1}\Lambda_0$, so (using $\ker M = \ker N$) we have $M^{-1}\Lambda_0 = N^{-1}\Lambda_0$. □

**Proposition 5.1.5.** *The morphism* $J_d : \widehat{\mathbb{A}^m} \to L(H_d)$ *is closed and continuous map if* $L(H_d)$ *is endowed with the linear topology.*

*Proof.* Write $J = J_d$ and $H = H_d$. Let $X \subseteq L(H)$ be a closed definable set. Let $p$ be a definable type on $J^{-1}(X)$, with limit point $a \in \widehat{\mathbb{A}^m}$. Since $J$ is continuous towards the linear pre-topology, $J(a)$ is a limit point of $J_*p$. By definition of a closed set it follows that $J(a) \in X$; so $a \in J^{-1}(X)$. It follows that the intersection of $J^{-1}(X)$ with any bounded subset of $\widehat{\mathbb{A}^m}$ is itself definably compact, and since $\widehat{\mathbb{A}^m}$ is the union of a family of bounded open sets it follows that $J^{-1}(X)$ is closed. Thus $J$ is continuous.

To show that $J$ is closed, let $Y$ be a closed subset of $\widehat{\mathbb{A}^m}$. Let $q$ be a definable type on $J(Y)$, and let $b$ be a limit point of $q$ for the linear pre-topology. The case $d = 0$ is easy as $J_0$ is a constant map, so assume $d \geq 1$. We have in $H_d$ the monomials $x_i$. For some nonzero $c'_i \in K$ we have $c'_i x_i \in b$, since $b$ generates $H_d$ as a vector space. Choose a nonzero $c_i$ such that $c_i x_i \in \mathcal{M}b$. Let $U = \{b' : c_i x_i \in \mathcal{M}b', i = 1, \dots, m\}$. Then $U$ is a pre-basic open neighborhood of $b$; as $b$ is a limit point of $q$, it follows that $q$ concentrates on $U$. Note that $J^{-1}(U)$ is contained in $\widehat{B}$ where $B$ is the polydisc $\mathrm{val}(x_i) \geq -\mathrm{val}(c_i), i = 1, \dots, m$. Thus $J^{-1}(U)$ is bounded. Lift $q$ to a definable type $p$ on $Y \cap \widehat{B}$ (Lemma 4.2.6). Then as $Y \cap \widehat{B}$ is closed and bounded, $q$ has a limit point $a$. By continuity we have $J(a) = b$, hence $b \in J(Y)$. □

5.2. **A representation of $\widehat{\mathbb{P}^n}$.** Let us define the tropical projective space $\mathrm{Trop}\,\mathbb{P}^n$, for $n \geq 0$, as the quotient $\Gamma_\infty^{n+1} \smallsetminus \{\infty\}^{n+1}/\Gamma$ where $\Gamma$ acts diagonally by translation. This space may be embedded in $\Gamma_\infty^{n+1}$ since it can be identified with

$$\{(a_0, \dots, a_n) \in \Gamma_\infty^{n+1} : \min a_i = 0\}.$$

Over a valued field $L$, we have a canonical definable map $\tau : \mathbb{P}^n \to \mathrm{Trop}\,\mathbb{P}^n$, sending $[x_0 : \cdots : x_n]$ to $[v(x_0) : \cdots : v(x_n)] = ((v(x_0) - \min_i v(x_i), \cdots, v(x_n) - \min_i v(x_i))$.

Let us denote by $H_{n+1;d,0}$ the set of homogeneous polynomials in $n+1$ variables of degree $d$ with coefficients in the valued field sort. Again we view $n$ as fixed and omit it from the notation, letting $H_{d,0} = H_{n+1;d,0}$. Denote by by $H_{d,m}$ the definable subset of $H_{d,0}^{m+1}$ consisting of $m+1$-uplets of homogeneous polynomials with no common zeroes other than the trivial zero. Hence, one can consider the image $PH_{d,m}$ of $H_{d,m}$ in the projectivization $P(H_{d,0}^{m+1})$. We have a morphism $c : \mathbb{P}^n \times H_{d,m} \to \mathbb{P}^m$, given by $c([x_0 : \cdots : x_n], (h_0, \cdots, h_m)) = [h_0(x) : \ldots : h_m(x]$. Since $c(x, h)$ depends only on the image of $h$ in $PH_{d,m}$, we obtain a morphism $c : \mathbb{P}^n \times PH_{d,m} \to \mathbb{P}^m$. Composing $c$ with the map $\tau : \mathbb{P}^m \to \operatorname{Trop} \mathbb{P}^m$, we obtain $\tau : \mathbb{P}^n \times PH_{d,m} \to \operatorname{Trop} \mathbb{P}^m$. For $h$ in $PH_{d,m}$ (or in $H_{d,m}$), we denote by $\tau_h$ the map $x \mapsto \tau(x, h)$. Thus $\tau_h$ extends to a map $\widehat{\tau_h} : \widehat{\mathbb{P}^n} \to \operatorname{Trop} \mathbb{P}^m$.

Let $T_{d,m}$ denote the set of functions $PH_{d,m} \to \operatorname{Trop} \mathbb{P}^n$ of the form $h \mapsto \widehat{\tau_h}(x)$ for some $x \in \widehat{\mathbb{P}^n}$. Note that $T_{d,m}$ is definable.

**Proposition 5.2.1.** *The space $\widehat{\mathbb{P}^n}$ may be identified via the canonical mappings $\widehat{\mathbb{P}^n} \to T_{m,d}$ with the projective limit of the spaces $T_{m,d}$. If one endows $T_{d,m}$ with the topology induced from the Tychonoff topology, this identification is a homeomorphism.* □

**Remark 5.2.2.** By composing with the embedding $\operatorname{Trop} \mathbb{P}^m \to \Gamma_\infty^{m+1}$, one gets a definable map $\widehat{\mathbb{P}^n} \to \Gamma_\infty^{m+1}$. The topology on $\widehat{\mathbb{P}^n}$ can be defined directly using the above maps into $\Gamma_\infty$, without an affine chart.

5.3. **Paths and homotopies.** By an interval we mean a subinterval of $\Gamma_\infty$. Note that intervals of different length are in general not definably homeomorphic and that the gluing of two intervals may not result in an interval. We get around the latter issue by formally introducing a more general notion, that of a generalized interval. First we consider the compactification $\{-\infty\} \cup \Gamma_\infty$ of $\Gamma_\infty$. (This is used for convenience; in practice all functions defined on $\{-\infty\} \cup \Gamma_\infty$ will be constant on some semi-infinite interval $[-\infty, a]$, $a \in \Gamma$.) If $I$ is an interval $[a, b]$, we may consider it either with the natural order of with the opposite order. The choice of one of these orders will be an orientation of $I$. By a *generalized* interval $I$ we mean a finite union of oriented copies $I_1, \ldots, I_n$ of $\{-\infty\} \cup \Gamma_\infty$ glued end-to-end in a way respecting the orientation, or a sub-interval of such an ordered set.

If $I$ is closed, we denote by $i_I$ the smallest element of $I$ and by $e_I$ its largest element. If $I = [a, b]$ is a sub-interval of $\Gamma_\infty$ and $\varphi$ is a function $I \times V \to W$, one may extend $\varphi$ to a function $\tilde{\varphi} : \{-\infty\} \cup \Gamma_\infty \times V \to W$ by setting $\tilde{\varphi}(t, x) = \varphi(a, x)$ for $x < a$ and $\tilde{\varphi}(t, x) = \varphi(b, x)$ for $x > b$. We shall say $\tilde{\varphi}$ is definable, resp. continuous, resp. v+g-continuous, if $\varphi$ is. Similarly if $I$ is obtained by gluing $I_1, \ldots, I_n$, we shall say a function $I \times V \to W$ is definable, resp. continuous, resp. v+g-continuous, if it is obtained by gluing definable, resp. continuous, resp. v+g-continuous, functions $\tilde{\varphi}_i : I_i \times V \to W$.

Let $V$ be a definable set. By a path on $\widehat{V}$ we mean a continuous definable map $I \to \widehat{V}$ with $I$ some generalized interval.

**Example 5.3.1.** Generalized intervals may in fact be needed to connect points of $\widehat{V}$. For instance let $V$ be a cycle of $n$ copies of $\mathbb{P}^1$, with consecutive pairs meeting in a point. We will see that a single homotopy with interval $[0, \infty)$ reduces $V$ to a cycle made of $n$ copies of $[0, \infty] \subset \Gamma_\infty$. However it is impossible to connect two points at extreme ends of this topological circle without glueing together some $n/2$ intervals.

**Definition 5.3.2.** Let $X$ be a pro-definable subset of $\widehat{V} \times \Gamma_\infty^n$. A *homotopy* is a continuous pro-definable map $h : I \times X \to X$ with $I$ a closed generalized interval.

If $W$ is a definable subset of $V \times \Gamma_\infty^n$, we will also refer to a v+g-continuous pro-definable map $h_0 : I \times W \to \widehat{W}$ as a homotopy; by Lemma 3.7.2, $h_0$ extends uniquely to a homotopy $h : \widehat{W} \to \widehat{W}$.

A homotopy $h : I \times V \to \widehat{V}$ or $h : I \times \widehat{V} \to \widehat{V}$ is called a deformation retraction to $A \subseteq \widehat{V}$ if $h(i_I, x) = x$ for all $x$, $h(t, a) = a$ for all $t$ in $I$ and $a$ in $A$ and furthermore $h(e_I, x) \in A$ for each $x$. (In the literature, this is sometimes referred to as a *strong* deformation retraction.) If $h : I \times V \to \widehat{V}$ is a deformation retraction, and $\varrho(x) = h(e_I, x)$, we say that $\varrho(V)$ is the image of $h$, and that $(\varrho, \varrho(X))$ is a deformation retract. Sometimes, we shall also call $\varrho$ or $\varrho(X)$ a deformation retract, the other member of the pair being understood implicitly.

A homotopy $h$ is said to satisfy condition $(*)$ if $h(e_I, h(t, x)) = h(e_I, x)$ for every $t$ and $x$.

Let $h_1 : I_1 \times \widehat{V} \to \widehat{V}$ and $h_2 : I_2 \times \widehat{V} \to \widehat{V}$ two homotopies. Denote by $I_1 + I_2$ the (generalized) interval obtained by gluing $I_1$ and $I_2$ at $e_{I_1}$ and $i_{I_2}$. Assume $h_2(i_{I_2}, h_1(e_{I_1}, x)) = h_1(e_{I_1}, x)$ for every $x$ in $\widehat{V}$. Then one denotes by $h_2 \circ h_1$ the homotopy $I_2 + I_1 \times \widehat{V} \to \widehat{V}$ given by $h_1(t, x)$ for $t \in I_1$ and by $h_2(t, H_1(e_{I_1}, x))$ for $t$ in $I_2$.

**Lemma 5.3.3.** *Let $X, X_1$ pro-definable subsets, $f : X_1 \to X$ a closed, surjective pro-definable map. Let $h_1 : I \times X_1 \to X_1$ be a homotopy, and assume $h_1$ leaves invariant $f^{-1}(e)$ for any $e \in X$. Then $h_1$ descends to a homotopy of $X$.*

*Proof.* Define $h : I \times X \to X$ by $h(t, f(x)) = f(h_1(t, x))$ for $x \in X_1$; then $h$ is well-defined and pro-definable. We denote the map $(t, x) \mapsto (t, f(x))$ by $f_2$. Clearly, $f_2$ is a closed, surjective map. (The topology on $I \times X_1$, $I \times X$ being the product topology.) To show that $h$ is continuous, it suffices therefore to show that $h \circ f_2$ is continuous. Since $h \circ f_2 = f \circ h_1$ this is clear. □

In particular, let $f : V_1 \to V$ be a proper surjective morphism of algebraic varieties over a valued field. Let $h_1$ be a homotopy $h_1 : I \times \widehat{V_1} \to \widehat{V_1}$, and assume

$h_1$ leaves invariant $\widehat{f^{-1}(e)}$ for any $e \in \widehat{V}$. Then $\widehat{f}$ is surjective by Lemma 4.2.6), and closed by Lemma 4.2.23; so $h_1$ descends to a homotopy of $X$.

## 6. Γ-internal spaces

6.1. **Preliminary remarks.** Our aim in this section is to show that a subspace of $\widehat{V}$, definably isomorphic to a subset of $\Gamma^n$ (after base change), is *homeomorphic* to a subset of $\Gamma_\infty^n$ (after base change).

A number of delicate issues arise here. We say $X$ is $\Gamma$-*parameterized* if there exists a (pro)-definable surjective map $g : Y \to X$, with $Y \subseteq \Gamma^n$. We do not know if a Γ-parameterized set is Γ-internal.

Note that $X$ is Γ-internal if and only if it is Γ-parameterized, and in addition one of the projections $\pi : \widehat{V} \to H$ to a definable set $H$, is injective on $X$. Even in this case however, if we give $H$ the induced topology so that $\pi$ is closed and continuous, the restriction of $\pi$ to $X$ need not be a homeomorphism. If it can be taken to be one, we say that $X$ is definably separated. The Γ-internal sets we will obtain in our theorems are Γ-separated, and the results of this section are applicable to such sets. Note that definably compact sets $X$ are automatically definably separated, since the image of a closed subset of $X$ is a definably compact and hence closed subset of $\Gamma_\infty^n$.

We first discuss briefly the role of parameters.

We fix a valued field $F$. The term "definable" refers to $\mathrm{ACVF}_F$. Varieties are assumed defined over $F$. At the level of definable sets and maps, Γ has elimination of imaginaries. Moreover, this is also true topologically, in the sense that if $X \subseteq \Gamma_\infty^n$ and $E$ is a closed, definable equivalence relation on $X$ in an o-minimal expansion of the theory ARCF of real closed fields, then there exists a definable map $f : X \to \Gamma_\infty^n$ inducing a homeomorphism between the topological quotient $X/E$, and $f(X)$ with the topology induced from $\Gamma_\infty^n$.

In another direction, the pair $(\mathrm{k}, \Gamma)$ also eliminates imaginaries (where k is the residue field, with induced structure), and so does $(\mathrm{RES}, \Gamma)$, where RES denotes the generalized residue structure of [17].

However, $(\mathrm{k}, \Gamma)$ or $(\mathrm{RES}, \Gamma)$ do not eliminate imaginaries topologically. One reason for this, due to Eleftheriou [10] and valid already for Γ, is that the theory DOAG of divisible ordered abelian groups is not sufficiently flexible to identify simplices of different sizes. A more essential reason for us is the existence of quotient spaces with nontrivial Galois action on cohomology. For instance take $\pm\sqrt{-1} \times [0,1]$ with $\pm\sqrt{-1} \times \{1\}$ and $\pm\sqrt{-1} \times \{1\}$, $\pm\sqrt{-1} \times \{0\}$ each collapsed to a point. However for connected spaces embedded in $\mathrm{RES}^m \times \Gamma^n$, the Galois action on cohomology is trivial. Hence the above circle cannot be embedded in $\Gamma_\infty^n$. The best we can hope for is that it be embedded in a twisted form $\Gamma_\infty^w$, for some finite set $w$; after base change to $w$, this becomes isomorphic to $\Gamma_\infty^n$.

Theorem 6.3.6 will shows that such an embedding is in fact exists for separated $\Gamma$-internal sets.

It would be interesting to study more generally the definable spaces occurring as closed iso-definable subsets of $\widehat{V}$ parametrized by a subset of $\mathrm{VF}^n \times \Gamma^m$. In the case of $\mathrm{VF}^n$ alone, a key example should be the set of generic points of subvarieties of $V$ lying in some constructible subset of the Hilbert scheme. This includes the variety $V$ embedded with the valuation topology via the simple points functor (Lemma 3.5.1); possibly other components of the Hilbert scheme obtain the valuation topology too, but the different components (of distinct dimensions) are not topologically disjoint.

## 6.2. **Guessing definable maps by regular algebraic maps.**

**Lemma 6.2.1.** *Let $V$ be a normal, irreducible, complete variety, $Y$ an irreducible variety, $g : Y \to X \subseteq V$ a dominant constructible map with finite fibers, all defined over a field $F$. Then there exists a pseudo-Galois covering $f : \widetilde{V} \to V$ such that each component $U$ of $f^{-1}(X)$ dominates $Y$ rationally, i.e. there exists a dominant rational map $g : U \to Y$ over $X$.*

*Proof.* First an algebraic version. Let $K$ be a field, $R$ an integrally closed subring, $G : R \to k$ a ring homomorphism onto a field $k$. Let $k'$ be a finite field extension. Then there exists a finite pseudo-Galois field extension $K'$ and a homomorphism $G' : R' \to k''$ onto a field, where $R'$ is the integral closure of $R$ in $K'$, such that $k''$ contains $k'$.

Indeed we may reach $k'$ as a finite tower of 1-generated field extensions, so we may assume $k' = k(a)$ is generated by a single element. Lift the monic minimal polynomial of $a$ over $k$ to a monic polynomial $P$ over $R$. Then since $R$ is integrally closed, $P$ is irreducible. Let $K'$ be the splitting field of $P$. The kernel of $G$ extends to a maximal ideal $M'$ of the integral closure $R'$ of $R$ in $K'$, and $R'/M'$ is clearly a field containing $k'$.

To apply the algebraic version let $K = F(V)$ be the function field of $V$. Let $R$ be the local ring of $X$, i.e. the ring of regular functions on some Zariski open set not disjoint from $X$, and let $G : R \to k$ be the evaluation homomorphism to the function field $k = F(X)$ of $X$. Let $k' = F(Y)$ the function field of $Y$, and $K'$, $R'$, $G'$, $M'$ and $k''$ be as above. Let $f : \widetilde{V} \to V$ be the normalization of $V$ in $K'$. Then $k''$ is the function field of a component $X'$ of $f^{-1}(X)$, mapping dominantly to $X$. Since $k'$ is contained in $k''$ as extensions of $k$ there exists a dominant rational map $g : X' \to Y$ over $X$. But $\mathrm{Aut}(K'/K)$ acts transitively on the components of $f^{-1}(X)$ mapping dominantly to $X$, proving the lemma. □

**Lemma 6.2.2.** *Let $V$ be an algebraic variety over a field $F$, $X_i$ a finite number of subvarieties, $g_i : Y_i \to X_i$ a surjective constructible map with finite fibers. Then there exists a surjective finite morphism of varieties $f : \widetilde{V} \to V$ such that for any* *field extension $F'$, any $i$, $a \in X_i(F')$, $b \in Y_i(F')$, $c \in \widetilde{V}(F')$ with $g_i(b) = a$ and $f(c) = a$, we have $b \in F'(c)$.*

*Hence there exists a finite number of Zariski open subsets $U_{ij}$ of of $\widetilde{V}$, morphisms $g_{ij} : U_{ij} \to Y_i$ such that for every $a$, $b$, and $c$ as above we have $c \in U_{ij}$ and $b = g_{ij}(c)$ for some $j$.*

*If $V$ is normal, we may take $f : \widetilde{V} \to V$ to be a pseudo-Galois covering.*

*Proof.* If the lemma holds for each irreducible subvariety $V_j$ of $V$, with $X_{j,i} = X_j \cap X_i$ and $Y_{j,i} = g_i^{-1}(X_{j,i})$, then it holds for $V$ with $X_i, Y_i$: assuming $f_j : \widetilde{V_j} \to V_j$ is as in the conclusion of the lemma, let $f$ be the disjoint union of the $f_j$. In this way we may assume that $V$ is irreducible. Clearly we may assume $V$ is complete. Finally, we may assume $V$ is normal, by lifting the $X_i$ to the normalization $V_n$ of $V$, and replacing $Y_i$ by $Y_i \times_{g_i} V_n$. We thus assume $V$ is irreducible, normal and complete.

Let $X_1, \ldots, X_\ell$ be the varieties of maximal dimension $d$ among the subvarieties $X_1, \ldots, X_n$. We use induction on $d$. By Lemma 6.2.1 there exist finite pseudo-Galois coverings $f_i : \widetilde{V_i} \to V$ such that each component of $f_i^{-1}(X_i)$ of dimension $d$ dominates $Y_i$ rationally. Let $V^*$ be an irreducible subvariety of the fiber product $\Pi_V \widetilde{V_i}$ with dominant (hence surjective) projection to each $\widetilde{V_i}$. (The function field of $V^*$ is an amalgam of the function fields of the $\widetilde{V_i}$, finite extensions of the function field of $V$.) Let $f = (f_1, \ldots, f_n)$ restricted to $V^*$. If $a$, $b$, $F'$ and $X_i$ are as above, with $a$ sufficiently generic in $X_i$, then there exists $c \in V^*((F')^{alg})$ with $f_i(c) = a$ and $b \in F'(c)$. Since $f_i$ is a pseudo-Galois covering, for any $c' \in V^*((F')^{alg})$ with $f_i(c') = a$ we have $c' \in F'(c)$, so $b \in F'(c)$. So there exists a dense open subset $W_i \subseteq X_i$ such that for any $a$, $b$, $F'$ and $X_i$ as above, with $a \in W_i$, $f_i(c) = a$, $g_i(b) = a$, we have $b \in F'(c)$.

It follows that there exists a finite number of rational functions $g_{ij}$ defined on Zariski open subsets of $f_i^{-1}(W_i)$, such that for any such $a$, $b$ and $F'$ for some $j$ we have $b = g_{ij}(c)$. By shrinking $W_i$ we may assume that $W_i$ is contained in some affine open subset of $V$, and that $g_{ij}$ is regular above $W_i$. Now we may extend $g_{ij}$ to a regular function on a Zariski open subset $U_{ij}$ of $V^*$.

Let $C_i$ be the complement of $W_i$ in $X_i$; so $\dim(C_i) < d$. Let $\{Y'_\nu\}$ be the pullbacks to $V^*$ of $Y_j$ for $j > \ell$, as well as the pullbacks of $C_i$ ($i \leq \ell$). By induction, there exists a finite morphism $f' : \widetilde{V}' \to V^*$ dominating the $Y'_\nu$ in the sense of the lemma. Let $\widetilde{V}$ be the normalization of $\widetilde{V}'$ in the normal hull over $F(V)$ of the function field $F(V^*)$. By the remark above Lemma 6.2.1, $\widetilde{V} \to V$ is pseudo-Galois, and clearly satisfies the conditions of the lemma. □

Note that since finite morphisms are projective (cf. [12] 6.1.11), if $V$ is projective then so is $\widetilde{V}$.

**Lemma 6.2.3.** *Let $V$ be a normal projective variety and $L$ an ample line bundle on $V$. Let $H$ be a finite dimensional vector space, and let $h : V \to H$ be a rational map. Then for any sufficiently large integer $m$ there exists sections $s_1, \dots, s_k$ of $\mathfrak{L} = L^{\otimes m}$ such that there is no common zero of the $s_i$ outside the domain of definition of $h$, and such that for each $i$, $s_i \otimes h$ extends to a morphism $V \to \mathfrak{L} \otimes H$.*

*Proof.* Say $H = \mathbb{A}^n$. We have $h = (h_1, \dots, h_n)$. Let $D_i$ be the polar divisor of $h_i$ and $D = \sum_{i=1}^n D_i$. Let $L_D$ be the associated line bundle. Then $h \otimes 1$ extends to a section of $H \otimes L_D$. Since $L$ is ample, for some $m$, $L^{\otimes m} \otimes L_D^{-1}$ is generated by global sections $\sigma_1, \dots, \sigma_k$. Since 1 is a global section of $L_D$, $s_i = 1 \otimes \sigma_i$ is a section of $L_D \otimes (L^{\otimes m} \otimes L_D^{-1}) \cong L^{\otimes m}$. Since away from the support of the divisor $D$, the common zeroes of the $s_i$ are also common zeroes of the $\sigma_i$, they have no common zeroes there. Now $h \otimes s_i = (h \otimes 1) \otimes (1 \otimes s_i)$ extends to a section of $(H \otimes L_D) \otimes (L_D^{-1} \otimes L^{\otimes m}) \cong H \otimes L^{\otimes m}$. ☐

A theory of fields is called an algebraically bounded theory, cf. [31] or [28], if for any subfield $F$ of a model $M$, $F^{alg} \cap M$ is model-theoretically algebraically closed in $M$. By Proposition 2.6.1 (4), ACVF is algebraically bounded. The following lemma is valid for any algebraically bounded theory. We work over a base field $F = \mathrm{dcl}(F)$.

**Lemma 6.2.4.** *Let $F$ be a valued field. Let $V$ and $H$ be $F$-varieties, with $V$ irreducible and normal. Let $\phi$ be an* ACVF*-definable subset of $V \times H$ whose projection to $V$ has finite fibers, all defined over $F$. Then there exists a finite pseudo-Galois covering $\pi : \widetilde{V} \to V$, a finite family of Zariski open subsets $U_i \subseteq V$, $\widetilde{U}_i = \pi^{-1}(U_i)$, and morphisms $\psi_i : \widetilde{U}_i \to H$ such that for any $\tilde{v} \in \widetilde{V}$, if $(\pi(\tilde{v}), h) \in \phi$ then $\tilde{v} \in \widetilde{U}_i$ and $h = \psi_i(\tilde{v})$ for some $i$.*

*Proof.* For $a$ in $V$ write $\phi(a) = \{b : (a, b) \in \phi\}$; this is a finite subset of $H$. Let $p$ be an ACVF-type over $F$ (located on $V$) and $a \models p$. By the algebraic boundedness of ACVF, $\phi(a)$ is contained in a finite normal field extension $F(a')$ of $F(a)$. Let $q = \mathrm{tp}_{ACF}(a'/F)$, and let $h_p : q \to V$ be a rational map with $h_p(a') = a$.

We can also write each element $c$ of $\phi(a)$ as $c = \psi(a')$ for some rational function $\psi$ over $F$. This gives a finite family $\Psi = \Psi(p)$ of rational functions $\psi$; enlarging it, we may take it to be Galois invariant. For any $c' \models q$ with $h_p(c') = a$, we have $\phi(a) \subseteq \Psi(c') := \{\psi(c') : \psi \in \Psi\}$.

The type $q$ can be viewed as a type of elements of an algebraic variety $W$, and after shrinking $W$ we can take $h_p$ to be a quasi-finite morphism on $W$, and assume each $\psi \in \Psi : W \to H$ is defined on $W$; moreover we can find $W$ such that:

$(*)$ for any $c' \in W$ with $h(c') = a \models p$, we have $\phi(a) \subseteq \Psi(c')$.

By compactness, there exist finitely many triples $(W_j, \Psi_j, h_j)$ such that for any $p$, some triple has $(*)$ for $p$. By Lemma 6.2.2, we may replace the $W_j$ by a single pseudo-Galois $\widetilde{V}$. $\square$

If $H$ is a vector space, or a vector bundle over $V$, let $H^n$ be the $n$-th direct power of $H$, and let $P(H^n)$ denote the projectivization of $H^n$. Let $h \mapsto: h:$ denote the natural map $H \smallsetminus \{0\} \to PH$. Let $r_k : P(H^n) \to PH$ be the natural rational map, $r_k(h_1 : \ldots : h_n) = (: h_k :)$. For any vector bundle $L$ over $V$, there is a canonical isomorphism $L \otimes H^n \cong (L \otimes H)^n$. When $L$ is a line bundle, we have $P(L \otimes E) \cong P(E)$ canonically for any vector bundle $E$. Composing, we obtain an identification of $P((L \otimes H)^n)$ with $P(H^n)$.

**Lemma 6.2.5.** *Let $F$ be a valued field. Let $V$ be a normal irreducible $F$-variety, $H$ a vector space with a basis of $F$-definable points, and $\phi$ an* $\mathrm{ACVF}_F$*-definable subset of $V \times (H \smallsetminus (0))$ whose projection to $V$ has finite fibers. Then there exist a finite Galois covering $\pi : \widetilde{V} \to V$, a regular morphism $\theta : \widetilde{V} \to P(H^m)$ for some $m$, such that for any $\tilde{v} \in \widetilde{V}$, if $(\pi(\tilde{v}), h) \in \phi$ then for some $k$, $r_k(\theta(\tilde{v}))$ is defined and equals $: h :$.*

*Proof.* Replacing $V$ by the normalization of the closure of $V$ in some projective embedding, we may assume $V$ is projective and normal. Let $\psi_i$ be as in Lemma 6.2.4. Let $L, s_{ij}$ be as in Lemma 6.2.3, applied to $\widetilde{V}$, $\psi_i$; choose $m$ that works for all $\psi_i$. Let $\theta_{ij}$ be the extension to $\widetilde{V}$ of $s_{ij} \otimes \psi_i$. Define $\theta = (\cdots : \theta_{ij} : \cdots)$, using the identification above the lemma. $\square$

## 6.3. $\Gamma$-internal subsets of $\widehat{V}$.

**Lemma 6.3.1.** *Let $V$ be a quasi-projective variety over an infinite valued field $F$, and let $f : \Gamma^n \to \widehat{V}$ be definable. There exists an affine open $V' \subseteq V$ with $f : \Gamma \to \widehat{V}'$. If $V = \mathbb{P}^n$, there exists a linear hyperplane $H$ such that $f(\Gamma^n) \cap \widehat{H} = \emptyset$.*

*Proof.* Since $V$ embeds into $\mathbb{P}^n$, we can view $f$ as a map into $\widehat{\mathbb{P}^n}$; so we may assume $V = \mathbb{P}^n$. For $\gamma \in \Gamma^n$, let $V(\gamma)$ be the linear Zariski closure of $f(\gamma)$; i.e.the intersection of all hyperplanes $H$ such that $f(\gamma)$ concentrates on $H$. The intersection of $V(\gamma)$ with any $\mathbb{A}^n$ is the zero set of all linear polynomials $g$ on on $\mathbb{A}^n$ such that $f(\gamma)_*(h \circ g) = 0$. So $V(\gamma)$ is definable uniformly in $\gamma$. Now $V(\gamma)$ is an $ACF_F$-definable set, with canonical parameter $e(\gamma)$; by elimination of imaginaries in $ACF_F$, we can take $e(\gamma)$ to be a tuple of field elements. But functions $\Gamma^n \to \mathrm{VF}$ have finitely many values (every infinite definable subset of VF contains an open subset, and admits a definable map onto k). So there are finitely many sets $V(\gamma)$. Let $H$ be any hyperplane containing none of these. Then no $f(\gamma)$ can concentrate on $H$. $\square$

We shall now make use of the spaces $L(H)$ of semi-lattices of §5.1. Given a basis $v_1, \ldots, v_n$ of $H$, we say that a semi-lattice is diagonal if it is a direct sum $\sum_{i=1}^{n} I_i v_i$, with $I_i$ an ideal of $K$ or $I_i = K$.

**Lemma 6.3.2.** *Let $Y$ be a $\Gamma$-internal subset of $L(H)$. Then there exists a finite number of bases $b^1, \ldots, b^\ell$ of $H$ such that each $y \in Y$ is diagonal for some $b^i$. If $Y$ is defined over a valued field $F$, these bases can be found over $F^{alg}$.*

*Proof.* For $y \in Y$, let $U_y = \{h \in H : Kh \subseteq y\}$. Then $U_y$ is a subspace of $H$, definable from $Y$. The Grassmanian of subspaces of $H$ is an algebraic variety, and has no infinite $\Gamma$-internal definable subsets. Hence there are only finitely many values of $U_y$. Partitioning $Y$ into finitely many sets we may assume $U_y = U$ for all $y \in Y$. Replacing $H$ by $H/U$, and $Y$ by $\{y/H : y \in Y\}$, we may assume $U = (0)$. Thus $Y$ is a set of lattices.

Now the lemma follows from Theorem 2.4.13 (iii) of [13], except that in this theorem one considers $f$ defined on $\Gamma$ (or a finite cover of $\Gamma$) whereas $Y$ is the image of $\Gamma^n$ under some definable function $f$. In fact the proof of 2.4.13 works for functions from $\Gamma^n$; however we will indicate how to deduce the $n$-dimensional case from the statement there, beginning with 2.4.11. We first formulate a relative version of 2.4.11. Let $U = G_i$ be one of the unipotent groups considered in 2.4.11 (we only need the case of $U = U_n$, the full strictly upper triangular group). Let $X$ be a definable set, and let $g$ be a definable map on $X \times \Gamma$, with $g(x, \gamma)$ a subgroup of $U$, for any $(x, \gamma)$ in the domain of $g$. Let $f$ be another definable map on $X \times \Gamma$, with $f(x, \gamma) \in U/g(\gamma)$. Then there exist finitely many definable functions $p_j : X \to \Gamma$, with $p_j \leq p_{j+1}$, definable functions $b$ on $X$, such that letting $g_j^*(x) = \cap_{p_j(x) < \gamma < p_{j+1}(x)} g(x, \gamma)$ we have $b_j(x) \in U/g_j^*(x)$, and

$$(*) \qquad f(x, \gamma) = b_j(x) g_j(x, \gamma)$$

whenever $p_j(x) < \gamma < p_{j+1}(x)$ This relative version follows immediately from 2.4.11 using compactness, and noting that (*) determines $b_j(x)$ uniquely as an element of $U/g_j^*(x)$.

Now by induction, we obtain the multidimensional version of 2.4.11:

Let $g$ be a definable map on a definable subset $I$ of $\Gamma^n$, with $g(\gamma)$ a subgroup of $U$ for each $\gamma \in I$. Suppose $f$ is also a definable map on $I$, with $f(\gamma) \in U/g(\gamma)$. Then there is a partition of $I$ into finitely many definable subsets $I'$ such that for each $I'$ there is $b \in U$ with $f(\gamma) = bg(\gamma)$ for all $\gamma \in I'$.

To prove this for $\Gamma^{n+1} = \Gamma^n \times \Gamma$, apply the case $\Gamma^n$ to the functions $b_j, g_j$ as well as $f, g(x, p_j(x))$ (at the endpoints of the open intervals).

Now the lemma follows as in 2.4.13 for the multidimensional case follows as in [13] 2.4.13. Namely, each lattice $\Lambda$ has a triangular $\mathcal{O}$-basis; viewed as a matrix, it is an element of the triangular group $B_n$. So there exists an element $A \in U_n$ such that $\Lambda$ is diagonal for $A$, i..e $\Lambda$ has a basis $DA$ with $D \in T_n$ a diagonal matrix. If $D'A'$ is another basis for $\Lambda$ of the same form, we have

$DA = ED'A'$ for some $E' \in B_n(\mathcal{O})$. Factoring out the unipotent part, we find that $D^{-1}D' \in T_n(\mathcal{O})$. So $D/T_n(\mathcal{O})$ is well-defined, the group $D^{-1}B_n(\mathcal{O})D$ is well-defined, we have $D^{-1}ED' \in D^{-1}B_n(\mathcal{O})D \cap U_n$, and the matrix $A$ is well-defined up to translation by an element of $g(\Lambda) = D^{-1}B_n(\mathcal{O})D \cap U_n$. By the multidimensional 2.4.11, since $Y$ is $\Gamma$-internal, it admits a finite partition into definable subsets $Y_i$, such that for each $i$, there exists a basis $A$ diagonalizing each $y \in Y_i$.

Moreover, $A$ is uniquely defined up to $\cap_{y \in Y_i} g(y)$. The rationality statement now follows from Lemma 6.3.3. □

**Lemma 6.3.3.** *Let $F$ be a valued field, let $h$ be an $F$-definable subgroup of the unipotent group $U_n$, and let $c$ be an $F$-definable coset of $h$. Then $c$ has a point in $F^{alg}$. If $F$ has residue characteristic 0, or if $F$ is trivially valued and perfect, $c$ has a point in $F$.*

*Proof.* In the non-trivially valued case the statement is clear for $F^{alg}$, since $F^{alg}$ is a model. As in [13], 2.4.11, the lemma can be proved for all unipotent algebraic groups by induction on dimension, so we are reduced to the case of the one-dimensional unipotent group $G_a$. In this case, in equal characteristic 0 we know that any definable ball has a definable point (by averaging a finite set of points). If $F$ is trivially valued, the subgroup must be $G_a, (0), \mathcal{O}$ or $\mathcal{M}$. The group $\mathcal{O}$ has no other $F$-definable cosets. As for $\mathcal{M}$ the definable cosets correspond to elements of the residue field; but each element of the residue field of $F$ is the residue of a (unique) point of $F$. □

**Remark 6.3.4.** Is the rationality statement in Lemma 6.3.3 valid in positive characteristic, for the groups encountered in Lemma 6.3.2, i.e. interesections of conjugates of $B_n(\mathcal{O})$ with $U_n$? This is not important for our purposes since the partition of $Y$ may require going to the algebraic closure at all events.

**Corollary 6.3.5.** *Let $X \subseteq \widehat{\mathbb{A}^N}$ be iso-definable and $\Gamma$-internal over an algebraically closed valued field $F$. Then for some $d$, and finitely many polynomials $h_i$ of degree $\leq d$, the map $p \mapsto (p_*(\mathrm{val}(h_i)))_i$ is injective on $X$.*

*Proof.* By Lemma 5.1.3, the maps

$$p \mapsto J_d(p) = \{h \in H_d : p_*(\mathrm{val}(h)) \geq 0\}$$

separate points on $\widehat{\mathbb{A}^N}$ and hence on $X$. So for each $x \neq x' \in X$, for some $d$, $J_d(x) \neq J_d(x')$. Since $X$ is iso-definable, for some fixed $d$, $J_d$ is injective on $X$. Let $F$ be a finite set of bases as in Lemma 6.3.2, and let $\{h_i\}$ be the set of elements of these basis. Pick $x$ and $x'$ in $X$; if $x_*(h_i) = x'_*(h_i)$ for all $i$, we have to show that $x = x'$, or equivalently that $J_d(x) = J_d(x')$; by symmetry it suffices to show that $J_d(x) \subseteq J_d(x')$. Choose a basis, say $b = (b^1, \ldots, b^m)$, such that $J_d(x)$ is diagonal with respect to $b$; the $b^i$ are among the $h_i$, so $x_*(b^i) = x'_*(b^i)$ for each $i$. It follows that $J_d(x) \cap Kb^i = J_d(x') \cap Kb^i$. But since $J_d(x)$ is diagonal for $b$, it is generated by $\cup_i (J_d(x) \cap Kb^i)$; so $J_d(x) \subseteq J_d(x')$ as required. $\square$

**Proposition 6.3.6.** *Let $V$ be a quasi-projective variety over a valued field $F$. Let $X \subseteq \widehat{V}$ be $\Gamma$-internal as an iso-definable set. Then there exist $m$, $d$ and $h \in H_{d,m}(F^{alg})$ such that, with the notations of §5.2, $\widehat{\tau_h}$ is injective on $X$. If $V$ is projective and $X$ is closed, $\widehat{\tau_h}$ is a homeomorphism onto its image.*

*Proof.* We may take $V = \widehat{\mathbb{P}^N}$. Note that if $\widehat{\tau_h}$ is injective, and $g \in \mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}(N+1)$, it is clear that $\widehat{\tau_{h \circ g}}$ is injective too. By Lemma 6.3.1, there exists a linear hyperplane $H$ with $\widehat{H}$ disjoint from $X$. We may assume $H$ is the hyperplane $x_0 = 0$. Let $X_1 = \{(x_1, \dots, x_N) : [1 : x_1 : \dots : x_N] \in X\}$. By Corollary 6.3.5, there exist finitely many polynomials $h_1, \dots, h_r$ such that $p \mapsto (p_*(h_i))_i$ is injective on $X_1$. Say $h_i$ has degree $\leq d$. Let $H_i(x_0, \dots, x_d) = x_0^d h_i(x_1/x_0, \dots, x_d/x_0)$, and let $h = (x_0^d, \dots, x_N^d, H_1, \dots, H_r)$, $m = N + r$. Then $h \in H_{d,m}$, and it is clear that $\widehat{\tau_h}$ is injective on $X$. $\square$

**Corollary 6.3.7.** *Let $V$ be a quasi-projective variety over a valued field $F$. Let $X \subseteq \widehat{V}$ be $\Gamma$-internal as an iso-definable set. Then there exists an $F$-definable continuous injective map $\alpha : X \to [0, \infty]^w$, for some finite set $w$ definable over $F$.*

*Proof.* By Proposition 6.3.6, such map $\alpha'_a$ exists over a finite Galois extension $F(a)$ over $F$, but possibly with values in $\Gamma_\infty^n$. Replacing each coordinate $\alpha'_i$ by two maps, namely $\max(\alpha'_i, 0)$ and $-\min(\alpha'_i, 0)$, we may assume $\alpha'_i$ takes values in $[0, \infty]$. Let $w$ be the set of Galois conjugates of $a$ over $F$. Define $\alpha(x) \in [0, \infty]^w$ by $\alpha(x)(a) = \alpha_a(x)$. Then the statement is clear. $\square$

**Proposition 6.3.8.** *Let $A$ be a base structure consisting of a field $F$, and a set $S$ of elements of $\Gamma$. Let $V$ be a projective variety over $F$, $X$ a $\Gamma$-internal, $A$-definable subset of $\widehat{V}$. Then there exists a $A$-definable continuous injective map $\phi : X \to [0, \infty]^w$ for some finite set $A$-definable set $w$. If $X$ is closed, then $\phi$ is a topological embedding.*

*Proof.* We have $\mathrm{acl}(A) = \mathrm{dcl}(A \cup F^{alg}) = F^{alg}(S)$ (Lemma 2.7.2). It suffices to show that a continuous, injective $\phi : X \to [0, \infty]^n$ is definable over $\mathrm{acl}(A)$, for then the descent to $A$ can be done as in Corollary 6.3.7. So we may assume $F = F^{alg}$, hence $A = \mathrm{acl}(A)$. We may also assume $S$ is finite, since the data is defined over a finite subset. Say $S = \{\gamma_1, \dots, \gamma_n\}$. Let $q$ be the generic type of field elements $(x_1, \dots, x_n)$ with $\mathrm{val}(x_i) = \gamma_i$. Then $q$ is stably dominated. If $c \models q$, then by Lemma 6.3.6 there exists an $A(b)$-definable topological embedding $f_b : X \to \Gamma^n$ for some $n$ and some $b \in F(c)^{alg}$. Since $q$ is stably dominated, and $A = \mathrm{acl}(A)$, $\mathrm{tp}(b/A)$ extends to a stably dominated $A$-definable type $p$. If $(a, b) \models p^2|A$ then $f_a f_b^{-1} : X \to X$; but $\mathrm{tp}(ab/A)$ is orthogonal to $\Gamma$ while $X$ is $\Gamma$-internal, so the canonical parameter of $f_a f_b^{-1}$ is defined over $A \cup \Gamma$ and also over $A(a, b)$, hence over

$A$. Thus $f_a f_b^{-1} = g$. If $(a,b,c) \models p^3$ we have $f_b f_c^{-1} = f_a f_c^{-1} = g$ so $g^2 = g$ and hence $g = \mathrm{Id}_X$. So $f_a = f_b$, and $f_a$ is $A$-definable, as required. The last statement is clear since maps from definably compact spaces to $\Gamma_\infty^n$ are closed. $\square$

We proceed towards a relative version of Proposition 6.3.6.

Let $f : V \to U$ be a morphism of algebraic varieties over a valued field $F$. We denote by $\widehat{V/U}$ the subset of $\widehat{V}$ consisting of types $p \in \widehat{V}$ such that $\widehat{f}(p)$ is a simple point of $\widehat{U}$.

**Proposition 6.3.9.** *Let $V \to U$ be a projective morphism of algebraic varieties, with $U$ normal, over a valued field $F$. Let $X \subseteq \widehat{V/U}$ be iso-definable, and relatively $\Gamma$-internal, i.e. such that each fiber $X_u$ of $X$ over $u \in U$ is $\Gamma$-internal. Then there exists a finite pseudo-Galois covering $U' \to U$, such that letting $X' = U' \times_U X$ and $V' = U' \times_U V$, there exists a definable morphism $g : V' \to U' \times \Gamma_\infty^N$ over $U'$, such that the induced map $g : \widehat{V'} \to \widehat{U'} \times \Gamma_\infty^N$ is continuous, and $g|X'$ is injective. In fact Zariski locally each coordinate of $g$ is obtained as a composition of regular maps and the valuation map.*

*Proof.* By Proposition 6.3.6, for each $u \in U$, there exists $h \in H_{d,m}(F(u)^{alg})$ such that $\tau_h$ is injective on the fiber $X_u$ above $u$. By compactness, a finite number of pairs $(m,d)$ will work for all $u$; by taking a large enough $(m,d)$, we may take it to be fixed. Again by compactness, there exists a definable $\phi \subseteq U \times H_{d,m}$ whose projection to $U$ has finite fibers, such that if $(u,h) \in \phi$ then $\tau_h$ is injective on $X_u$. By Lemma 6.2.5, there is a finite pseudo-Galois covering $\pi : U' \to U$, and a regular morphism $\theta : U' \to P(H'^M_{d,m})$ for some $M$, with $H'_{d,m}$ the vector space generated by $H_{d,m}$, such that for any $u' \in U'$, if $(\pi(u'),h) \in \phi$ then, for some $k$, $r_k(\theta(u'))$ is defined and equals $: h :$. Note that since $h \in H_{d,m}$, it follows that $\theta(u') \in PH_{Mm,d}$. Let $g(u',v) = (u', \tau_{\theta(u')})(v)$. Then it is clear that $g$ is continuous and that its restriction to $X'$ is injective. $\square$

**Remark 6.3.10.** The normality hypothesis in Proposition 6.3.9 and Lemma 6.2.2 is unnecessary. If $V$ is any quasi-projective variety, it suffices to replace $V$ in Lemma 6.2.2 with the larger, normal variety $\mathbb{P}^n$ and pull back the data, and similarly for $U$ in 6.3.9.

Note that the proposition has content even when the fibers of $X/U$ are finite.

Under certain conditions, the continuous injection of Proposition 6.3.9 can be seen to be a homeomorphism. This is clear when $X$ is definably compact, but we will need it in somewhat greater generality.

**Definition 6.3.11.** If $\rho : X \to \Gamma_\infty$ is a v+g-continuous function, say $X$ is *compact at* $\rho = \infty$ if any definable type $q$ on $X$ with $\rho_* q$ unbounded has a limit point in $X$.

Compactness at $\rho = \infty$ implies that $\rho^{-1}(\infty)$ is definably compact. If $X$ is a subspace of a definably compact space $Y$, $\rho$ extends to a v+g-continuous definable function $\rho_Y$ on $Y$, and $\rho_Y^{-1}(\infty) \subset X$, then $X$ is compact at $\rho = \infty$. In the applications, this will be the case, with $Y = \widehat{V}$.

We say a pro-definable subset $X$ of $\widehat{V}$, for $V$ an algebraic variety, is $\sigma$-compact with respect to a v+g-continuous definable function $\xi : X \to \Gamma$, if for any $\gamma \in \Gamma$, $\{x \in X : \xi(x) \leq \gamma\}$ is definably compact.

More generally, let $\rho, \xi : X \to \Gamma_\infty$ be v+g-continuous functions. We say that $X$ is $\sigma$-compact via $(\rho, \xi)$ if $\xi^{-1}(\infty) \subseteq \rho^{-1}(\infty)$, $X$ is compact at $\rho = \infty$, and $X \smallsetminus \xi^{-1}(\infty)$ is $\sigma$-compact via $\xi$.

If $X$ is given over $U$ by means of a function $\pi : X \to U$ and $\xi : U \to \Gamma$, we say $X$ is $\sigma$-compact over $U$ via $\xi$ if it is so with respect to $\xi \circ \pi$ (and similarly for $(\rho, \xi)$).

**Lemma 6.3.12.** *In Proposition 6.3.9, assume $X$ is $\sigma$-compact over $U$ via $(\rho, \sigma)$, where $\rho : X \to \Gamma_\infty$ and $\sigma : U \to \Gamma$ are v+g-continuous. Then one can find $g$ as in the Proposition inducing a homeomorphism of $\widehat{X'}$ with its image in $\widehat{U'} \times \Gamma_\infty^N$.*

*Proof.* We add $\rho$ to the list of functions $\xi'$ in the construction of Proposition 6.3.9; the result is that $\rho = \rho' \circ g$ for some continuous $\rho'$ on $\Gamma_\infty^N$. We have $g$ injective and continuous, and must show that $g^{-1}$ is continuous too; equivalently that $g^{-1} \circ \phi$ is continuous for any continuous $\phi : \widehat{X'} \to \Gamma_\infty$. It suffices thus to show that if $W$ is a closed relatively definable subset of $\widehat{X'}$, then $g(W)$ is closed.

By Lemma 4.2.12, it suffices to show this: if $p$ is a definable type on $W$, and $g(w)$ is a limit of $g_* p$ in $\widehat{U'} \times \Gamma_\infty^N$, we have to show that $w$ is a limit of $p$ in $\widehat{X'}$. As $g$ is injective and continuous, it suffices to show that $p$ has a limit in $\widehat{X'}$.

If $\rho_* p$ is unbounded, then the limit point exists by compactness at $\rho = \infty$.

Otherwise, $\rho'$ is bounded on $g_* p$, hence as $\rho'$ is continuous, $\rho'(g(w)) < \infty$. So $\rho(w) \in \Gamma$. Hence $\sigma(\pi(w))$ is defined, and in $\Gamma$. By definition of a limit (say), $\pi(w') \in \Gamma$ and remains bounded for all $w'$ in some neighborhood of $w$, contained in $p$. Thus by $\sigma$-compactness via $\sigma$, $p$ contains a definably compact definable set, containing $w$; so $p$ has a limit in this set, hence in $\widehat{X'}$. $\square$

The following lemma shows that o-minimal covers may be replaced by finite covers carrying the same information, at least as far as homotopy lifting goes.

Given a morphism $g : U' \to U$ and homotopies $h : I \times U \to \widehat{U}$ and $h' : I \times U' \to \widehat{U'}$, we say $h$ and $h'$ are *compatible* or that $h'$ *lifts* $h$ if $\widehat{g}(h'(t, u')) = h(t, g(u'))$ for all $t \in I$ and $u' \in U'$. Here, $I$ refers to any closed generalized interval, with final point $e_I$. Let $H$ be the canonical homotopy $I \times \widehat{U} \to \widehat{U}$ lifting $h$, cf. Lemma 3.7.3. Note that if $h(e, U)$ is iso-definable and $\Gamma$-internal, then $h(e, U) = H(e, \widehat{U})$.

Assume now that $X \subseteq \widehat{V/U}$ is iso-definable and relatively $\Gamma$-internal. We use Lemma 3.4.1 (2) to identify $\widehat{X}$ with a subset of $\widehat{V}$; namely the set $\int_U X$ of $p \in \widehat{V}$ such that if $p$ is based on $A$ and $c \models p|A$, then $\mathrm{tp}(c/A(\pi(c))) = q|A(\pi(c))$ for some $q \in X$. It is really this set that we have in mind when speaking of $\widehat{X}$ below. In particular, it inherits a topology from $\widehat{V}$.

**Lemma 6.3.13.** *Let $\phi : V \to U$ be a morphism of algebraic varieties with $U$ normal, over a valued field $F$. Let $X \subseteq \widehat{V/U}$ be iso-definable over $F$ and relatively $\Gamma$-internal over $U$ (uniformly in $u \in U$).*

*Let $\rho : X \to \Gamma_\infty$, $X_0 = \rho^{-1}(\Gamma)$, $\sigma : U \to \Gamma$ be v+g-continuous. Assume $X$ is $\sigma$-compact over $U$ via $(\rho, \sigma)$.*

*Then there exists a pseudo-Galois covering $U'$ of $U$, and a definable function $j : X \times_U U' \to U' \times \Gamma_\infty^m$ over $U'$, inducing a homeomorphism of $\widehat{X \times_U U'}$ with the image in $\widehat{U'} \times \Gamma_\infty^m$. Moreover:*

1. *There exist a finite number of $F$-definable functions $\xi_i'' : U \to \Gamma_\infty$, such that, for any compatible pair of definable homotopies $h : I \times U \to \widehat{U}$ and $h' : I \times U' \to \widehat{U'}$, if $h$ respects the functions $\xi_i''$, then $h$ lifts to a definable homotopy $H_X : I \times \widehat{X} \to \widehat{X}$. If the image of $h$ is $\Gamma$-internal, the same is true of the image of $H_X$.*
2. *Given a finite number of $F$-definable functions $\xi : X \to \Gamma_\infty$ on $X$, and a finite group action on $X$ over $U$, one can choose the functions $\xi_i' : U' \to \Gamma_\infty$ such that the lift $I \times \widehat{X} \to \widehat{X}$ respects the given functions $\xi$ and the group action.*
3. *If $h'$ satisfies condition $(*)$ of 5.3, one may also impose that $H_X$ satisfies $(*)$.*

*Proof.* We take $U'$ and $j$ as given by Proposition 6.3.9 and Lemma 6.3.12 (that is, $j$ is the restriction of $g$). First consider the case when $X \subseteq U \times \Gamma_\infty^N$. There exists a finite number of $F$-definable functions $\xi_i'$ on $U$ such that the set of values $\xi_i'(u)$ determine the fiber $X_u = \{x : (u, x) \in X\}$, as well as the functions $\xi | X_u$ (with $\xi$ as in (2)), and the group action on $X_u$. In other words if $\xi_i'(u) = \xi_i'(u')$ for simple points $u, u'$ then $X_u = X_{u'}$, $\xi(u, x) = \xi(u', x)$ for $x \in X_u$ and $\xi$ from (2), and $g(u, x) = (u, x')$ iff $g(u', x) = (u', x')$ for $g$ a group element from the group acting in (2). Clearly any homotopy $h : I \times U \to \widehat{U}$ respecting the functions $\xi_i'$ lifts to a homotopy $H_X : I \times \widehat{X} \to \widehat{X} = \widehat{U} \times \Gamma_\infty^N$ given by $(t, (u, \gamma)) \mapsto (H(t, u), \gamma)$, where $H$ is the canonical homotopy $I \times \widehat{U} \to \widehat{U}$ lifting $h$ provided by Lemma 3.7.3. Moreover $H_X$ respects the functions of (2) and the group action.

This applies to $X' = X \times_U U'$, via the homeomorphism induced by $j$; so for any pair $(h, h')$ as in (1), if $h'$ respects the functions $\xi_i'$, then $h'$ lifts to a definable homotopy $H' : I \times \widehat{X'} \to \widehat{X'}$, respecting the data of (2). Note that $\int_{U'} X'$ is the pullback of $\int_U X$ under the natural map $\widehat{V'} \to \widehat{V}$, where $V' = V \times_U U'$. Since $V' \to V$ is a proper morphism of algebraic varieties, $\widehat{V'} \to \widehat{V}$ is closed by Lemma 4.2.23, so $\widehat{X'} = \int_{U'} X' \to \int_U X = \widehat{X}$ is closed, and it is surjective since $X' \to X$ is surjective (Lemma 4.2.6). Moreover $H'$ respects the fibers of $X' \to X$

in the sense of of Lemma 5.3.3 Hence by this lemma, $H'$ descends to a homotopy $H_X : I \times \widehat{X} \to \widehat{X}$.

By Lemma 8.6.5, the condition that $h'$ respects the $\xi'$ can be replaced with the condition that $h$ respects certain other definable functions $\xi''$ into $\Gamma$.

Since $X$ is iso-definable uniformly over $U$, Lemma 2.7.4 applies to the image of $H'$; so this image is iso-definable and $\Gamma$-internal. The image of $H$ is obtained by factoring out the action of the Galois group of $U'/U$; by Lemma 2.2.5, the image of $H$ is also iso-definable, and hence $\Gamma$-internal.

The statement regarding condition $(*)$ is verified by construction, using density of simple points and continuity. □

**Example 6.3.14.** In dimension $> 1$ there exist definable topologies on definable subsets of $\Gamma^n$, induced from function space topologies, for which Proposition 6.3.6 fails. For instance let $X = \{(s,t) : 0 \leq s \leq t\}$. For $(s,t) \in X$ consider the continuous function $f_{s,t}$ on $[0,1]$ supported on $[s,t]$, with slope 1 on $(s, s+\frac{s+t}{2})$, and slope $-1$ on $(s+\frac{s+t}{2}, t)$. The topology induced on $X$ from the Tychonoff topology on the space of functions $[0,1] \to \Gamma$ is a definable topology, and definably compact. Any neighborhood of the function 0 (even if defined with nonstandard parameters) is a finite union of bounded subsets of $\Gamma^2$, but contains a "line" of functions $f_{s,s+\varepsilon}$ whose length is at least $1/n$ for some standard $n$, so this topology is not induced from any definable embedding of $X$ in $\Gamma_\infty^m$. By Proposition 6.3.6, such topologies do not occur within $\widehat{V}$ for an algebraic variety $V$.

## 7. Curves

### 7.1. **Definability of $\widehat{C}$ for a curve $C$.** 

Recall that a pro-definable set is called iso-definable if it is isomorphic, as a pro-definable set, to a definable set.

**Proposition 7.1.1.** *Let $C$ be an algebraic curve defined over a valued field $F$. Then $\widehat{C}$ is an iso-definable set. The topology on $\widehat{C}$ is definably generated, that is, generated by a definable family of (iso)-definable subsets.*

*Proof.* Let $L$ be the function field of $C$ with genus $g$. Let $Y$ be the set of elements $f \in L$ with at most $g+2$ poles.

**Claim.** Any element of $L^\times$ is a product of finitely many elements of $Y$.

*Proof of the Claim.* We use induction on the number of poles of $f \in L^\times$. If this number is $\leq g+2$, then $f \in Y$. Otherwise, let $a_1, \dots, a_H$ be poles of $f$, not necessarily distinct, and let $b$ be a zero of $f$. Using Riemann-Roch, one finds $f_1$ with poles at most at $a_1, \dots, a_{g+2}$, and a zero at $b$. Then $f_1 \in Y$, and $f/f_1$ has fewer poles than $f$ (say $f$ has $m$ poles; they are all among the poles of $f$; and $f_1$ has at most $m-1$ zeroes other than $b$). The statement follows by induction. □

Choose an embedding of the smooth projective model of $C$ in some projective space. Let $W$ be the set of pairs of homogeneous polynomials of degree $N$. We consider the morphism $f : C \times W \to \Gamma_\infty$ mapping $(x, \varphi, \psi)$ to $v(\varphi(x)) - v(\psi(x))$ or to 0 if $x$ is a zero of both $\varphi, \psi$.

With notations from the proof of Theorem 3.1.1, $f$ induces a mapping $\widehat{C} \to Y_{W,f}$ with $Y_{W,f}$ definable. Now, let us remark that any type $p$ on $C$ induces a valuation on $L$ in the following way: let $c \models p$ send $g$ in $L$ to $v(g(c))$ (or say to the symbol $-\infty$ if $c$ is a pole of $g$), and that different types give rise to different valuations. It follows that the map $\widehat{C} \to Y_{W,f}$ is injective, since if two valuations agree on $Y$ they agree on $L^\times$. This shows that $\widehat{C}$ is iso-$\infty$-definable set. Since $\widehat{C}$ is strict pro-definable by Theorem 3.1.1 it follows it is iso-definable. The statement on the topology is clear. □

**Example 7.1.2.** : $\widehat{\mathbb{P}^1}$ may be decribed as the set of generic types of closed balls $B(x, \alpha) := \{y : \mathrm{val}(y - x) \geq \alpha\}$, for $x$ and $\alpha$ running over $F$ and $\Gamma_\infty(F)$, repectively, together with the type corresponding to the point $\infty$. Note that by definition, as sets, $\widehat{\mathbb{P}^1}$ consists of the point just mentioned and of $\widehat{\mathbb{A}^1}$. For the latter see [13], 2.3.6, 2.3.8, 2.5.5.

Let $f : C \to V$ be a relative curve over an algebraic variety $V$, that is, $f$ is flat with fibers of dimension 1. Let $\widehat{C/V}$ be the set of $p \in \widehat{C}$ such that $\widehat{f}(p)$ is a simple point of $\widehat{V}$. Then we have the following relative version of Proposition 7.1.1:

**Lemma 7.1.3.** *Let $f : C \to V$ be a relative curve over an algebraic variety $V$. Then $\widehat{C/V}$ is iso-definable.*

*Proof.* The proof is the obvious relativization of the proof of Proposition 7.1.1. We embed $C$ in $\mathbb{P}^m_V$. Note that the genus of the curves $C_a = f^{-1}(a)$ is bounded, and there exists a number $N$ such that for any $a \in V$, any function on $C_a$ with $\leq g+2$ poles is the quotient of two homogeneous polynomials of degree $N$. Denoting by $W$ the set of functions of the form $\mathrm{val}(f) - \mathrm{val}(g)$ (with $f, g$ two homogeneous polynomials of degree $N$) as well as the characteristic functions of points of $V$, we see that the map $\widehat{C} \to Y_{W,f}$ is injective, and proceed as in Proposition 7.1.1. □

7.2. **A question on finite covers.** To explain the use of Riemann-Roch in the previous subsection proof was roundabout, we pose a question that we can answer positively in characteristic zero. When the answer is positive, the definablity of $\widehat{C}$ follows from that of $C = \mathbb{P}^1$ which is clear by Example 7.1.2. This subsection will not be used in the sequel.

**Question 7.2.1.** If $f : U \to V$ is a finite morphism of algebraic varieties, is the inverse image of an iso-definable subset of $\widehat{V}$ iso-definable?

**Proposition 7.2.2.** *Assume the residue characteristic is 0. Let $f : U \to V$ be a definable map with finite fibers. Let $Y$ be an iso-definable subset of $\widehat{V}$. Let $Y' = f^{-1}(Y) \subseteq \widehat{U}$. Then $Y'$ is iso-definable.*

*Proof.* Since we assume residue characteristic 0, by [17], we may assume $U$ is a cover of the form $V \times_{g(V)} W$, with $g : V \to V'$ a definable morphism, $V'$ and $W$ both defined over RV, and $W$ a finite cover of $g(V)$.

It follows from Lemma 2.9.2 that $\widehat{V'}$ is a countable increasing union of definable sets $U_i$. Since $Y$ is the union of the relatively definable subsets $Y \cap \widehat{g}^{-1}(U_i)$, it follows by compactness that $Y \subseteq \widehat{g}^{-1}(U_i)$ for some $i$. Hence $\widehat{g}(Y)$ is an $\infty$-definable subset of $U_i$. Since by Lemma 2.2.3, $g_*(Y)$ is strict pro-definable, it is definable.

Thus we may assume $f : U \to V$ is defined over RV. Using the terminology above Lemma 2.9.2, an element of $Y$ is the generic type of an irreducible subvariety of some $V_{\bar{\gamma}}$, of some bounded degree $d$. Over the residue field, if $V$ is an algebraic variety, $\widehat{V}$ corresponds to the set of irreducible subvarieties; stratifiying and taking projective embeddings, and using a form of Bézout, it is clear that a degree bound on an algebraic variety $U$ gives a degree bound on any irreducible component of $f^{-1}(U)$. This immediately extends to the stable part of RV, as in Lemma 2.9.2. □

**Remark 7.2.3.** The proof of Proposition 7.2.2 also shows that if $f : U \to V$ is a morphism of algebraic varieties, tamely ramified above each each irreducible subvariety, i.e. above each valuation on the function field of an irreducible subvariety compatible with the valuation on the base field, then the inverse image of an iso-definable subset of $\widehat{V}$ is iso-definable.

7.3. **Definable types on curves.** Let $V$ be an algebraic variety. Two pro-definable functions $f, g : [a, b) \to \widehat{V}$ are said to have the same germ if $f|[a', b) = g|[a', b)$ for some $a'$.

**Remark 7.3.1.** The germ of a pro-definable function into $\widehat{V}$ is always the germ of a path. Indeed if $f : [a, b) \to \widehat{V}$ is pro-definable, there exists a unique smallest $a' > a$ such that $f|(a, b)$ is continuous. This is a consequence of the fact that we will see later, that the image of $f$, being a $\Gamma$-internal subset of $\widehat{V}$, is homeomorphic to a subset of $\Gamma_\infty^n$. It follows from o-minimal automatic continuity that $f$ is piecewise continuous. Moreover, the topology of $\widehat{V}$ restricted to $f([a, b))$ is a definable topology in the sense of Ziegler; so the set of $a'$ with $(a', b)$ continuous is definable, and so a least element exists.

**Proposition 7.3.2.** *Let $C$ be a curve, defined over $A$. There is a canonical bijection between:*

(1) *$A$-definable types on $C$.*
(2) *$A$-definable germs at $b$ of (continuous) paths $[a, b) \to \widehat{C}$, up to reparametrization.*

*Under this bijection, the stably dominated types on $C$ correspond to the germs of constant paths on $\widehat{C}$.*

*Proof.* A constant path, up to reparametrization, is just a point of $\widehat{C}$. In this way the stably dominated types correspond to germs of constant paths into $\widehat{C}$. Let $p$ be a definable type on $C$, which is not stably dominated. Then, by Lemma 2.10.2, for some definable $\delta : C \to \Gamma$, $\delta_*(p)$ is a non-constant definable type on $\Gamma$. Changing sign if necessary, either $\delta_*(p)$ is the type of very large elements of $\Gamma$, or else for some $b$, $\delta_*(p)$ concentrates on elements in some interval $[a, b]$; in the latter case there is a smallest $b$ such that $p$ concentrates on $[a, b)$, so that it is the type of elements just $< b$, or else dually. Thus we may assume $\delta_*(p)$ is the generic at $b$ of an interval $[a, b)$ (where possibly $b = \infty$).

By Proposition 2.10.5 there exists a $\delta_*(p)$-germ $f$ of definable function to $\widehat{C}$ whose integral is $p$. It is the germ of a definable function $f = f_{p,\delta} : [a_0, b) \to \widehat{C}$; since $\widehat{C}$ is definable and the topology is definably generated by Proposition 7.1.1, for some (not necessarily definable) $a$, the restriction $f = f_{p,\delta} : [a, b) \to \widehat{C}$ is continuous. The germ of this function $f$ is well-defined.

Conversely, given $f : [a, b) \to \widehat{C}$, we obtain a definable type $p_f$ on $C$; namely $p_f|E = \mathrm{tp}(e/E)$ if $t$ is generic over $E$ in $[a, b)$, and $e \models f(t)|E(t)$. It is clear that $p_f$ depends only on the germ of $f$, that $p = p_{f_{p,\delta}}$ and $\delta \circ f = \mathrm{Id}$. Hence if the germ of $f$ is $A$-definable, then each $\phi$-definition $d_{p_f}\phi$ is $A$-definable, and so $p_f$ is $A$-definable. A change in the choice of $\delta$ corresponds to reparametrization. □

**Remarks 7.3.3.**
1. The same proof gives a correspondence between invariant types on $C$, and germs at $b$ of paths to $\widehat{C}$, up to reparametrization, where now $b$ is a Dedekind cut in $\Gamma$.
2. Assume $C$ is $M$-definable, and $p$ a definable type over $C$. If $M$ is a maximally complete model, or in the definable case if $M = \mathrm{dcl}(F)$ for a field $F$, the germ in (2) is represented by an $M$-definable path.
3. Without the assumptions on $M$ in (2), the germ may not have an $M$-definable representative. For instance assume $M$ is the canonical code for an open ball of size $b$. The path in question takes $t \in (b, \infty)$ to the generic type of a closed sub-ball of $M$, of size $t$, containing a given point $p_0$. The germ at $b$ does not depend on $p_0$, but there is no definable representative over $M$.

7.4. **Lifting paths.** Let us start by an easy consequence of Hensel's lemma, valid in all dimensions; it will not be used, but may help indicate where the difficulties lie (by showing where they do not.)

**Lemma 7.4.1.** *Let $f : X \to Y$ be a finite morphism between smooth varieties, and let $x \in X$ be a closed point. Assume $f$ is unramified at $x \in X$. Then there exists neighborhoods $N_x$ of $x$ in $\widehat{X}$ and $N_y$ of $y$ in $\widehat{Y}$ such that $\widehat{f} : \widehat{X} \to \widehat{Y}$ induces a homeomorphism $N_x \to N_y$.*

*Proof.* By Hensel's lemma, there exist valuative neighborhoods $V_x$ of $x$ and $V_y$ of $y$ such that $f$ restricts to a bijection $V_x \to V_y$. We take $V_x$ and $V_y$ to be defined by weak inequalities; let $U_x$ and $U_y$ be defined by the corresponding strict inequalities. Then $f$ induces a continuous bijection $\widehat{V_x} \to \widehat{V_y}$ which is a homeomorphism by definable compactness. In particular, $f$ induces a homeomorphism $N_x \to N_y$, where $N_x = \widehat{U_x}$ and $N_y = \widehat{U_y}$. □

In fact this gives a notion of a small closed ball on a curve, in the following sense:

**Lemma 7.4.2.** *Let $F$ be a valued field, $C$ be a smooth curve over $F$, and let $a \in C(F)$ be a point. Then there exists an $\mathrm{ACVF}_F$-definable decreasing family $b(\gamma)$ of g-closed, v-clopen definable subsets of $C$, with intersection $\{a\}$. Any two such families agree eventually up to reparametrization, in the sense that if $b'$ is another such family then for some $\gamma_0, \gamma_1 \in \Gamma$ and $\alpha \in \mathbb{Q}_{>0}$, for all $\gamma \geq \gamma_1$ we have $b(\gamma) = b'(\alpha\gamma + \gamma_0)$.*

*Proof.* Choose $f : C \to \mathbb{P}^1$, étale at $a$. Then $f$ is injective on some v-neighborhood $U$ of $a$. We may assume $f(a) = 0$. Let $b_\gamma$ be the closed ball of radius $\gamma$ on $\mathbb{A}^1$ centered at 0. For some $\gamma_1$, for $\gamma \geq \gamma_1$ we have $b_\gamma \subseteq f(U)$ since $f(U)$ is v-open. Let $b(\gamma) = f^{-1}(b_\gamma) \cap U$. Note that $A = \{(x, y) \in C \times b_\gamma : f(x) = y\}$ is a v+g-closed and bounded subset of $C \times \mathbb{P}^1$. It follows from Proposition 4.2.18, Proposition 4.2.17 and Lemma 4.2.20 that $b(\gamma)$ is g-closed. Since $f$ is a local v-homeomorphism it is v-clopen.

Now suppose $b'(\gamma)$ is another such family. Let $b'_\gamma = f(b'(\gamma))$. Then by the same reasoning $b'_\gamma$ is a v-clopen, g-closed definable subset of $\mathbb{A}^1$, with $\cap_{\gamma \geq \gamma_2} b'_\gamma = \{0\}$. Each $b'_\gamma$ (for large $\gamma$) is a finite union $\cup_{i=1}^m c_i(\gamma) \smallsetminus d_i(\gamma)$, where $c_i(\gamma)$ is a closed ball and $d_i(\gamma)$ is a finite union of open sub-balls of $c_i(\gamma)$, cf. Holly Theorem, Theorem 2.1.2 of [13]. From [13] it is known that there exists an $F$-definable finite set $S$, meeting each $c_i(\gamma)$ (for large $\gamma$) in one point $a_i$. The valuative radius of $c_i(\gamma)$ must approach $\infty$, otherwise it has some fixed radius $\gamma_i$ for large $\gamma$, forcing the balls in $d_i(\gamma)$ to have eventually fixed radius and contradicting $\cap_\gamma b'_\gamma = \{0\}$. So, for every $i$ and large $\gamma$, $c_i(\gamma)$ are disjoint closed balls centered at $a_i$. It follows that $c_i(\gamma') \smallsetminus d_i(\gamma') \subseteq c_i(\gamma) \smallsetminus d_i(\gamma)$ for $\gamma \ll \gamma'$. We have $a_i \notin d_i(\gamma)$, or else for large $\gamma'$ we would have $c_i(\gamma') \subseteq d_i(\gamma)$. Hence $a_i \in \cap_\gamma c_i(\gamma) \smallsetminus d_i(\gamma)$ and $a_i = 0$.

Now the balls of $d_1(\gamma)$ must also be centered in a point of $S'$ for some finite set $S'$, and for large $\gamma$ we have $c_1(\gamma)$ disjoint from these balls; so $b(\gamma) = c_1(\gamma)$ is a closed ball around 0. For large $\gamma$ it must have valuative radius $\alpha\gamma + \gamma_0$, for some $\alpha \in \mathbb{Q}_{>0}, \gamma_0 \in \Gamma$. □

**Definition 7.4.3.** A continuous map $f : X \to Y$ between topological spaces with finite fibers is *topologically étale* if there exists a closed subset $Z$ of $X \times_Y X$ such that $\Delta_X \cup Z = X \times_Y X$, and $Z \cap \Delta_X = \emptyset$.

**Remark 7.4.4.** Let $f : U \to V$ be a continuous definable map with finite fibers. Let $p$ be an unramified point, i.e. suppose $p$ has a neighborhood above which $f$ is topologically étale. Then, viewing $p$ as a simple point of $\widehat{V}$, it has a neighborhood $W$ such that $f^{-1}(W) = \cup_{i=1}^m W_i$, with $f|W_i$ injective. For general $\widehat{V}$-points this may not be true, for instance for the generic point of ball.

**Lemma 7.4.5.** *Let $f : X \to Y$ be a finite morphism between varieties over a valued field. Let $c : I \to \widehat{Y}$ be a path, and $x_0 \in \widehat{X}$. If $\widehat{f} : \widehat{X} \to \widehat{Y}$ is topologically étale above $c(I)$, then $c$ has at most one lift to a path $c' : I \to \widehat{X}$, with $c'(i_I) = x_0$.*

*Proof.* Let $c'$ and $c''$ be two such lifts. So $\{t : c'(t) = c''(t)\}$ is definable. It contains the initial point, and is closed by continuity. So it suffices to show that if $c'(a) = c''(a)$ then $c'(a+t) = c''(a+t)$, for sufficiently small $t < 0$. This is clear since $(c', c'')$ maps into the (closed) complement of the diagonal. □

**Examples 7.4.6.** (1) In characteristic $p > 0$, let $f : \mathbb{A}^1 \to \mathbb{A}^1$, $f(x) = x^p - x$. Let $a \in \mathbb{A}^1$ be a closed point, and consider the standard path $c_a : (-\infty, \infty] \to \widehat{\mathbb{A}^1}$, with $c_a(t)$ the generic of the closed ball of valuative radius $t$ around $a$. Then $\widehat{f}^{-1}(c_a(t))$ consists of $p$ distinct points for $t > 0$, but of a single point for $t \leq 0$. In this sense $c_a(t)$ is backwards-branching. The set of backwards-branching points is the set of balls of valuative radius 0 which is not a $\Gamma$-internal set. The complement of the diagonal within $\widehat{\mathbb{A}^1} \times_f \widehat{\mathbb{A}^1}$ is the union over $0 \neq \alpha \in \mathbb{F}_p$ of the sets $U_\alpha = \{(c_a(t), c_b(t)) : a - b = \alpha, t > 0\}$. The closure (at $t = 0$) intersects the diagonal in the backwards branching points.
(2) In characteristic 0 the set of branching points is $\Gamma$-internal; namely the balls containing a ramification point.
(3) The generic of $\mathcal{O}$ is a forward branch point of the affine curve $C : y^2 = x(x-1)$, with respect to $x : C \to \mathbb{A}^1$.

Because of Example 7.4.6 (1), we will rely on the classical notion of étale only near initial simple points.

**Lemma 7.4.7.** *Let $C$ be a curve over $F$ and let $a$ be a closed point of $c$.*
(1) *Then there exists a path $c : [0, \infty] \to \widehat{C}$ with $c(\infty) = a$, but $c(t) \neq a$ for $t < \infty$.*
(2) *If $a$ is a smooth point, and $c$ and $c'$ are two such paths then they eventually agree, up to definable reparametrization.*
(3) *If $a$ is in the valuative closure of an $F$-definable $W$, then for large $t$ one has $c(t) \in \widehat{W}$.*

*Proof.* One first reduces to the case where $C$ is smooth. Let $n : \tilde{C} \to C$ be the normalization, and let $\tilde{a} \in \tilde{C}$ be a point such that, if a $W$ is given as above, then $\tilde{a}$ is a limit point of $n^{-1}(W)$. Then the lemma for $\tilde{C}$ and $\tilde{a}$ implies the same for $C$ and $a$. For $\mathbb{P}^1$ the lemma is clear by inspection. In general, find a morphism $p : C \to \mathbb{P}^1$, with $p(c) = 0$ which is unramified above 0. By Lemma

7.4.1 and its proof, there exists a definable homeomorphism for the valuation topology between a definable neighborhood $Y$ of $c$ and a definable neighborhood $W'$ of 0 in $\mathbb{P}^1$ which extends to a homeomorphism between $\widehat{Y}$ and $\widehat{W'}$. If $c$ and $c'$ are two paths to $a$ then eventually they fall into $\widehat{W'}$. This reduces to the case of $\mathbb{P}^1$. For (3) it is enough to notice that one assumes $p(W) \cup \{0\} = W'$. □

**Remark 7.4.8.** More generally let $p \in \widehat{C}$, where $C$ is a curve. If $c \models p$, let $\mathrm{res}(F)(\bar{c})$ be the set of points of $\mathrm{St}_F$ definable over $F(c)$. This is the function field of a curve $\bar{C}$ in $\mathrm{St}_F$. One has a definable family of paths in $\widehat{C}$ with initial point $p$, parameterized by $\bar{C}$. And any such path eventually agrees with some member of the family, up to definable reparametrization.

7.5. **Branching points.** Let $C$ be a (non complete curve) over $F$ together with a finite morphism of algebraic varieties $f : C \to \mathbb{A}^1$ defined over $F$. Given a closed ball $b \subseteq \mathbb{A}^1$, let $p_b \in \widehat{\mathbb{A}^1}$ be the generic type of $b$.

By an *outward path* on $\mathbb{A}^1$ we mean a path $c : I \to \widehat{\mathbb{A}^1}$ with $I$ a interval in $\Gamma_\infty$ such that $c(t) = p_{b(t)}$, with $b(t)$ a ball around some point $c_0$ of valuative radius $t$.

Let $X$ be a definable subset of $C$. By an *outward path* on $(X, f)$ we mean a germ of path $c : [a, b) \to \widehat{X}$ with $f_* \circ c$ an outward path on $\mathbb{A}^1$. We first consider the case $X = C$.

In the next lemma, we do not worry about the field of definition of the path; this will be considered later.

**Lemma 7.5.1.** *Let $p \in \widehat{C}$. Then $p$ is the initial point of at least one outward path on $(C, f)$.*

*Proof.* The case of simple $p$ was covered in Lemma 7.4.7, so assume $p$ is not simple. The point $\widehat{f}(p)$ is a non-simple element of $\widehat{\mathbb{A}^1}$, i.e. the generic of a closed ball $b_p$, of size $\alpha \neq \infty$. Fix a model $F$ of ACVF over which $C$, $p$ and $f$ are defined, $p(F) \neq \emptyset$, and $\alpha = \mathrm{val}(a_0)$ for some $a_0 \in F$. We will show the existence of an $F$-definable outward path with initial point $p$. For this purpose we may renormalize, and assume $b$ is the unit ball $\mathcal{O}$.

Let $c \models p|F$. Then $f(c)$ is generic in $\mathcal{O}$. Since $C$ is a curve, $k(F(c))$ is a function field over $k(F)$ of transcendence degree 1. Let $z : k(F(c)) \to k(F)$ be a place, mapping the image of $f(c)$ in $k(F(c))$ to $\infty$. We also have a place $Z : F(c) \to k(F(c))$ corresponding to the structural valuation on $F(c)$. The composition $z \circ Z$ gives a place $F(c) \to k(F)$, yielding a valuation $v'$ on $F(c)$. Since $z \circ Z$ agrees with $Z$ on $F$, we can take $v'$ to agree with val on $F$. We have an exact sequence:

$$0 \to \mathbb{Z}v'(f(c)) \to v'(F(c)^\times) \to \mathrm{val}(F^\times) \to 0$$

with $0 < -v'(f(c)) < \mathrm{val}(y)$ for any $y \in F$ with $\mathrm{val}(y) > 0$.

Let $q = \mathrm{tp}(c/F; (F(c), v'))$ be the quantifier-free type of $c$ over $F$ in the valued field $(F(c), v')$. In other words, find an embedding of valued fields $\iota : (F(c), v') \to$

$\mathbb{U}$ over $F$, and let $q = \text{tp}(\iota(c)/F)$. Similarly, set $p = \text{tp}(f(c)/F; (F(c), v')) := \text{tp}(\iota(f(c))/F)$. Since $p$ is definable, by Lemma 2.3.2 it follows that $q$ is a definable type over $F$, so we can extend it to a global $F$-definable type. Note that $q$ comes equipped with a definable map $\delta \to \Gamma$ with $\delta_*(q)$ non-constant, namely $\text{val}(f(x))$. According to Proposition 7.3.2, $q$ corresponds to a germ at 0 of a path $c : (-\infty, 0) \to C$. Since for any rational function $g \in F(C)$ regular on $p$, we have $v'(g(c)) = \text{val}(g(c)) \mod \mathbb{Z}v'(f(c))$, one may extend $c$ by continuity to $(-\infty, 0]$ by $c(0) = p$. It is easy to check that $c$ is an outward path, since $f_* \circ c$ is a standard outward path on $\mathbb{A}^1$. □

We note immediately that the number of germs at $a$ of paths as given in the lemma is finite. Fix an outward path $c_0 : [\infty, a] \to \widehat{\mathbb{A}^1}$, with $c_0(a) = f_*(p)$. Let $OP(p)$ be the set of paths $c : [-\infty, a] \to \widehat{C}$ with $c(a) = p$ and $f_* \circ c = c_0$ (on $(b, a)$ for some $b < a$). If $c_1, \ldots, c_N \in OP(p)$ have distinct germs at $a$, then for $a' < a$ and sufficiently close to $a$ the points $c_i(a')$ are distinct; in particular $N \leq \deg(f)$.

**Definition 7.5.2.** A point $p \in \widehat{C}$ is called forward-branching for $f$ if there exists more than one germ of outward paths $c : (b, a] \to \widehat{C}$ with $c(a) = p$, above a given outward path on $\mathbb{A}^1$. We will also say in this case that $f_*(p)$ is forward-branching for $f$, and even that $b$ is forward-branching for $f$ where $f_*(p)$ is the generic type of $b$.

Let $b$ be a closed ball in $\mathbb{A}^1$, $p_b$ the generic type of $b$. Let $M \models \text{ACVF}$, with $F \leq M$ and $b$ defined over $M$, and let $a \models p_b|M$. Define $n(f, b)$ to be the number of types

$$\{\text{tp}(c/M(a)) : f(c) = a\}.$$

This is also the number of types: $\{\text{tp}(c/\operatorname{acl}(F(b))(a)) : f(c) = a\}$ (where $M$ is not mentioned), using the stationarity lemma Proposition 3.4.13 of [13]. Equivalently it is the number of types $q(y, x)$ over $M$ extending $p_b(x)|M$. In other words $n(f, b)$ is the cardinal of the fiber of $\widehat{f}^{-1}(p_b)$, where $\widehat{f} : \widehat{C} \to \widehat{\mathbb{A}^1}$. In particular, the function $b \mapsto n(f, b)$ is definable.

If $b$ is a closed ball of valuative radius $\alpha$, and $\lambda > \alpha$, both defined over $F$, we define a *generic closed sub-ball of $b$ of size $\lambda$* (over $F$) to be a ball of size $\lambda$ around $c$, where $c$ is generic in $b$ over $F$. Equivalently, $c$ is contained in no proper $\operatorname{acl}(F)$-definable sub-ball of $b$.

**Lemma 7.5.3.** *Assume $b$ and $\lambda$ are in* $\operatorname{dcl}(F)$, *and let $b'$ be a generic closed sub-ball of $b$ of size $\lambda$, over $F$. Then $n(f, b') \geq n(f, b)$.*

*Proof.* Let $F(b) \leq M \models \text{ACVF}$, and $M(b') \leq M' \models \text{ACVF}$. Take $a$ generic in $b'$ over $M'$. Then $a$ is also a generic point of $b$ over $F$. Now $n(f, b)$ is the number of types $\{\text{tp}(c/M(a) : f(c) = a\}$, while $n(f, b')$ is the number of types $\{\text{tp}(c/M'(a)) : f(c) = a\}$. The restriction map sending of type over $M'(a)$ to its restriction to $M(a)$ being surjective, we get $n(f, b) \leq n(f, b')$. □

**Lemma 7.5.4.** *The set $FB'$ of closed balls $b$ such that, for some closed $b' \supsetneqq b$, for all closed $b''$ with $b \subsetneqq b'' \subsetneqq b'$, we have $n(f,b) < n(f,b'')$, is a finite definable set, uniformly with respect to the parameters.*

*Proof.* The statements about definability of $FB'$ are clear since $b \mapsto n(f,b)$ is definable. Let us prove that for $\alpha \in \Gamma$, the set $FB'_\alpha$ of balls in $FB'$ of size $\alpha$ is finite. Otherwise, by the Swiss cheese description of 1-torsors in Lemma 2.3.3 of [13], $FB'$ would contain a closed ball $b^*$ of size $\alpha' < \alpha$ such that every sub-ball of $b^*$ of size $\alpha$ is in $FB'$. For each such sub-ball $b'$, for some $\lambda = \lambda(b')$ with $\alpha' < \lambda < \alpha$, we have $n(f,b') < n(f,b'')$ where $b''$ is the ball of size $\lambda(b')$ around $b'$. Recall that the generic type of $b^*$ is generated by $b^*$ and the complements of all proper sub-balls, and that this is a stably dominated type. Now $\lambda$ is a definable function into $\Gamma$, so it is constant generically on $b^*$. Replacing $b^*$ with a slightly smaller ball, we may assume $\lambda$ is constant; so we find $b$ of size $\lambda$ such that for any sub-ball $b'$ of $b$ of size $\alpha$, we have $n(f,b') < n(f,b)$. But this contradicts Lemma 7.5.3.

Hence $FB'$ has only finitely many balls of each size, so it can be viewed as a function from a finite cover of $\Gamma$ into the set of closed balls. Suppose $FB'$ is infinite. Then it must contain all closed balls of size $\gamma$ containing a certain point $c_0 \in C$, for $\gamma$ in some proper interval $\alpha < \gamma < \alpha'$ (again by Lemma 2.3.3 of [13]). But then by definition of $FB'$ we find $b_1 \subset b_2 \subset \dots$ with $n(f,b_1) < n(f,b_2) < \dots$, a contradiction. □

**Proposition 7.5.5.** *The set of forward-branching points for $f$ is finite.*

*Proof.* By Lemma 7.5.4 it is enough to prove that if $p_b$ is forward-branching, then $b \in FB'$. Let $n = n(f,b) = |\widehat{f}^{-1}(p_b)|$. Let $c$ be an outward path on $\widehat{\mathbb{A}^1}$ beginning at $p_b$. For each $q \in \widehat{f}^{-1}(p_b)$ there exists at least one path starting at $q$ and lifting $c$ by Lemma 7.5.1, and for some such $q$, there exist more than one germ of such path. So in all there are $> n$ distinct germs of paths $c_i$ lifting $c$. For $b''$ along $c$ sufficiently close to $b$, the $c_i(b'')$ are distinct; so $n(f,b'') > n$. □

**Proposition 7.5.6.** *Let $f : C \to \mathbb{A}^1$ be a finite morphism of curves over a valued field $F$. Let $x_0 \in C$ be a closed point where $f$ is unramified, $y_0 = f(x_0)$, and let $c$ be an outward path on $\widehat{\mathbb{A}^1}$, with $c(\infty) = y_0$. Let $t_0$ be maximal such that $c(t_0)$ is a forward-ramification point of $f$, or $t_0 = -\infty$ if there is no such point. Then there exists a unique $F$-definable path $c' : [t_0,\infty) \to \widehat{C}$ with $\widehat{f} \circ c' = c$, and $c'(\infty) = x_0$.*

*Proof.* Let us first prove uniqueness . Suppose $c'$ and $c''$ are two such paths. By Lemma 7.4.1 and Lemma 7.4.5, $c'(t) = c''(t)$ for sufficiently large $t$. By continuity, $\{t : c'(t) = c''(t)\}$ is closed. Let $t_1$ be the smallest $t$ such that $c'(t) = c''(t)$. Then we have two germs of paths lifting $c$ beginning with $c'(t)$, namely the continuations of $c', c''$. So $c'(t)$ is a forward-branching point, and hence $t \le t_0$. This proves uniqueness on $[t_0,\infty)$.

Now let us prove existence. Since we are aiming to show existence of a unique and definable object, we may increase the base field; so we may assume the base field $F \models$ ACVF.

**Claim 1.** Let $P \subseteq (t_0, \infty]$ be a complete type over $F$, with $n(f, a) = n$ for $a \in c(P)$. Then there exist continuous definable $c_1, \ldots, c_n : P \to \widehat{C}$ with $f_* \circ c_i = c$, such that $c_i(\alpha) \neq c_j(\alpha)$ for $\alpha \in P$ and $i \neq j \leq n$.

*Proof of the Claim.* The proof is like that of Proposition 7.3.2, but we repeat it. Let $\alpha \in P$, and let $b_1, \ldots, b_n$ be the distinct points of $\widehat{C}$ with $f(b_i) = c(\alpha)$. Then since $\dim(C) = 1$, $\mathrm{rk}_{\mathbb{Q}}\Gamma(F(b_i))/\Gamma(F) \leq 1$, so $\alpha$ generates $\Gamma(F(b_i))/\Gamma(F)$. Hence by Theorem 2.8.2, $\mathrm{tp}(b_i/\operatorname{acl}(F(\alpha))$ is stably dominated. By [13], Corollary 3.4.3 and Theorem 3.4.4, $\operatorname{acl}(F(\alpha)) = \operatorname{dcl}(F(\alpha))$. Thus $\mathrm{tp}(b_i/F(\alpha)) \in \widehat{C}$ is $\alpha$-definable over $F$, and we can write $\mathrm{tp}(b_i/F(\alpha)) = c_i(\alpha)$. $\square$

**Claim 2.** For each complete type $P \subseteq (t_0, \infty]$, over $F$, there exists a neighborhood $(\alpha_P, \beta_P)$ of $P$, and for each $y \in \widehat{f}^{-1}(c(\beta_P))$ a (unique) $F(y)$-definable (continuous) path $c' : (\alpha_P, \beta_P) \to \widehat{C}$ with $\widehat{f} \circ c' = c$ and $c'(\beta_P) = y$.

*Proof of the Claim.* For $P = \{\infty\}$ this again follows from Lemma 7.4.1 and Lemma 7.4.4. For $P$ a point, but not $\infty$, it follows from Lemma 7.5.1. There remains the case that $P$ does not reduce to a point. Say $n(f, a) = n$ for $a \in c(P)$. By Claim 1 there exist disjoint $c_1, \ldots, c_n$ on $P$ with $\widehat{f} \circ c_i = c$. By definability of the space $\widehat{C}$, and compactness, they may be extended to an interval $(\alpha_P, \beta_P]$ around $P$, such that moreover $n(c(\beta_P)) = n$, and the $c_i(\beta_P)$ are distinct. So $\{c_i(\beta_P) : i = 1, \ldots, n\} = \widehat{f}^{-1}(c(\beta_P))$; and the claim follows. $\square$

Now by compactness of the space of types, $(t_0, \infty]$ is covered by a finite union of intervals $(\alpha, \beta)$ where the conclusion of Claim 2 holds. It is now easy to produce $c'$, beginning at $\infty$ and glueing along these intervals. $\square$

**Remark 7.5.7.** Here we continue the path till the first time $t$ such that some point of $C$ above $c(t)$ is forward ramified. It is possible to continue the path $c'$ a little further, to the first point such that $c'(t)$ itself is forward-ramified. However in practice, with the continuity with respect to nearby starting points in mind, we will stop short even of $t_0$, reaching only the first $t$ such that $c(t)$ *contains* a forward-ramified ball.

7.6. **Construction of a deformation retraction.** Let $\mathbb{P}^1$ have the standard metric of Lemma 3.8.1, dependent on a choice of open embedding $\mathbb{A}^1 \to \mathbb{P}^1$. Define $\psi : [0, \infty] \times \mathbb{P}^1 \to \widehat{\mathbb{P}^1}$ by letting $\psi(t, a)$ be the generic of the closed ball around $a$ of valuative radius $t$, for this metric. By definition of the metric, the homotopy preserves $\widehat{\mathcal{O}}$ (in either of the standard copies of $\mathbb{A}^1$). We will refer to $\psi$ as the *standard homotopy* of $\mathbb{P}^1$.

Given a Zariski closed subset $D \subset \mathbb{P}^1$, let $\rho(a, D) = \max\{\rho(a, d) : d \in D\}$. Define $\psi_D : [0, \infty] \times \mathbb{P}^1 \to \widehat{\mathbb{P}^1}$ by $\psi_D(t, a) = \psi(\max(t, \rho(a, D)), a)$. In case $D = \mathbb{P}^1$ this is the identity homotopy, $\psi_D(t, a) = a$; but we will be interested in the case of finite $D$. In this case $\psi_D$ has $\Gamma$-internal image.

Let $C$ be a projective curve over $F$ together with a finite morphism $f : C \to \mathbb{P}^1$ defined over $F$. Working in the two standard affine charts $A_1$ and $A_2$ of $\mathbb{P}^1$, one may extend the definition of forward-branching points of $f$ to the present setting. The set of forward-branching points of $f$ is contained in a finite definable set, uniformly with respect to the parameters.

**Proposition 7.6.1.** *Fix a finite $F$-definable subset $G_0$ of $\widehat{C}$, including all forward-branching points of $f$, all singular points of $C$ and all ramifications points of $f$. Set $G = \widehat{f}(G_0)$ and fix a divisor $D$ in $\mathbb{P}^1$ having a non empty intersection with all balls in $G$ (i.e. all balls of either affine line in $\mathbb{P}^1$, whose generic point lies in $G$). Then $\psi_D : [0, \infty] \times \mathbb{P}^1 \to \widehat{\mathbb{P}^1}$ lifts to a $v + g$-continuous $F$-definable function $[0, \infty] \times C \to \widehat{C}$ extending to a deformation retraction $H : [0, \infty] \times \widehat{C} \to \widehat{C}$ onto a $\Gamma$-internal subset of $\widehat{C}$.*

*Proof.* Fix $y \in \mathbb{P}^1$. The function $c'_y : [0, \infty] \to \widehat{\mathbb{P}^1}$ sending $t$ to $\psi_D(t, y)$ is v+g-continuous. By Proposition 7.5.6, for every $x$ in $C$ there exists a unique (continuous) path $c_x : [0, \infty] \to \widehat{C}$ lifting $c'_{f(x)}$. This path remains within the preimage of either copy of $\mathbb{A}^1$. By Lemma 9.1.1 with $X = \mathbb{P}^1$, it follows that the function $h : [0, \infty] \times C \to \widehat{C}$ defined by $(t, x) \mapsto c_x(t)$ is v+g-continuous. By Lemma 3.7.3, $h$ extends to a deformation retraction $H : [0, \infty] \times \widehat{C} \to \widehat{C}$. To show that $H(0, C)$ is $\Gamma$-internal, it is enough to check that $\widehat{f}(H(0, C))$ is $\Gamma$-internal, which is clear. □

## 8. Specializations and $ACV^2F$

8.1. **g-topology and specialization.** Let $F$ be a valued field, and consider pairs $(K, \Delta)$, with $(K, v_K)$ a valued field extension of $F$, and $\Delta$ a proper convex subgroup of $\Gamma(K)$, with $\Delta \cap \Gamma(F) = (0)$. Let $\pi : \Gamma(K) \to \Gamma(K)/\Delta$ be the quotient homomorphism. We extend $\pi$ to $\Gamma_\infty(K)$ by $\pi(\infty) = \infty$. Let $\mathbf{K}$ be the field $K$ with valuation $\pi \circ v_K$. We will refer to this situation as a g-pair over $F$.

**Lemma 8.1.1.** *Let $F$ be a valued field, $V$ an $F$-variety, and let $U \subseteq V$ be* $\mathrm{ACVF}_F$*-definable. Then $U$ is g-open if and only if for any g-pair $K, \mathbf{K}$ over $F$, we have $U(\mathbf{K}) \subseteq U(K)$. The field $K$ may be taken to have the form $F(a)$, with $a \in U$.*

*Proof.* One verifies immediately that each of the conditions is true if and only if it holds on every $F$-definable open affine. So we may assume $U$ is affine.

Assume $U$ is g-open, and let $K, \mathbf{K}$ be a g-pair over $F$. If $a \in V(K)$ and $a \in U(\mathbf{K})$, we have to show that $a \in U(K)$. If $F$ is trivially valued, let $t$ be such that $\mathrm{val}(t) > \mathrm{val}(K)$; then $K(t), \mathbf{K}(t)$ form a g-pair over $F(t)$; so we may assume $F$ is not trivially valued. Further, $KF^{alg}, \mathbf{K}F^{alg}$ form a g-pair over $F^{alg}$, so we may assume $F \models ACVF$. As $U$ is g-open, it is defined by a positive Boolean combination of strict inequalities $\mathrm{val}(f) < \mathrm{val}(g)$, and algebraic equalities and inequalities over $F$. Since $\pi$ is order-preserving on $\Gamma_\infty$, if $\pi \circ v_K(f) < \pi \circ v_K(g)$ then $v_K(f) < v_K(g)$. The algebraic equalities and inequalities are preserved since the fields are the same. Hence $U(\mathbf{K}) \subseteq U(K)$.

In the reverse direction, let $W = V \smallsetminus U$. Assume $W \subseteq \mathrm{VF}^n$ is $\mathrm{ACVF}_F$-definable, and for any g-pair $K, \mathbf{K}$ over $F$, $W(K) \subseteq W(\mathbf{K})$. We must show that $W$ is g-closed, that is, defined by a finite disjunction of finite conjunctions of weak valuation inequalities $v(f) \leq v(g)$, equalities $f = g$ and inequalities $f \neq g$.

It suffices to show that any complete type $q$ over $F$ extending $W$ implies a finite conjunction of this form, which in turn implies $W$. Let $q'$ be the set of all equalities, inequalities and weak valuation inequalities in $q$; by compactness, it suffices to show that $q'$ implies $W$. Let $a \models q'$, and let $\mathbf{K}$ be the valued field $F(a)$. (We are done if $a \in W$, so we may take $a \in U$.) Let $b \models q$, and let $K = F(b)$. Since $q'$ is complete inasfar as ACF formulas go, $F(a), F(b)$ are $F$-isomorphic, and we may assume $a = b$ and $K, \mathbf{K}$ coincide as fields. Any element $c$ of $K$ can be written as $f(a)/g(a)$ for some polynomials $f, g$. Let $c, c' \in K$; say $c = f(a)/g(a), c' = f'(a)/g'(a)$. If $v_K(c) \geq v_K(c')$ then $v_K(f(a)g'(a)) \geq v_K(f'(a)g(a))$; the weak valuation inequality $v_K(f(x)g'(x)) \geq v_K(f'(x)g(x))$ is thus in $q$, hence in $q'$, so $v_{\mathbf{K}}(f(a)g'(a)) \geq v_{\mathbf{K}}(f'(a)g(a))$, and hence $v_{\mathbf{K}}(c) \geq v_{\mathbf{K}}(c')$. It follows that the map $v_K(c) \mapsto v_{\mathbf{K}}(c)$ is well-defined, and weak order-preserving; it is clearly a group homomorphism $\Gamma(K) \to \Gamma(\mathbf{K})$, and is the identity on $\Gamma(F)$. By the hypothesis, $W(K) \subseteq W(\mathbf{K})$. Since $b \in W(K)$, we have $a \in W(\mathbf{K})$. But $a$ was an arbitrary realization of $q'$, so $q'$ implies $W$. □

**Lemma 8.1.2.** *Let $F_0$ be a valued field, $V$ an $F_0$-variety, and let $W \subseteq V$ be $\mathrm{ACVF}_{F_0}$-definable. Then $W$ is g-closed if and only if for any $F \geq F_0$ with $F$ maximally complete and algebraically closed, and any g-pair $K, \mathbf{K}$ over $F$ such that $\Gamma(K) = \Gamma(F) + \Delta$ with $\Delta$ convex and $\Delta \cap \Gamma(F) = (0)$, we have $W(K) \subseteq W(\mathbf{K})$.*

*When $V$ is an affine variety, $W$ is g-closed iff $W \cap E$ is g-closed for every bounded, g-closed, $F_0$-definable subset of $V$.*

*Proof.* The "only if" direction follows from Lemma 8.1.1. In the "if" direction, suppose $W$ is not g-closed. By Lemma 8.1.1 there exists a g-pair $K, \mathbf{K}$ over $F_0$ with $W(K) \not\subseteq W(\mathbf{K})$; further, $K$ is finitely generated over $F_0$, so $\Gamma(K) \otimes \mathbb{Q}$ is finitely generated over $\Gamma(F_0) \otimes \mathbb{Q}$ as a $\mathbb{Q}$-space. Let $c_1, \ldots, c_k \in K$ be such that $\mathrm{val}(c_1), \ldots, \mathrm{val}(c_k)$ form a $\mathbb{Q}$-basis for $\Gamma(K_0) \otimes \mathbb{Q}/(\Delta + \Gamma(F_0)) \otimes \mathbb{Q}$. Let $F = F_0(c_1, \ldots, c_k)$. Then $K, \mathbf{K}$ is a g-pair over $F$, $\Gamma(K) = \Gamma(F) + \Delta$, and $W(K) \not\subseteq W(\mathbf{K})$. We continue to modify $F, K, \mathbf{K}$. As above we may replace $F$ by $F^{alg}$. Next, let $K'$ be a maximally complete immediate extension of $K$, $F'$ a maximally complete immediate extension of $F$, and embed $F'$ in $K'$ over $F$. Let $\mathbf{K}'$ be the same field as $K'$, with valuation obtained by composing $\mathrm{val} : K' \to \mathrm{val}K' = \mathrm{val}K$ with the quotient map $\mathrm{val}K \to \mathrm{val}K/\Delta$. Then $\mathbf{K}$ embeds in $\mathbf{K}'$ as a valued field. We have now the same situation but with $F$ maximally complete. This proves the criterion.

For the statement regarding bounded sets, suppose again that $W$ is not g-closed; let $K, \mathbf{K}$ be a g-pair as above, $a \in W(K)$, $a \notin W(\mathbf{K})$. Then $a \in V \subset \mathbb{A}^n$; say $a = (a_1, \dots, a_n)$ and let $\gamma = \max_{i \le n} -\mathrm{val}(a_i)$. Then $\gamma \in \Delta + \Gamma(F)$ so $\gamma \le \gamma'$ for some $\gamma' \in \Gamma(F)$. Let $E = \{(x_1, \dots, x_n) \in V : \mathrm{val}(x_i) \ge -\gamma'\}$. Then $E$ is $F$-definable, bounded, g-closed, and $W \cap E$ is not g-closed, by the criterion. $\square$

**Corollary 8.1.3.** *Let $W$ be a definable subset of a variety $V$. Assume whenever a definable type $p$ on $W$, viewed as a set of simple points on $\widehat{W}$, has a limit point $p' \in \widehat{V}$, then $p' \in \widehat{W}$. Then $W$ is g-closed.*

*Proof.* We will verify the criterion of Lemma 8.1.2. Let $(K, \Delta), \mathbf{K}$ be a g-pair, over $F$ with $K$ finitely generated over $F$, and $\Gamma(K) = \Delta + \Gamma(F)$, $F$ maximally complete. Let $a \in W(K)$. Let $a'$ be the same point $a$, but viewed as a point of $V(\mathbf{K})$. We have to show that $a' \in W(\mathbf{K})$. Let $d = (d_1, \dots, d_n)$ be a basis for $\Delta$. Note $\mathrm{tp}(d/F)$ has $0 = (0, \dots, 0)$ as a limit point, in the sense of Lemma 4.2.11. Hence $\mathrm{tp}(d/F)$ extends to an $F$-definable type $q$. Now $\mathrm{tp}(a/F(d))$ is definable by metastability; hence $p = \mathrm{tp}(a/F)$ is definable. Since $F$ is maximally complete and $\Gamma(\mathbf{K}) = \Gamma(F)$, $p' = \mathrm{tp}(a'/F(d'))$ is stably dominated by Theorem 2.8.2, where $d'$ is $d$ viewed in $\Gamma(\mathbf{K})$. Furthermore, $p'$ is a limit of $p$. To check this, since $M$ is an elementary submodel and $p, p'$ are $M$-definable, it suffices to consider $M$-definable open subsets of $\widehat{V}$, of the form $\mathrm{val}(g) < \infty$, $\mathrm{val}(g) < 0$ or $\mathrm{val}(g) > 0$ with $g$ a regular function on a Zariski open subset of $V$. If $p'$ belongs to such an open set, the strict inequality holds of $g(a')$, and hence clearly of $g(a)$; so $p$ belongs to it too. By assumption, $p' \in \widehat{W}$, so $a' \in W$. $\square$

**Lemma 8.1.4.** *Let $F$ be a valued field, $V$ an $F$-variety, and let $U \subseteq V \times \Gamma^\ell$ be* $\mathrm{ACVF}_F$*-definable. Then $U$ is g-closed if and only if for any g-pair $K, \mathbf{K}$ over $F$, $\pi(U(K)) \subseteq U(\mathbf{K})$.*

*Proof.* If $U$ is g-closed then the condition on g-pairs is also clear, since $\pi$ is order-preserving. In the other direction, let $\widetilde{U}$ be the pullback of $U$ to $V \times \mathrm{VF}^\ell$. Then $U$ is g-closed if and only if $\widetilde{U}$ is g-closed. The condition $\pi(U(K)) \subseteq U(\mathbf{K})$ implies $\widetilde{U}(K) \subseteq \widetilde{U}(\mathbf{K})$. By Lemma 8.1.1, since this holds for any g-pair $(K, \mathbf{K})$, $\widetilde{U}$ is indeed g-closed. $\square$

8.2. **v-topology and specialization.** Let $F$ be a valued field, and consider pairs $(K, \Delta)$, with $(K, v_K)$ a valued field extension of $F$, and $\Delta$ a proper convex subgroup of $\Gamma(K)$, with $\Gamma(F) \subseteq \Delta$. Let $R = \{a \in K : v_K(a) > 0 \text{ or } v_K(a) \in \Delta\}$. Then $M = \{a \in R : v_K(a) \notin \Delta\}$ is a maximal ideal of $R$ and we may consider the field $\widetilde{K} = R/M$, with valuation $v_{\widetilde{K}}(r) = v_K(a)$ for nonzero $r = a + M \in \widetilde{K}$. We will refer to $(K, \widetilde{K})$ and the related data as a v-pair over $F$. For an affine $F$-variety $V \subseteq \mathbb{A}^n$, let $V(R) = V(K) \cap R^n$. If $h : V \to V'$ is an isomorphism between $F$-varieties, defined over $F$, then since $F \subseteq R$ we have $h(V(R)) = V'(R)$. Hence $V(R)$ can be defined independently of the embedding in $\mathbb{A}^n$, and the notion can be extended to an arbitrary $F$-variety. We have a residue map $\pi : V(R) \to V(\widetilde{K})$. We will write $\pi(x') = x$ to mean: $x' \in V(R)$ and $\pi(x') = x$, and say: $x'$ specializes to $x$. Note that $\Gamma(\widetilde{K}) = \Delta$. If $\gamma = v_K(x)$ with $x \in R$, we also write $\pi(\gamma) = \gamma$ if $v_K(x) \in \Delta$, and $\pi(\gamma) = \infty$ if $\gamma > \Delta$.

**Lemma 8.2.1.** *Let $V$ be an $F$-variety, $W$ an $\mathrm{ACVF}_F$-definable subset of $V$. Then $W$ is v-closed if for any (or even one nontrivial) v-pair $(K, \widetilde{K})$ over $F$ with $\widetilde{K} = F$, $\pi(W(R)) \subseteq W(\widetilde{K})$. The converse is also true, at least if $F$ is nontrivially valued.*

*Proof.* Since $\mathrm{ACVF}_F$ is complete and eliminates quantifiers, we may assume $W$ is defined without quantifiers. By the discussion above, we may take $V$ to be affine; hence we may assume $V = \mathbb{A}^n$.

Assume the criterion holds. Let $b \in V(\widetilde{K}) \smallsetminus W(\widetilde{K})$. If $a \in V(R), b = \pi(a)$, then $a \notin W$. Thus there exists a $K^{alg}$-definable open ball containing $a$ and disjoint from $W$. Since $F = \widetilde{K}$, we may view $\widetilde{K}$ as embedded in $R$, hence take $a = b$. It follows that the complement of $W$ is open, so $W$ is closed.

Conversely, assume $W$ is v-closed, and let $a \in W(R)$, $b = \pi(a)$. Then $b \in V(\widetilde{K})$. If $b \notin W$, there exists $\gamma \in \Gamma(F)$ such that, in $\mathrm{ACVF}_F$, the $\gamma$-polydisk $D_\gamma(b)$ is disjoint from $W$. However we have $a \in D_\gamma(b)$, and $a \in W$, a contradiction. $\square$

**Lemma 8.2.2.** *Let $U$ be a variety over a valued field $F$, let $f : U \to \Gamma_\infty$ be an $F$-definable function, and let $e \in U(F)$. Then $f$ is v-continuous at $e$ if and only if for any v-pair $K, \widetilde{K}$ over $F$ and any $e' \in U(R)$, with $\pi(e') = e$, we have $f(e) = \pi(f(e'))$.*

*If $F$ is nontrivially valued, one can take $\widetilde{K} = F$.*

*If $f(e) \in \Gamma$ then in fact $f$ is v-continuous at $e$ if and only if it is constant on some v-neighborhood of $e$.*

*Proof.* Embed $U$ in affine space; then we have a basis of v-neighborhoods $N(e, \delta)$ of $e$ in $U$ parameterized by elements of $\Gamma$, with $\delta \to \infty$.

First suppose $\gamma = f(e) \in \Gamma$. Assume for some nontrivial $v$-pair $K, F$ and for every $e' \in U(R)$ with $\pi(e') = e$, we have $f(e) = \pi(f(e'))$. To show that $f^{-1}(\gamma)$ contains an open neighborhood of $e$, it suffices, since $f^{-1}(\gamma)$ is a definable set, to show that it contains an open neighborhood defined over some set of parameters. Now if we take $\delta > \Gamma(F)$, $\delta \in \Gamma(K)$, then any element $e'$ of $N(e, \delta)$ specializes to $e$, i.e. $\pi(e') = e$, hence $f(e) = f('e)$ and $f^{-1}(g)$ contains an open neighborhood.

Conversely if $f^{-1}(\gamma)$ contains an open neighborhood of $e$, this neighborhood can be taken to be $N(e,\delta)$ for some $\delta \in \mathbb{Q}\otimes\Gamma(F)$. It follows that the criterion holds, i.e. $\pi(e') = e$ implies $e' \in N(e,\delta)$ so $f(e') = f(e)$, for any $v$-pair $K, \widetilde{K}$.

Now suppose $\gamma = \infty$. Assume for some nontrivial $v$-pair $K, F$ and for every $e' \in U(R)$ with $\pi(e') = e$, we have $f(e) = \pi(f(e'))$. We have to show that for any $\gamma'$, $f^{-1}((\gamma',\infty))$ contains an open neighborhood of $e$. It suffices to take $\gamma' \in \Gamma(F)$. Indeed as above, any element $e'$ of $N(e,\delta)$ must satisfy $f(e') > \gamma'$, since $\pi f(e') = \infty$. Conversely, if continuity holds, then some definable function $h : \Gamma \to \Gamma$, if $e' \in N(e, h(\gamma'))$ then $f(e') > \gamma'$; so if $\pi(e') = e$, i.e. $e' \in N(e,\delta)$ for all $\delta > \Gamma(F)$, then $f(e') > \Gamma(F)$ so $\pi(f(e')) = \infty$. ☐

**Remark 8.2.3.** Let $f : U \to \Gamma$ be as in Lemma 8.2.2, but suppose it is merely (v-to-g-)-continuous at $e$, i.e. the inverse image of any interval around $\gamma = f(e) \in \Gamma$ contains a v-open neighborhood of $e$. Then $f$ is v-continuous at $e$.

*Proof.* It is easy to verify that under the conditions of the lemma, the criterion holds: $\pi(f(e'))$ will be arbitrarily close to $f(e)$, hence they must be equal. But here is a direct proof. We have to show that $f^{-1}(\gamma)$ contains an open neighborhood of $e$. If not then there are points $u_i$ approaching $e$ with $f(u_i) \neq \gamma$. By curve selection we may take the $u_i$ along a curve; so we may replace $U$ by a curve. By pulling back to the resolution, it is easy to see that we may take $U$ to be smooth. By taking an étale map to $\mathbb{A}^1$ we find an isomorphism of a v-neighborhood of $e$ with a neighborhood of 0 in $\mathbb{A}^1$; so we may assume $e = 0 \in U \subseteq \mathbb{A}^1$. For some neighborhood $U_0$ of 0 in $U$, and some rational function $F$, we have $f(0) = \mathrm{val}(F)$ for $u \in U_0 \smallsetminus 0$. By (v-to-g-)-continuity we have $f(0) = \infty$ or $f(0) = \mathrm{val}(F) \neq \infty$ also. But by assumption $\gamma \neq \infty$. Now $f = \mathrm{val}(F)$ is v-continuous, a contradiction. ☐

**Lemma 8.2.4.** *Let $V$ be an $F$-variety, $W \subseteq W'$ two* $\mathrm{ACVF}_F$*-definable subsets of $V$. Then $W'$ is v-dense in $W$ if and only if for any $a \in W(F)$, for some v-pair $(K,F)$ and $a' \in W'(K)$, $\pi(a') = a$.*

*Proof.* Straightforward, but this and Lemma 8.2.5 will not be used and are left as remarks. ☐

**Lemma 8.2.5.** *Let $U$be an algebraic variety over a valued field $F$, and let $Z$ be an $F$-definable family of definable functions $U \to \Gamma$. Then the following are equivalent:*

*(1) There exists an* $\mathrm{ACVF}_F$*-definable, v-dense subset $U'$ of $U$ such that each $f \in Z$ is continuous.*

*(2) For any $K, \widetilde{K}$ such that $(\widetilde{K}, F)$ and $(K, \widetilde{K})$ are both v-pairs over $F$, for any $e \in U(F)$, for some $e' \in U(\widetilde{K})$ specializing to $e$, for any $f \in Z(\widetilde{K})$ and any $e'' \in U(K)$ specializing to $e'$, we have $f(e'') = f(e')$.*

*Proof.* Let $U'$ be the set of points where each $f \in Z$ is continuous. Then $U'$ is $\mathrm{ACVF}_F$-definable, and by Lemma 8.2.2, for $\widetilde{K} \models \mathrm{ACVF}_F$ we have:
$e' \in U'(\widetilde{K})$ if and only if for any $f \in Z(\widetilde{K})$, any v-pair $(K, \widetilde{K})$ and any $e'' \in U(K)$ specializing to $e'$, $f(e'') = f(e')$.

Thus (2) says that for any v-pair $(\widetilde{K}, F)$, and any $e \in U(F)$, some $e' \in U'(\widetilde{K})$ specializes to $e$. By Lemma 8.2.4 this is equivalent to $U'$ being dense. □

**Lemma 8.2.6.** *Let $U$ be an $F$-definable v-open subset of a smooth quasi-projective variety $V$ over a valued field $F$, let $W$ be an $F$-definable open subset of $\Gamma^m$, let $Z$ be an algebraic variety over $F$, and let $f : U \times W \to \widehat{Z}$ or $f : U \times W \to \Gamma_\infty^k$ be an $F$-definable function. We consider $\Gamma^m$ and $\Gamma_\infty^k$ with the order topology. We say $f$ is (v,o)-continuous at $(a, b) \in U \times W$ if the preimage of every open set containing $f(a, b)$ contains the product of a v-open containing $a$ and an open containing $b$. Then $f$ is (v, o)-continuous if and only if it is continuous separately in each variable. More precisely $f$ is (v, o)-continuous at $(a, b) \in U \times W$ provided that $f(x, b)$ is v-continuous at $a$, and $f(a', y)$ is continuous at $b$ for any $a' \in U$, or dually that $f(a, y)$ is continuous at $b$, and $f(x, b')$ is v-continuous at $a$ for any $b' \in W$.*

*Proof.* Since a base change will not affect continuity, we may assume $F \models \mathrm{ACVF}$. The case of maps into $\widehat{Z}$ reduces to the case of maps into $\Gamma_\infty$, by composing with continuous definable maps into $\Gamma_\infty$, which determine the topology on $\widehat{Z}$. For maps into $\Gamma_\infty^k$, since the topology on $\Gamma_\infty^k$ is the product topology, it suffices also to check for maps into $\Gamma_\infty$. So assume $f : U \times W \to \Gamma_\infty$ and $f(a, b) = \gamma_0$. Suppose $f$ is not continuous at $(a, b)$. So for some neighborhood $N_0$ of $\gamma_0$ (defined over $F$) there exist $(a', b')$ arbitrarily close to $(a, b)$ with $f(a', b') \notin N_0$. Fix a metric on $V$ near $a$, and write $\nu(u)$ for the valuative distance of $u$ from $a$. Also write $\nu'(v)$ for $\min |v_i - b_i|$, where $v = (v_1, \ldots, v_m), b = (b_1, \ldots, b_m)$. For any $F' \supseteq F$, let $r_0^+|F'$ be the type of elements $u$ with $\mathrm{val}(a) < \mathrm{val}(u)$ for every non zero $a$ in $F'$, and let $r_1^-|F'$ be the type of elements $v$ with $0 < \mathrm{val}(v) < \mathrm{val}(b)$ for every $b$ in $F'$ with $\mathrm{val}(b) > 0$. Then $r_0^+, r_1^-$ are definable types, and they are orthogonal to each other, that is, $r_0^+(x) \cup r_1^-(y)$ is a complete definable type. Consider $u, v \in \mathbb{A}^1$ with $u \models r_0^+|F, v \models r_1^-|F$. Since $F(u, v)^{alg} \models \mathrm{ACVF}$, there exist $a' \in U(F(u, v)^{alg})$ and $b' \in W(F(u, v)^{alg})$ such that $\nu(a') \geq \mathrm{val}(u)$, $\nu'(b') \leq \mathrm{val}(v)$, and $f(a', b') \notin N_0$. Note that any nonzero coordinate of $a' - a$ realizes $r_0^+$; since $r_0^+$ is orthogonal to $r_1^-$ and $v \models r_1^-|F(u)$, we have $a' - a \in F(u)^{alg}$, so $a' \in F(u)^{alg}$. Similarly $b' \in \Gamma(F(v)^{alg})$. Say two points of $\Gamma_\infty$ are very close over $F$ if the interval between them contains no point of $\Gamma(F)$. By the continuity assumption (say the first version), $f(a', b')$ is very close to $f(a', b)$ (even over $F(u)$) and $f(a', b)$ is very close to $f(a, b)$ over $F$. So $f(a', b')$ is very close to $f(a, b)$ over $F$. But then $f(a', b') \in N_0$, a contradiction. □

**Corollary 8.2.7.** *More generally, let $f : U \times \Gamma_\infty^\ell \times \Gamma^m \to \widehat{Z}$ be $F$-definable, and let $a \in U \times \{\infty\}^\ell, b \in \Gamma^m$. Then $f$ is (v, o)-continuous at $(a, b)$ if $f(a, y)$ is continuous at $b$, and $f(x, b')$ is (v, o)-continuous at $a$ for any $b' \in W$.*

*Proof.* Pre-compose with $\mathrm{Id}_U \times \mathrm{val} \times \mathrm{Id}_W$. □

**Remark 8.2.8.** It can be shown that a definable function $f : \Gamma^n \to \Gamma$, continuous in each variable, is continuous. But this is not the case for $\Gamma_\infty$. For instance, $|x - y|$ is continuous in each variable, if it is given the value $\infty$ whenever $x = \infty$ or $y = \infty$. But it is not continuous at $(\infty, \infty)$, since on the line $y = x + \beta$ it takes the value $\beta$. By pre-composing with $\mathrm{val} \times \mathrm{Id}$ we see that Lemma 8.2.6 cannot be extended to $W \subseteq \Gamma_\infty^m$.

8.3. **ACV²F.** We consider the theory $ACV^2F$ of triples $(K_2, K_1, K_0)$ of fields with surjective, non-injective places $r_{ij} : K_i \to K_j$ for $i > j$, $r_{20} = r_{10} \circ r_{21}$, such that $K_2$ is algebraically closed. We will work in $ACV^2F_{F_2}$, i.e. over constants for some subfield of $K_2$, but will suppress $F_2$ from the notation. The lemmas below should be valid over imaginary constants too, at least from $\Gamma$.

We let $\Gamma_{ij}$ denote the value group corresponding to $r_{ij}$. Then we have a natural exact sequence

$$0 \to \Gamma_{10} \to \Gamma_{20} \to \Gamma_{21} \to 0.$$

The inclusion $\Gamma_{10} \to \Gamma_{20}$ is given as follows: for $a \in \mathcal{O}_{21}$, $\mathrm{val}_{10}(r_{21}(a)) \mapsto \mathrm{val}_{20}(a)$. Note that if $\mathrm{val}_{10}(r_{21}(a)) = 0$ then $a \in \mathcal{O}_{20}^*$ so $\mathrm{val}_{20}(a) = 0$. The surjection on the right is $\mathrm{val}_{20}(a) \mapsto \mathrm{val}_{21}(a)$.

Note that $(K_2, K_1, K_0)$ is obtained from $(K_2, K_0)$ by expanding the value group $\Gamma_{20}$ by a predicate for $\Gamma_{10}$. On the other hand it is obtained from $(K_2, K_1)$ by expanding the residue field $K_1$.

**Lemma 8.3.1.** *The induced structure on $(K_1, K_0)$ is just the valued field structure; moreover $(K_1, K_0)$ is stably embedded. Hence the set of stably dominated types $\widehat{V}$ is unambiguous for $V$ over $K_1$, whether interpreted in $(K_1, K_0)$ or in $(K_2, K_1, K_0)$.*

*Proof.* Follows from quantifier elimination, cf. [13] Proposition 2.1.3. □

**Lemma 8.3.2.** *let $W$ be a definable set in $(K_2, K_1)$ (possibly in an imaginary sort).*

1. *Let $f : W \to \Gamma_{2,\infty}$ be a definable function in $(K_2, K_1, K_0)$. Then there exist $(K_2, K_1)$-definable functions $f_1, \ldots, f_k$ such that on any $a \in \mathrm{dom}(f)$ we have $f_i(a) = f(a)$ for some $i$.*
2. *Let $f : \Gamma_{21} \to W$ be a $(K_2, K_1, K_0)$-definable function. Then $f$ is $(K_2, K_1)$-definable (with parameters; see remark below on parameters).*

*In fact this is true for any expansion of $(K_2, K_1)$ by relations $R \subseteq K_1^m$.*

*Proof.* We may assume $(K_2, K_1, K_0)$ is saturated.

(1) It suffices to show that for any $a \in W$ we have $f(a) \in \mathrm{dcl}_{21}(a)$, where $\mathrm{dcl}_{21}$ refers to the structure $M_{21} = (K_2, K_1)$. We have at all events that $f(a)$ is fixed by $\mathrm{Aut}(M_{21}/K_1, a)$. By stable embeddedness of $K_1$ in $M_{21}$, we have $f(a) \in \mathrm{dcl}_{21}(e, a)$ for some $e \in K_1$. But by orthogonality of $\Gamma_{21}$ and $K_1$ in $M_{21}$ we have $f(a) \in \mathrm{dcl}_{21}(a)$.

(2) Let $A$ be a base structure, and consider a type $p$ over $A$ of elements of $\Gamma_{21}$. Note that the induced structure on $\Gamma_{21}$ is the same in $(K_2, K_1, K_0)$ as in $(K_2, K_1)$, and that $\Gamma_{21}$ is orthogonal to $K_1$ in both senses. For $a \models p$, $b = f(a)$, let $g(b)$ be an enumeration of the $(K_2, K_1)$-definable closure of $b$ within $K_1$ (over $A$). By orthogonality, $g \circ f$ must be constant on $p$; say it takes value $e$ on $p$. Now $\mathrm{tp}_{21}(ab/e) \models \mathrm{tp}_{21}(ab/K_1)$ by stable embeddedness of $K_1$ within $(K_2, K_1)$. By considering automorphisms it follows that $\mathrm{tp}_{21}(ab/e) \models \mathrm{tp}_{210}(ab/e)$, so $\mathrm{tp}_{21}(ab/e)$ is the graph of a function on $p$; this function must be $f|p$. By compactness, $f$ is $(K_2, K_1)$-definable. □

**Remark 8.3.3.** Let $D$ be definable in $(K_2, K_1, K_0)$ over an algebraically closed substructure $(F_2, F_1, F_0)$ of constants. If $D$ is $(K_2, K_1)$ definable with additional parameters, then $D$ is $(K_2, K_1)$-definable over $(F_2, F_1)$. This can be seen by considering that the canonical parameter must be fixed by $\mathrm{Aut}(K_2, K_1, K_0/F_2, F_1, F_0)$.

**Lemma 8.3.4.** *Let $W$ be a $(K_2, K_1)$-definable set and let $I$ be a definable subset of $\Gamma_{21}$ and let $f : I \times W \to \Gamma_{21,\infty}$ be a $(K_2, K_1, K_0)$-definable function such that for fixed $t \in I$, $f_t(w) = f(t, w)$ is $(K_2, K_1)$-definable. Then $f$ is $(K_2, K_1)$-definable.*

*Proof.* Applying compactness to the hypothesis, we see that there exist finitely many functions $g_k, h_k$ such that $g_k$ is $(K_2, K_1)$-definable, $h_k$ is definable, and that for any $t \in I$ for some $k$ we have $f(t, w) = g_k(h_k(t), w))$. Now by Lemma 8.3.2 (2), $h_k$ is actually $(K_2, K_1)$-definable too. So we may simplify to $f(t, w) = G_k(t, w)$ with $G_k$ a $(K_2, K_1)$-definable function. But every definable subset of $I$ is $(K_2, K_1)$-definable, in particular $\{t : (\forall w)(f(t, w) = G_k(t, w))\}$. From this it follows that $f(t, w)$ is $(K_2, K_1)$-definable. □

**Lemma 8.3.5.** *Let $T$ be any theory, $T_0$ the restriction to a sublanguage $L_0$, and let $\mathbb{U} \models T$ be a saturated model, $\mathbb{U}_0 = \mathbb{U}|L_0$. Let $V$ be a definable set of $T_0$. Let $\widehat{V}$, $\widehat{V_0}$ denote the spaces of generically stable types in $V$ of $T, T_0$ respectively. Then there exists a map $r_0 : \widehat{V} \to \widehat{V_0}$ such that $r_0(p)|\mathbb{U}_0 = (p|\mathbb{U})|L_0$. If $A = \mathrm{dcl}(A)$ (in the sense of $T$) and $p$ is $A$-definable, then $r_0(p)$ is $A$-definable.*

*Proof.* In general, a definable type $p$ of $T$ over $\mathbb{U}$ need not restrict to a definable type of $T_0$. However, when $p$ is generically stable, for any formula $\phi(x, y)$ of $L_0$ the $p$-definition $(d_p x)\phi(x, y)$ is equivalent to a Boolean combination of formulas $\phi(x, b)$. Hence $(d_p x)\phi(x, y)$ is $\mathbb{U}_0$-definable. The statement on the base of definition is clear by Galois theory. □

Returning to $ACV^2F$, we have:

**Lemma 8.3.6.** *Let $V$ be an algebraic variety over $K_1$. Then the restriction map of Lemma 8.3.5 from the stably dominated types of $V$ in the sense of $(K_2, K_1, K_0)$ to those in the sense of $(K_1, K_0)$ is a bijection.*

*Proof.* This is clear since $(K_1, K_0)$ is embedded and stably embedded in $(K_2, K_1, K_0)$. □

We can thus write unambiguously $\widehat{V}_{10}$ for $V$ an algebraic variety over $K_1$.

Now let $V$ be an algebraic variety over $K_2$. Note that $K_1$ may be interpreted in $(K_2, K_0, \Gamma_{20}, \Gamma_{10})$ (the enrichment of $(K_2, K_0, \Gamma_{20})$ by a predicate for $\Gamma_{10}$).

**Lemma 8.3.7.** *Any stably dominated type of $(K_2, K_0)$ in $V$ over $\mathbb{U}$ generates a complete type of $(K_2, K_1, K_0)$. More generally, assume $T$ is obtained from $T_0$ by expanding a linearly ordered sort $\Gamma$ of $L_0$, and that $p_0$ is a stably dominated type of $T_0$. Then $p_0$ generates a complete definable type of $T$; over any base set $A = \mathrm{dcl}(A) \leq M \models T$, $p_0|A$ generates a complete $T$-type over $A$.*

*Proof.* We may assume $T$ has quantifier elimination. Then $\mathrm{tp}(c/A)$ is determined by the isomorphism type of $A(c)$ over $A$. Now since $\Gamma(A(c)) = \Gamma(A)$, any $L_0$-isomorphism $A(c) \to A(c')$ is automatically an $L$-isomorphism. □

**Lemma 8.3.8.** *Assume $T$ is obtained from $T_0$ by expanding a linearly ordered sort $\Gamma$ of $L_0$, and that in $T_0$, a type is stably dominated if and only if it is orthogonal to $\Gamma$. Then the following properties of a type on $V$ over $\mathbb{U}$ are equivalent:*

(1) *$p$ is stably dominated.*
(2) *$p$ is generically stable.*
(3) *$p$ is orthogonal to $\Gamma$.*
(4) *The restriction $p_0$ of $p$ to $L_0$ is stably dominated.*

*Proof.* The implication (1) to (2) is true in any theory, and so is (2) to (3) given that $\Gamma$ is linearly ordered. Also in any theory (3) implies that $p_0$ is orthogonal to $\Gamma$, so by the assumption on $T_0$, $p$ is stably dominated, hence (4). Finally, given that $p_0$ is stably dominated and generates a type $p$ of $L$ (Lemma 8.3.7), it is clear that this type is also stably dominated. Using the terminology from [14] p. 37, say $p$ is dominated via some definable $*$-functions $f : V \to D$, with $D$ a stable ind-definable set of $T_0$.

Since $T$ is obtained by expanding $\Gamma$, which is orthogonal to $D$, the set $D$ remains stable in $T$. Now for any base $A$ of $T$ we have that $p|A$ is implied by $p_0|A$, hence by $(f_*(p_0)|A)(f(x))$, hence by $(f_*(p)|A)(f(x))$. So (4) implies (1). □

**Lemma 8.3.9.** *For $ACV^2F$, the following properties of a type on $V$ over $\mathbb{U}$ are equivalent:*

(1) *$p$ is stably dominated.*
(2) *$p$ is generically stable.*

(3) *$p$ is orthogonal to $\Gamma_{20}$.*

(4) *The restriction $p_{20}$ of $p$ to the language of $(K_2, K_0)$ is stably dominated.*

*Proof.* Follows from Lemma 8.3.8 upon letting $T_0$ be the theory of $(K_2, K_0)$. $\square$

8.4. **The map** $r_{21} : \widehat{V}_{20} \to \widehat{V}_{21}$**.** Let $V$ be an algebraic variety over $K_2$. We have on the face of it three spaces: $\widehat{V}_{2j}$ the space of stably dominated types for $(K_2, K_j)$ for $j = 0$ and 1, and $\widehat{V}_{210}$ the space of stably dominated types with respect to the theory $(K_2, K_1, K_0)$. But in fact $\widehat{V}_{20}$ can be identified with $\widehat{V}_{210}$, as Lemmas 8.3.7 and 8.3.8 show. We thus identify $\widehat{V}_{210}$ with $\widehat{V}_{20}$.

By Lemma 8.3.5, we have a restriction map $r_{21} : \widehat{V}_{20} = \widehat{V}_{210} \to \widehat{V}_{21}$. Note that $r_{21}$ is the identity on simple points. Note also that $r_{21}$ is continuous.

We move towards the $(K_2, K_1)$-definability of the image of $(K_2, K_1, K_0)$-definable paths in $\widehat{V}$.

**Lemma 8.4.1.** *Let $f : \Gamma_{20} \to \widehat{V}_{20}$ be $(K_2, K_1, K_0)$-(pro-) definable. Assume $r_{21} \circ f = \bar{f} \circ \pi$ for some $\bar{f} : \Gamma_{21} \to \widehat{V}_{21}$ with $\pi : \Gamma \to \Gamma_{21}$ be the natural projection. Then $\bar{f}$ is $(K_2, K_1)$-(pro-) definable.*

*Proof.* Let $U$ be a $(K_2, K_1)$-definable set, and let $g : V \times U \to \Gamma_{21}$ be definable. We have to prove the $(K_2, K_1)$-definability of the map: $(\gamma, u) \mapsto g(\bar{f}(\alpha), u)$, where $g(q, u)$ denotes here the $q$-generic value of $g(v, u)$. For fixed $\gamma$, this is just $u \mapsto g(q, u)$ for a specific $q = r_{21}(p)$, which is certainly $(K_2, K_1)$-definable. By Lemma 8.3.4, the map : $(\gamma, u) \mapsto g(\bar{f}(\alpha), u)$ is $(K_2, K_1)$-definable. $\square$

**Lemma 8.4.2.** *Let $f : \Gamma_{20} \to \widehat{V}_{20}$ be a path. Then there exists a path $\bar{f} : \Gamma_{21} \to \widehat{V}_{21}$ such that $r_{21} \circ f = \bar{f} \circ \pi$.*

*Proof.* Let us first prove the existence of $\bar{f}$ as in Lemma 8.4.1. Fixing a point of $\Gamma_{21}$, with a preimage $a$ in $\Gamma$, it suffices to show that $r_{21} \circ f$ is constant on $\{\gamma + a : \gamma \in \Gamma_{10}\}$. Hence, for any $\phi(x, y)$ we need to show that $\gamma \mapsto \pi(f(\gamma + a)_* \phi)$ is constant in $\gamma$; or again that for any $b$, the map $\gamma \mapsto \pi(f(\gamma + a)_* \phi(b))$ is constant in $\gamma$. This is clear since any definable map $\Gamma_{10} \to \Gamma_{21}$ has finite image, and by continuity. By Lemma 8.4.1 $\bar{f}$ is definable, it remains to show it is continuous. This amounts, as the topology on $\widehat{V}$ is determined by continuous functions into $\Gamma$, to checking that if $g : \Gamma \to \Gamma$ is continuous and $(K_2, K_1, K_0)$-definable, then the induced map $\Gamma_{21} \to \Gamma_{21}$ is continuous, which is easy. $\square$

**Example 8.4.3.** Let $a \in \mathbb{A}^1$ and let $f_a : [0, \infty] \to \widehat{\mathbb{A}^1}$ be the map with $f_a(t) =$ the generic of the closed ball around $a$ of valuative radius $t$. Then $r_{21} \circ f_a(t) = f_a(\pi(t))$, where on the right $f_a$ is interpreted in $(K_2, K_1)$ and on the left in $(K_2, K_0)$. Also, if $f_a^\gamma(t)$ is defined by $f_a^\gamma(t) = f_a(\max(t, \gamma))$ for then $r_{21} \circ f_a^\gamma(\bar{t}) = f_a^{\pi(\gamma)}(\bar{t})$.

Let $\mathbb{P}^1$ have the standard metric of Lemma 3.8.1. Given a Zariski closed set $D \subset \mathbb{P}^1$ of points, recall the standard homotopy $\psi_D : [0, \infty] \times \mathbb{P}^1 \to \widehat{\mathbb{P}^1}$ defined in 7.6.

**Lemma 8.4.4.** *For every $(t, a)$ we have $r_{21} \circ \psi_D(t, a) = \psi_D(\pi(t), a)$, where on the right $\psi$ is interpreted in $(K_2, K_1)$ and on the left in $(K_2, K_0)$.*

*Proof.* Clear, since $\pi\rho(a, D) = \rho_{21}(a, D)$. □

**Lemma 8.4.5.** *Let $f : V \to V'$ be an ACF-definable map of varieties over $K_2$. Then $f$ induces $f_{20} : \widehat{V}_{20} \to \widehat{V'}_{20}$ and also $f_{21} : \widehat{V}_{21} \to \widehat{V'}_{21}$. We have $r_{21} \circ f_{20} = f_{21} \circ r_{21}$.*

*Proof.* Clear from the definition of $r_{21}$. □

8.5. **Relative versions.** Let $V$ be an algebraic variety over $U$, with $U$ an algebraic variety over $K_2$, that is, a morphism of algebraic varieties $f : V \to U$ over $K_2$. We have already defined the relative space $\widehat{V/U}$. It is the subset of $\widehat{V}$ consisting of types $p \in \widehat{V}$ such that $\widehat{f}(p)$ is a simple point of $\widehat{U}$. A map $h : W \to \widehat{V/U}$ will be called pro-definable (or definable) if the composite $W \to \widehat{V}$ is. We endow $\widehat{V/U}$ with the topology induced by the topology of $\widehat{V}$. In particular one can speak of continuous, v-, g-, or v+g-continuous maps with values in $\widehat{V/U}$. Exactly as above we obtain $r_{21} : \widehat{V/U}_{20} \to \widehat{V/U}_{21}$, so that the restriction to a definable element $v_0 \in V$ is the $r_{21}$ previously defined on $\widehat{U_0}$, with $U_0$ the fiber over $v_0$.

The relative version of all the above lemmas holds without difficulty:

**Lemma 8.5.1.** *Let $f : \Gamma_{20} \to \widehat{V/U}_{20}$ be $(K_2, K_1, K_0)$-definable. Assume $r_{21} \circ f = \bar{f} \circ \pi$ for some $\bar{f} : \Gamma_{21} \to \widehat{V/U}_{21}$. Then $\bar{f}$ is $(K_2, K_1)$-definable.*

*Proof.* Same proof as Lemma 8.4.1, or by restriction. □

**Lemma 8.5.2.** *For (continuous) paths $f : \Gamma_{20} \to \widehat{V/U}_{20}$ the assumption that $r_{21} \circ f$ factors through $\Gamma_{21}$ is automatically verified.*

*Proof.* This follows from Lemma 8.4.2 since a function on $U \times \Gamma_{20}$ factors through $U \times \Gamma_{21}$ if and only if this is true for the section at a fixed $u$, for each $u$. □

Example 8.4.3 goes through for the relative version $\widehat{\mathbb{A}^1 \times U/U}$, where now $a$ may be taken to be a section $a : U \to \mathbb{A}^1$.

The standard homotopy on $\mathbb{P}^1$ defined in 7.6 may be extended fiberwise to a homotopy $\psi : [0, \infty] \times \mathbb{P}^1 \times U \to \widehat{\mathbb{P}^1 \times U/U}$, which we still call standard. Consider now an ACF-definable (constructible) set $D \subset \mathbb{P}^1 \times U$ whose projection to $U$ has finite fibers. One may consider as above the standard homotopy with stopping time defined by $D$ at each fiber $\psi_D : [0, \infty] \times \mathbb{P}^1 \times U \to \widehat{\mathbb{P}^1 \times U/U}$.

In this framework Lemma 8.4.4 still holds, namely:

**Lemma 8.5.3.** *For every $(t, a)$ we have $r_{21} \circ \psi_D(t, a) = \psi_D(\pi(t), a)$, where on the right $\psi$ is interpreted in $(K_2, K_1)$ and on the left in $(K_2, K_0)$.*

Finally Lemma 8.4.5 also goes through in the relative setting:

**Lemma 8.5.4.** *Let $f : V \to V'$ be an ACF-definable map of varieties over $U$ (and over $K_2$). Then $f$ induces $f_{20} : \widehat{V/U}_{20} \to \widehat{V'/U}_{20}$ and also $f_{21} : \widehat{V/U}_{21} \to \widehat{V'/U}_{21}$. We have $r_{21} \circ f_{20} = f_{21} \circ r_{21}$.* □

8.6. **g-continuity criterion.** Let $F \leq K_2$. Assume $v_{20}(F) \cap \Gamma_{10} = (0)$; so $(F, v_{20}|F) \cong (F, v_{21}|F)$. In this case any $\mathrm{ACVF}_F$-definable object $\phi$ can be interpreted with respect to $(K_2, K_1)_F$ or to $(K_2, K_0)_F$. We refer to $\phi_{20}, \phi_{21}$. In particular if $V$ is an algebraic variety over $F$, then $V_{20} = V_{21} = V$; $\widehat{V}$ is $\mathrm{ACVF}_F$-pro-definable, and $\widehat{V}_{20}, \widehat{V}_{21}$ have the meaning considered above. If $f : W \to \widehat{V}$ is a definable function with $W$ a g-open $\mathrm{ACVF}_F$-definable subset fo $V$, we obtain $f_{2j} : W \to \widehat{V}_{2j}$, $j = 0, 1$. Let $W_{21}, W_{20}$ be the interpretations of $W$ in $(K_2, K_1)$, $(K_2, K_0)$. By Lemma 8.1.1 we have $W_{21} \subseteq W_{20}$.

**Lemma 8.6.1.** *Let $V$ be an algebraic variety over $F$ and $W$ be a g-open $\mathrm{ACVF}_F$-definable subset of $V$. Assume $v_{20}(F) \cap \Gamma_{10} = (0)$.*

(1) *An $\mathrm{ACVF}_F$-definable map $g : W \to \Gamma_\infty$ is g-continuous if and only if $g_{21} = \pi \circ g_{20}$ on $W_{21}$.*
(2) *An $\mathrm{ACVF}_F$-definable map $g : W \times \Gamma_\infty \to \Gamma_\infty$ is g-continuous if and only if $g_{21} \circ \pi_2 = \pi \circ g_{20}$ on $W_{21} \times (\Gamma_{20})_\infty$, where $\pi_2(u, t) = (u, \pi(t))$, $\pi$ being the projection $\Gamma_{20} \to \Gamma_{21}$.*
(3) *An $\mathrm{ACVF}_F$-definable map $f : W \to \widehat{V}$ is g-continuous if and only if $f_{21} = r_{21} \circ f_{20}$ on $W_{21}$.*
(4) *An $\mathrm{ACVF}_F$-definable map $f : W \times \Gamma_\infty \to \widehat{V}$ is g-continuous if and only if $f_{21} \circ \pi_2 = r_{21} \circ f_{20}$ $W_{21} \times (\Gamma_{20})_\infty$.*

*Proof.* (1) The function $g$ is g-continuous with respect to $\mathrm{ACVF}_F$ if and only if for any open interval $I$ of $\Gamma_{21}$, $U = g^{-1}(I)$ is g-open. Let us start with an interval of the form $I = \{x : x > \mathrm{val}_{21}(a)\}$, with $a \in K_2$.

By increasing $F$ we may assume $a \in F$. (We may assume $F = F^{alg}$. There is no problem replacing $F$ by $F(a)$ unless $v_{20}(F(a)) \cap \Gamma_{10} \neq (0)$. In this case it is easy to see that $v_{21}(a) = v_{21}(a')$ for some $a' \in F$, so we may replace $a$ by $a'$.)

We view $U$ as defined by $g(u) > \mathrm{val}(a)$ in $\mathrm{ACVF}_F$. By Lemma 8.1.1, $U$ is g-open if and only if $U_{21} \subseteq U_{20}$, that is, $g_{21}(u) > \mathrm{val}_{21}(a)$ implies $g_{20}(u) > \mathrm{val}_{20}(a)$, or, equivalently, $g_{21}(u) \leq \pi(g_{20}(u))$. By considering intervals of the form $I = \{x : x < \mathrm{val}_{21}(a)\}$ one gets the similar statement for the reverse equality.

(2) Let $G(u, a) = g(u, \mathrm{val}(a))$. Then $g$ is g-continuous if and only if $G$ is g-continuous. The statement follows from (1) applied to $G$ together with Lemma 8.4.1 and Lemma 8.4.2.

For (3) and (4), we pass to affine $V$, and consider a regular function $H$ on $V$. Let $g(u) = f(u)_*(\mathrm{val} H)$. Then $f_{21} = r_{21} \circ f_{20}$ if and only if for each such $H$ we have $g_{21} = \pi \circ g_{20}$; and $f$ is g-continuous if and only if, for each such $H$, $g$ is g-continuous. Thus (3) follows from (1), and similarly (4) from (2). □

**Remark 8.6.2.** A similar criterion is available when $W$ is g-closed rather than g-open; in this case we have $W_{20} \subseteq W_{21}$, and the equalities must be valid on $W_{21}$. In practice we will apply the criterion only with g-clopen $W$.

As an example of using the continuity criteria, assume $h : V \to W$ is a finite morphism of degree $n$ between algebraic varieties of pure dimension $d$, with $W$ normal. For $w \in W$, one may endow $h^{-1}(w)$ with the structure of a multi-set (i.e. a finite set with multiplicities assigned to points) of constant cardinality $n$ as follows. One consider a pseudo-Galois covering $h' : V' \to W$ of degree $n'$ with Galois group $G$ factoring as $h' = h \circ p$ with $p : V' \to V$ finite of degree $m$. If $y' \in V'$, one sets $m(y') = |G|/|\mathrm{Stab}(y')|$ and for $y \in V$, one sets $m(y) = 1/m \sum_{p(y')=y} m(y')$. The function $m$ on $V$ is independent from the choice of the pseudo-Galois covering $h'$ (if $h''$ is another pseudo-Galois covering, consider a pseudo-Galois covering dominating both $h'$ and $h''$). Also, the function $m$ on $V$ is ACF-definable. Let $R$ be a regular function on $V$ and set $r = \mathrm{val} \circ R$. More generally, $R$ may be a tuple of regular functions $(R^1, \ldots, R^m)$, and $r = (\mathrm{val} \circ R^1, \ldots, \mathrm{val} \circ R^m)$. The push-forward $r(h^{-1}(w))$ is also a multi-set of size $n$, and a subset of $\Gamma_\infty^m$. Given a multi-set $Y$ of size $n$ in a linear ordering, we can uniquely write $Y = \{y_1, \ldots, y_n\}$ with $y_1 \leq \ldots \leq y_n$ and with repetitions equal to the multiplicities in $Y$. Thus, using the lexicographic ordering on $\Gamma_\infty^n$, we can write $r(h^{-1}(w)) = \{r_1(w), \ldots, r_n(w)\}$; in this way we obtain definable functions $r_i : W \to \Gamma_\infty$, $i = 1, \ldots, n$. In this setting we have:

**Lemma 8.6.3.** *The functions $r_i$ are v+g-continuous.*

*Proof.* Note that if $g : A \to B$ is a weakly order preserving map of linearly ordered set, $X$ is a multi-subset of $A$ of size $n$ and $Y = g(X)$, then $g(x_i) = y_i$ for $i \leq n$. It follows that both the v-criterion Lemma 8.2.2 and the g-criterion Lemma 8.6.1 (a) hold in this situation. □

**Corollary 8.6.4.** *Let $h : V \to W$ be a finite morphism between algebraic varieties of pure dimension $d$ over a valued field, with $W$ normal. Then $\widehat{h} : \widehat{V} \to \widehat{W}$ is an open map.*

*Proof.* We may assume that $W$ and hence $V$ are affine. A basic open subset of $\widehat{V}$ may be written as $G = \{p : (r(p)) \in U\}$ for some $r = (\mathrm{val} \circ R^1, \ldots, \mathrm{val} \circ R^m)$, $R^i$ regular functions on $V$, and some v+g-open definable subset $U$ of $\Gamma_\infty^n$. Consider the functions $r_i$ as in Lemma 8.6.3. By Lemma 8.6.3 they are v+g-continuous. By Lemma 3.7.1, they extend to continuous functions $\widehat{r_i} : \widehat{W} \to \Gamma_\infty$. Since $w \in \widehat{h}(G)$ if and only if for for some $i$ we have $\widehat{r_i}(w) \in U$, it follows that $\widehat{h}(G)$ is open. □

Note the necessity of the assumption of normality. If $h$ is a a pinching of $\mathbb{P}^1$, identifying two points $a \neq b$, the image of a small valuative neighborhood of $a$ is not open.

**Corollary 8.6.5.** *Let $h : V \to W$ be a finite morphism of algebraic varieties of pure dimension $d$ over a valued field, with $W$ normal. Let $\xi : V \to \Gamma_\infty^n$ be a definable function. Then there exists a definable function $\xi' : W \to \Gamma_\infty^m$ such that for any path $p : I \to \widehat{V}$, still denoting by $\xi$ and $\xi'$ their canonical extensions to $\widehat{V}$ and $\widehat{W}$, if $\xi' \circ h \circ p$ is constant on $I$, then so is $\xi \circ p$.*

*Proof.* Any definable function $\xi : V \to \Gamma_\infty^n$ can be written $\xi = d \circ \xi^*$, with $\xi^* : V \to \Gamma_\infty^N$ a v+g continuous function. (On $\mathbb{P}^n$, the valuation of a rational function $f/g$ factors through $\mathrm{val}(f) - \min(\mathrm{val}(f), \mathrm{val}(g), \mathrm{val}(g) - \min(\mathrm{val}(f), \mathrm{val}(g))$.) So we may assume $\xi$ is continuous. Also, we can treat the coordinate functions separately, so we may as well take $\xi : V \to \Gamma_\infty$. Let $d = \deg(h)$, and define $\xi_1, \ldots, \xi_d$ on $W$ as above, so that the canonical extension of $\xi_i$ (still denoted by $\xi_i$) is continuous on $\widehat{W}$ and $\xi(v) \in \{\xi_1(h(v)), \ldots, \xi_d(h(v))\}$. Let $\xi' = (\xi_1, \ldots, \xi_d)$. Now if $\xi' \circ h \circ p$ is constant on $I$, then $\xi \circ p$ takes only finitely many values, so by definable connectedness of $I$ it must be constant too. □

8.7. **The map $r_{10}$.** Let $V$ be an algebraic variety defined over a field $F_2 \subseteq \mathcal{O}_{21}$. This means that $v_{21}(a) \geq 0$ for $a \in F_2$, so $v_{21}(a) = 0$ for $a \in F_2$, equivalently $v_{20}(F_2^\times) \subseteq \Gamma_{10}$. This is the condition of the v-criterion, cf. Lemma 8.2.1 and the definitions above it, and Lemma 8.2.2. Let $F_1 = r_{21}(F_2)$, hence $r_{21}$ induces a field isomorphism $\mathrm{res} : F_2 \to F_1$. Let $V_1$ be the conjugate of $V$ under the field isomorphism res, so $(F_2, V) \cong (F_1, V_1)$. We can also view $V_1$ as the special fiber of the $\mathcal{O}_{21}$-scheme $V_2 \otimes_{F_2} \mathcal{O}_{21}$. As noted earlier, $\widehat{V_1}$ is unambiguous for varieties over $F_1$.

Recall $\widehat{V}_{20} = \widehat{V}_{210}$. Now $\widehat{V}_{210}$ has a subset $\widehat{V}_{\mathcal{O}} = \widehat{V(\mathcal{O}_{21})}$ consisting of types concentrating on $V(\mathcal{O}_{21})$. We have a definable map $\mathrm{res} : V(\mathcal{O}_{21}) \to V(K_1)$. This induces a map

$$r_{10} = \mathrm{res}_* : \widehat{V}_{\mathcal{O}} \to \widehat{V}_1.$$

Let $\Gamma_{20}^+ = \{x \in \Gamma_{20\infty} : x \geq 0 \vee x \in \Gamma_{10}\}$. Define a retraction $r_{10} : \Gamma_{20}^+ \to \Gamma_{10\infty}$ by letting $r_{10}(x) = \infty$ for $x \in \Gamma_{20}^+ \smallsetminus \Gamma_{10}$. Note that $r_{10}$ is the same as the map $\pi$ of Lemma 8.2.2, and the "only if" direction of that lemma implies that $r_{10}$ is functorial with respect to maps of the form $\mathrm{val} \circ g : V \to \Gamma_\infty$.

**Lemma 8.7.1.** *Let $W$ be an $\mathrm{ACVF}_{F_2}$-definable subset of $\mathbb{P}^n \times \Gamma_\infty^m$. Let $V$ be an algebraic variety over $F_2$, let $X$ be an $\mathrm{ACVF}_{F_2}$-definable subset of $V$ and consider an $\mathrm{ACVF}_{F_2}$-definable map $f : V \to \widehat{W}$. Assume $r_{10} \circ f_{20} = f_{10} \circ r_{10}$ at $x$ whenever $x \in V(\mathcal{O}_{21})$ and $r_{10}(x) \in X$. Then $f$ is v-continuous at each point of $X$. Hence if $f$ is also g-continuous, then the canonical extension $F : \widehat{V} \to \widehat{W}$ is continuous at each point of $\widehat{X}$.*

*Proof.* As in the proof of Lemma 3.7.1, to show that $f$ is v-continuous at each point of $X$ it is enough to prove that for any continuous definable function $c : \widehat{W} \to \Gamma_\infty^n$, $c \circ f$ is v-continuous at each point of $X$. Since, by the functoriality noted above, the equation holds for $c \circ f$, we may assume $f : V \to \Gamma_\infty$. In this case the statement follows from Lemma 8.2.2. The last statement follows directly from Lemma 3.7.1. □

**Remark 8.7.2.** Let $F(X) \in \mathcal{O}_{21}[X]$ be a polynomial in one variable, and let $f(X)$ be the specialization to $K_1[X]$. Assume $f \neq 0$. Then the map $r_{21}$ takes the roots of $F$ onto the roots of $f$. Indeed, consider a root of $f$; we may take it to be 0. Then the Newton polygon of $f$ has a vertical edge. So the Newton polygon of $F$ has a very steep edge compared to $\Gamma_{10}$. Hence it has a root of that slope, specializing to 0.

The following lemma states that a continuous map on $X$ remains continuous relative to a set $U$ that it does not depend on; i.e. viewed as a map on $X \times U$ with dummy variable $U$, it is still continuous. This sounds trivial, and the proof is indeed straightforward if one uses the continuity criteria; it seems curiously nontrivial to prove directly.

For $U$ a definable set and $b \in U$, let $s_b$ denote the corresponding simple point of $\widehat{U}$, i.e. the definable type $x = b$. For $q \in \widehat{V}$, $q \otimes s_b$ is the unique definable type $q(v, u)$ extending $q(v)$ and $s_b(u)$.

**Lemma 8.7.3.** *Let $U$ and $V$ be varieties, $X$ a v+g-open definable subset of a variety $V'$, or of $V' \times \Gamma_\infty$. Let $f : X \to \widehat{V}$ be v+g-continuous, and let $\bar{f}(x, u) = f(x) \otimes s_u$. Then $\bar{f} : X \times U \to \widehat{V \times U}$ is v+g-continuous.*

*Proof.* For g-continuity, we use Lemma 8.6.1 (3) and (4). We have $f_{21} = r_{21} \circ f_{20}$ on $X_{21}$. Also for $x \in X_{21}, u \in U_{21}$, we have $\bar{f}_{21}(x, u) = f_{21}(x) \otimes s_u$, and $\bar{f}_{20}(x, u) = f_{20}(x) \otimes s_u$. Moreover we noted that $r_{21}$ is the identity on simple points, so $r_{21}(p \otimes s_b) = r_{21}(p) \otimes s_b$ in the natural sense. The criterion follows.

For v-continuity, Lemma 8.7.1 applies. Assume $r_{10}(x) \in X$, so $x \in X$. Let $u \in U(\mathcal{O}_{21})$. We have $r_{10} \circ f_{20}(x) = f_{10} \circ r_{10}(x)$. Now $r_{10}(q \otimes s_u) = r_{10}(q) \otimes s_{\bar{u}}$, where $\bar{u} = r_{10}(u)$, and $r_{10}(x, u) = (r_{10}(x), \bar{u})$, so the criterion follows. □

Recall that the map $\otimes : \widehat{U} \times \widehat{V} \to \widehat{U \times V}$ is well-defined but not continuous. If $f : I \times \widehat{V} \to \widehat{V}$ is a homotopy, let $\phi : I \times V \to \widehat{V}$ be the restriction to simple points, and let $(\phi \otimes \mathrm{Id})(t, v, u) = \phi(t, v) \otimes u$. By Lemma 8.7.3, $(\phi \otimes \mathrm{Id})$ is v+g-continuous. By Lemma 3.7.2, it extends to a homotopy $I \times \widehat{V \times U} \to \widehat{V \times U}$, which we denote $\widehat{f \times \mathrm{Id}}$. We easily compute: $\widehat{f \times \mathrm{Id}}(t, p \otimes q) = f(t, p) \otimes q$.

Let $X$ and $Y$ be definable subsets of $U$ and $V$.

**Corollary 8.7.4.** *Let $f : I \times \widehat{X} \to \widehat{X}$, $g : I' \times \widehat{Y} \to \widehat{Y}$ be two homotopies from $\mathrm{Id}_X, \mathrm{Id}_Y$ to $f_0, g_0$, respectively, whose images $S, T$ are $\Gamma$-internal. Then there exists a homotopy $h : (I + I') \times \widehat{X \times Y} \to \widehat{X \times Y}$ whose image equals $S \otimes T$.*

*The canonical map $\pi : \widehat{X \times Y} \to \widehat{X} \times \widehat{Y}$ is a homotopy equivalence.*

*Proof.* Recall $I + I'$ is obtained from the disjoint union of $I, I'$ by identifying the endpoint $0_I$ of $I$ with the initial point of $I'$. Let $h$ be the concatenation composition of $\widehat{f \times \mathrm{Id}}$ with $\widehat{\mathrm{Id} \times g}$:

$$h(t,z) = \widehat{f \times \mathrm{Id}} \text{ for } t \in I, \quad h(t,z) = \widehat{\mathrm{Id} \times g}(t, \widehat{f \times \mathrm{Id}}(0_I, z)) \text{ for } t \in I'$$

So $h(t, p \otimes q) = f(t,p) \otimes q$ for $t \in I$, and $= f(0_I, p) \otimes g(t,q)$ for $t \in I'$. In particular, $h(0_{I'}, p \otimes q) = f(0_I, p) \otimes g(0_{I'}, q)$.

Since any simple point of $\widehat{X \times Y}$ has the form $a \otimes b$, we see that $h(0_{I'}, X \times Y) \subseteq S \otimes T$. Hence for any $r \in \widehat{X \times Y}$, $h(0_{I'}, r)$ is an integral over $r$ of a function into $S \otimes T$. But as $S \otimes T$ is $\Gamma$-internal, and $r$ is stably dominated, this function is generically constant on $r$, and the integral is an element of $S \otimes T$. Thus the final image of $h$ is contained in $S \otimes T$.

Using again the expression for $h(t, p \otimes q)$ we see that if $f(t,s) = s$ for $s \in S$ and $g(t,y) = y$ for $y \in T$, then $h(t,z) = z$ for all $t$ and all $z \in S \otimes T$. So the final image is exactly equal to $S \otimes T$.

For the final statement, we have a homotopy equivalence $h_{0'} : \widehat{X \times Y} \to S \times T$. Moreover $\pi \circ h_{0'} = (f_0 \times g_{0'}) \circ \pi$. Since $(f_0 \times g_{0'})$ is a homotopy equivalence and $\pi | (S \times T)$ is a homeomorphism, $\pi$ is the composition of three homotopy equivalences (or their inverses), and so is a homotopy equivalence. □

**Remark 8.7.5.** When $X, Y$ are v+g-closed and bounded, so that $\widehat{X \times Y}$ is definably compact, we conclude that the image $S \otimes T$ is definably compact. Since the map $\widehat{X \times Y} \to \widehat{X} \times \widehat{Y}$ is continuous, and the restriction to $S \otimes T$ is bijective, it follows that $S \otimes T \to S \times T$ is a homeomorphism, in other words the restriction of $\otimes$ to $S \times T$ is continuous. Now according to Theorem 10.1.1, homotopies can be found so that $S, T$ contain any given $\Gamma$-internal subset of $X, Y$. Hence the restriction of $\otimes$ to $S' \times T'$ is continuous whenever $S', T'$ are $\Gamma$-internal subsets of $\widehat{X}, \widehat{Y}$. This can also be shown more directly using the semi-latttice representation of $\Gamma$-internal sets.

## 9. Continuity of homotopies

9.1. **Preliminaries.** The following lemma will be used both for the relative curve homotopy, and for the inflation homotopy. In the former case, $X$ will be $V \setminus D_{ver} \cup D_0$. Points of $D_{ver} \cap D_0$ are fixed by the homotopy; over these points unique lifting is clear, since a path with finite image must be constant.

**Lemma 9.1.1.** *Let $f : W \to U$ be a morphism of varieties over some valued field $F$. Let $h : [0, \infty] \times U \to \widehat{U}$ be $F$-definable. Let $H : [0, \infty] \times W \to \widehat{W}$ be an $F$-definable lifting of $h$. Let $H_w(t) = H(t,w)$ and $h_u(t) = h(t,u)$. Assume for all $w \in W$, $H_w$ and $h_{f(w)}$ are (continuous) paths and that $H_w$ is unique path lifting $h_{f(w)}$ with $H_w(\infty) = w$. Let $X$ be a g-open definable subset of $U$. Assume $h$ is g-continuous, and v-continuous on (respectively, at each point of) $[0, \infty] \times X$.*

*Then $H$ is g-continuous, and is v-continuous on (respectively at each point of) $[0,\infty] \times f^{-1}(X)$ (we say a function is v-continuous on a subset, if its restriction to that subset is v-continuous).*

*Proof.* We first use the criterion of Lemma 8.6.1 (4) to prove g-continuity. We may assume the data are defined over a subfield $F$ of $K_2$, such that $v_{20}(F) \cap \Gamma_{10} = (0)$; so $(F, v_{20}) \cong (F, v_{21})$.

To show that $H_{21} \circ \pi_2 = r_{21} \circ H_{20}$, we fix $w \in W$. By Lemma 8.4.2, $r_{21} \circ H_{20}(w,t) = H'_w \circ \pi$ for some path $H'_w$. To show that $H_{21}(w,t) = H'_w(t)$, it is enough to show that $f \circ H'_w = h_{f(w)}$. It is clear that $H'_w(\infty) = H_{20}(\infty) = w$ since $r_{21}$ preserves simple points. To see that $f \circ H'_w = h_{f(w)}$ it suffices to check that $f \circ H'_w \circ \pi = h_{f(w)} \circ \pi$, i.e. $f \circ r_{21} \circ H_{20}(w,t) = r_{21} \circ h_{f(w)}$. Now $f \circ r_{21} \circ H_{20} = r_{21} \circ h_{20} = h_{21} \circ \pi_2$. It follows that the g-continuity criterion for $H$ is satisfied.

Let now use the v-continuity criterion in Lemma 8.7.1 above $X$, $(r_{10} \circ H_{20})(t,v) = (H_{10} \circ r_{10})(t,v)$ whenever $(f \circ r_{10})(v) \in X$. Fixing $w = r_{10}(v)$, $H_{10}(t,w)$, for $t \in \Gamma_{10}$, is the unique path lifting $h_{f(w)}$ and starting at $w$, hence to conclude it is enough to prove that $r_{10} \circ H_{20}(t,v)$ also has these properties. But continuity follows from Lemma 9.1.2 and the lifting property from Lemma 9.1.3. □

In the next two lemmas we shall use the notations and assumptions in 8.7. In particular we will assume that $v_{20}(F_2^\times) \subseteq \Gamma_{10,\infty}$.

**Lemma 9.1.2.** *Let $V$ be a quasi-projective variety over $F_2$. Let $f : [0,\infty] \subset \Gamma_{20\infty} \to \widehat{V}_{20}$ be a $(K_2, K_0)$-definable path defined over $F_2$, with $f(\infty)$ a simple point $p_0$ of $\widehat{V}_{\mathcal{O}}$. Then:*

1. *For all $t$, $f(t) \in \widehat{V}_{\mathcal{O}}$.*
2. *We have $r_{10}(f(t)) = r_{10}(p_0)$ for positive $t \in \Gamma_{20} \setminus \Gamma_{10}$.*
3. *The restriction of $r_{10} \circ f$ to $[0,\infty] \subset \Gamma_{10\infty}$ is a continuous $(K_1, K_0)$-definable path $[0,\infty] \subset \Gamma_{10\infty} \to \Gamma_{10\infty} \to \widehat{V}_1$.*

*Proof.* Using base change if necessary and Lemma 6.3.1 we may assume $V \subseteq \mathbb{A}^n$ is affine. So $f : [0,\infty] \subset \Gamma_{20\infty} \to \widehat{\mathbb{A}^n}_{20}$ and we may assume $V = \mathbb{A}^n$.

To prove (1) and (2), by using the projections to the coordinates, one reduces to the case $V = \mathbb{A}^1$. Let $\rho(t) = v(f(t) - p_0)$. Then $\rho$ is a continuous function $[0,\infty] \to \Gamma_\infty$, which is $F$-definable (in $(K_2, K_0)$), and sends $\infty$ to $\infty$. If $\rho$ is constant, there is nothing to prove, since $f$ is constant, so suppose not. As $\Gamma$ is stably embedded, it follows that there is $\alpha \in \Gamma_{20}(F) \subset \Gamma_{10}$ such that for all$t \in [0,+\infty], \alpha \leq \rho(t)$. Hence, if $t \in [0,+\infty]_{20}$, then $v_{20}(f(t) - p_0) \geq \alpha$, which implies that $f(t) \in \widehat{O_{21}}$ as desired, and gives (1). Again, by $F$-definability and since $f$ is not constant, for some $\mu > 0$ and $\beta \in \Gamma_{20}(F)$, if $t > \beta$, then $\rho(t) > \mu t$. Thus, when $t > \Gamma_{10}$, then $\pi(\rho(t)) = 0$, i.e., $r_{10}(f(t)) = r_{10}(p_0)$.

(3) Definability of the restriction of $r_{10} \circ f$ to $[0, \infty] \subset \Gamma_{10\infty}$ follows directly from Lemma 8.3.1. For continuity, note that if $h$ is a polynomial on $V = \mathbb{A}^n$, over $K_1$ and if $H$ is a polynomial over $\mathcal{O}_{21}$ lifting $h$, then $v_{20}(H(a)) = v_{10}(h(\mathrm{res}(a))$. It follows that for $t \neq \infty$ in $[0, \infty] \subset \Gamma_{10\infty}$ continuity of $f$ at $t$ implies continuity of $r_{10} \circ f$.

In fact since $(r_{10} \circ f(t))_* h$ factors through $\pi_{10}(t)$ as we have shown in (2), the argument in (3) shows continuity at $\infty$ too. To see this directly, one may again consider a polynomial $h$ on $V = \mathbb{A}^n$ over $K_1$ and a lift $H$ over $\mathcal{O}_{21}$, and also lift an open set containing $r_{10}(p_0)$ to one defined over a subfield $F_2'$ contained in $\mathcal{O}_{21}$. The inverse image contains an interval $(\gamma, \infty)$, and since $\gamma$ is definable over $F_2'$ we necessarily have $\gamma \in \Gamma_{10}$. The pushforward by $\pi_{10}$ of $(\gamma, \infty)$ contains an open neighborhood of $\infty$. □

**Lemma 9.1.3.** *Let $f : V \to V'$ be a morphism of varieties defined over $F_2$. Then $f$ induces $f_{20} : \widehat{V}_{20} \to \widehat{V}'_{20}$ and also $f_{10} : \widehat{V}_1 \to \widehat{V}'_1$. We have $r_{10} \circ f_{20} = f_{10} \circ r_{10}$.*

*Proof.* In fact $f_{20}, f_{10}$ are just induced from restriction of the morphism $f \otimes_{F_2} \mathcal{O}_{21}$ : $V \times_{F_2} \mathrm{Spec}\mathcal{O}_{21} \to V' \times_{F_2} \mathrm{Spec}\mathcal{O}_{21}$, to the general and special fiber respectively, and the statement is clear. □

**Lemma 9.1.4.** *Let $U$ be a projective variety over a valued field, $D$ a divisor. Let $m$ be a metric on $U$, cf. Lemma 3.8.1. Then $\rho(u, D) = \sup\{m(u, d) : d \in D\}$ is v+g-continuous.*

*Proof.* Let $\rho(u) = \rho(u, D)$. It is clearly v-continuous. Indeed, if $\rho(u, D) = \alpha \in \Gamma$, then $\rho(u', D) = \alpha$ for any $u'$ with $m(u, u') > \alpha$. If $\rho(u, D) = \infty$ then $\rho(u', D) > \alpha$ for any $u'$ with $m(u, u') > \alpha$. Let us show g-continuity by using the criterion in Lemma 8.6.1. Let $(K_2, K_1, K_0), F$ be as in that criterion. Let $u \in U(K_2)$. We have to show that $\rho_{21}(u) = (\pi \circ \rho_{20})(u)$. Say $\rho_{20}(u) = m(u, d)$ with $d \in D(K_2)$. Then $m_{21}(u, d) = \pi(m(u, d))$ by g-continuity of $m$. Let $\alpha = \pi(m(u, d))$ and suppose for contradiction that $\rho_{21}(u) \neq \alpha$. Then $m_{21}(u, d') > \alpha$ for some $d'$. We have again $m_{21}(u, d') = \pi(m_{20}(u, d'))$ so $m_{20}(u, d') > m_{20}(u, d)$, a contradiction. □

**Remark 9.1.5.** In the proof of Lemma 9.1.4, semi-continuity can be seen directly as follows. Indeed, $\rho^{-1}(\infty) = D$ which is g-clopen. It remains to show $\{u : \rho(u, D) \geq \alpha\}$ and $\{u : \rho(u, D) \leq \alpha\}$ are g-closed. Now $\rho(u, D) \geq \alpha$ if and only if $(\exists y \in D)(\rho(u, y) \geq \alpha)$ ; this is the projection of a v+g-closed subset of $U$, hence v+g-closed. The remaining inequality seems less obvious without the criterion, which serves in effect as a topological refinement of quantifier elimination.

**Lemma 9.1.6.** *Let $h : \widehat{U} \times I \to \widehat{U}$ be a homotopy. Let $\gamma : \widehat{U} \to I$ be a definable continuous function. Let $h[\gamma]$ be the cut-off, defined by $h[\gamma](u, t) = h(u, \max(t, \gamma(u)))$. Then $h[\gamma]$ is a homotopy.*

*Proof.* Clear. □

**Lemma 9.1.7.** *Let $U$ be an affine variety over some valued field $F$, $f : U \to \Gamma$ be an $F$-definable function. Assume $f$ is locally bounded on $U$, i.e. any $u \in U$ has a neighborhood in the valuation topology where $f$ is bounded. Then there exists a v+g-continuous $F$-definable function $F : \mathbb{A}^n \to \Gamma_\infty$ such that $f(x) \leq F(x) \in \Gamma$ for $x \in U$.*

*Proof.* Replacing $f(u)$ by the infimum over all neighborhoods $w$ of $u$ of the supremum of $f$ on $w$, we may assume $f$ is semi-continuous, i.e. $\{u : f(u) < \alpha\}$ is open. Let $(g_1, \ldots, g_l)$ be generators of the coordinate ring of $U$, $g(u) = \min -(\mathrm{val}(g_i(u)))$ Note that $g(u) \in \Gamma$ for $u \in U$. Now

$$U_\alpha = \{x : g(x) \leq \alpha\}$$

is v+g-closed and bounded, hence $\widehat{U_\alpha}$ is definably compact by Lemma 4.2.4. By Lemma 4.2.16, since $U_\alpha$ is covered by the union over all $\gamma \in \Gamma$ of the open sets $\{x : f(x) < \gamma\}$, $f$ is bounded on $U_\alpha$; let $f_1(\alpha)$ be the least upper bound. Piecewise in $\Gamma$, $f_1$ is an affine function. It is easy to find $m \in \mathbb{N}$ and $c_0 \in \Gamma$ such that $f_1(\alpha) \leq m\alpha + c_0$ for all $\alpha \geq 0$. Let $F(x) = mg(x) + c_0$. □

9.2. **Continuity on relative $\mathbb{P}^1$.** To define the relative $\mathbb{P}^1$ homotopy over the variety $U$, we require the following data: a line bundle $L$ over $U$; a trivialization $t_0$ of $L$ over an open subset $U_0$ of $U$; $E = \mathbb{P}(L \oplus 1)$; so $E$ is a $\mathbb{P}^1$-bundle over $U$, and the pullback $E_0$ to $U_0$ is trivial, $E_0 = U_0 \times \mathbb{P}^1$. We are further given a divisor $D_0$ on $E$, finite over $U$, that we take to contain the divisor at $\infty$ at each fiber, $E \smallsetminus L$; and another divisor $D$ on $E$, containing $D_0$. Let $D_{ver}$ be the vertical part of $D$, i.e. a divisor on $U$ whose pullback to $E$ is contained in $D$. We assume $D_{ver}$ contains $U \smallsetminus U_0$. Moreover we make use of a distinguished 0 and $\infty$ in $\mathbb{P}^1$ so that the notion of a ball and the standard homotopy are well-defined, cf. Lemma 3.8.1, Lemma 7.6.1.

In practice we will have $U = \mathbb{P}^{n-1}$, $E$ will be the blowup of $\mathbb{P}^n$ at one point, and $D_0$ will be a divisor away from which inflation is possible.

Again consider the metric of Lemma 3.8.1 on $\mathbb{P}^1 \supset \mathbb{A}^1$. Let $\psi_D : [0, \infty] \times E_0 \to \widehat{E_0/U_0}$ be the standard homotopy with stopping time defined by $D$ at each fiber, as defined above Lemma 8.5.3. We extend $\psi_D$ to $[0, \infty] \times E$ by $\psi_D(t, x) = x$ for $x \in E \smallsetminus E_0$. Of course $\psi_D$ is not continuous at a general point of $E \smallsetminus E_0$, but we wish to show that it is g-continuous, and v-continuous at $X = E_0 \cup D_0$.

Let $L_u$ be the fiber of $L$ at $u$. Note that the restriction of $\psi_D$ to $\mathbb{P}^1 = \mathbb{P}(L_u \oplus 1)$ fixes the point at $\infty$, and so leaves invariant the affine line $L_u$. Furthermore, if $h_u(x)$ is a polynomial whose roots are $D_0 \cap L_u$, then $\mathrm{val}(h_u(x)) = \mathrm{val}(h_u(\psi_D(t, x)))$ for all $t$ (that is, $\psi_D$ does not increase schematic distance from $D_0$ on $L_u$). Indeed, it suffices to show this for all $t$ up to the stopping time of the homotopy $\psi_u$. Up to this time, either $\psi_u(t, x) = x$ or else $\psi_u(t, x)$ is the generic of a certain ball $b$ around $x$, not containing any point of $D$ and in particular of $D_0$. Hence $\mathrm{val}(h_u)$ is constant on $b$.

**Lemma 9.2.1.** *The pro-definable map $\psi_D : [0,\infty] \times E \to \widehat{E}$ is g-continuous on $[0,\infty] \times E$ and v-continuous at each point of $[0,\infty] \times X$ for $X = E_0 \cup D_0$.*

*Proof.* Since $D_{ver}$ is g-clopen g-continuity may be shown separately on $D_{ver}$ and away from $D_{ver}$. On $D_{ver}$ it is trivial since $\psi_D$ is constant there. Away from $D_{ver}$ it follows from Lemma 8.5.3 and Lemma 8.5.4, applying the g-criterion Lemma 8.6.1.

Let us note that v-continuity for the basic homotopy on $\mathbb{P}^1$, applied fiberwise on $\mathbb{P}^1 \times U_0$, is clear and that, by Lemma 9.1.6, $\psi_D$ is also v-continuous over $U_0$, i.e. on $E_0$. There remains to show v-continuity on $D_0$. Let $F_2$, $r_{10}$, res be as in the v-continuity criterion Lemma 8.7.1. Let $a \in E(F_2)$ with $\mathrm{res}(a) \in D_0$. If $a \notin E_0$ then $\psi_D$ fixes $a$, so assume $a \in E_0(F_2)$. As noted above, schematic distance from $D_0$ does not grow as $\psi_D(t,\ )$ is applied. Hence $r_{10}$ maps this distance to $\infty$, so $r_{10} \circ \psi_D(t,a)$ remains on $D_0$ for all $t$. But by assumption $D_0$ is finite on every fiber; hence a (continuous) path on the fiber at $\mathrm{res}(a)$ remaining on $D_0$, must be constant. So $r_{10} \circ \psi_D(t,a) = \mathrm{res}(a) = \psi_D(t, \mathrm{res}(a))$. □

**Lemma 9.2.2.** (1) *Let $f : W \to U$ be a generically finite morphism of varieties over a valued field $F$, with $U$ a normal variety, and $\xi : W \to \Gamma_\infty$ an $F$-definable map. Then there exists a divisor $D_\xi$ on $U$ and $F$-definable maps $\xi_1, \ldots, \xi_n : U \to \Gamma_\infty$ such that any homotopy of $W$ lifting a homotopy of $U$ fixing $D_\xi$ and the levels of the functions $\xi_i$ also preserves $\xi$.*

(2) *Let $\xi : \mathbb{P}^1 \times U \to \Gamma_\infty$ be a definable map, with $U$ an algebraic variety over a valued field. Then there exists a divisor $D_\xi$ on $U$ such that if $D_\xi \subseteq D$ then the standard homotopy with stopping time defined by $D$ preserves $\xi$.*

*Proof.* (1) There exists a divisor $D_0$ of $U$ such that $f$ is finite above the complement of $D_0$, and such that $U \smallsetminus D_0$ is affine. By making $D_0$ a component of $D_\xi$, we reduce to the case that $U$ is affine, and $f$ is finite. So $W$ is also affine, and $\xi$ factorizes through functions of the form $\mathrm{val}(g)$, with $g$ regular; hence $\xi$ can be assumed v+g-continuous, so that it induces a continuous function on $\widehat{U}$. Let $\xi_i(u)$, $i = 1, \ldots, n$, list the values of $\xi$ on $f^{-1}(u)$. Let $h$ be a homotopy of $W$ lifting a homotopy of $U$ fixing $D_\xi$ and the levels of the $\xi_i$. Then for fixed $w \in W$, $\xi(h(t,w))$ can only take finitely many values as $t$ varies. On the other hand $t \mapsto \xi(h(t,w))$ is continuous, so it must be constant.

(2) As in (1) we may assume $U$ is affine, and that $\xi | \mathbb{A}^1 \times U$ has the form $\xi(u) = \mathrm{val} g$, $g$ regular on $\mathbb{A}^1 \times U$. Here we take $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$; by adding $\{\infty\} \times U$ to $D_\xi$ we can ensure that $\xi$ is preserved there, and so it suffices to preserve $\xi | \mathbb{A}^1 \times U$. Write $g = g(x,u)$, so for fixed $u \in U$ we have a polynomial $g(x,u)$; let $D_\xi$ include the divisor of zeroes of $g$. Now it suffices to see for each fiber $\mathbb{P}^1 \times \{u\}$ separately, that the standard homotopy fixing a divisor containing the roots of $g$ must preserve $\mathrm{val} g$. This is clear since this standard homotopy fixes

any ball containing a root of $g$; while on a ball containing no root of $g$, $\mathrm{val} g$ is constant. $\square$

### 9.3. **The inflation homotopy.**

**Lemma 9.3.1.** *Let $V$ be a quasiprojective variety over a valued field $F$, $W$ be closed and bounded $F$-pro-definable subset of $\widehat{V}$. Let $D$ and $D'$ be closed subvarieties of $V$, and suppose $W \cap \widehat{D'} \subseteq \widehat{D}$. Then there exists a v+g-closed, bounded $F$-definable subset $Z$ of $V$ with $Z \cap D' \subseteq D$, and $W \subseteq \widehat{Z}$.*

*Proof.* We assume $V$ is affine. (We may take $V = \mathbb{P}^n$; then find finitely many affine open $V_i \subset V$ and closed bounded $B_i \subset V_i$ such that $V = \cup_i V_i$; given $Z_i$ solving the problem for $V_i$, let $Z = \cup_i (B_i \cap Z_i)$.)

Choose a finite generating family $(f_i)$ of the ideal of regular functions vanishing on $D$ and set $d(x, D) \inf \mathrm{val}(f_i(x))$ for $x$ in $V$. Similarly, fixing a finite generating family of the ideal of regular functions vanishing on $D'$, one defines a distance function $d(x, D')$ to $D'$. Note that the functions $d(x, D)$ and $d(x, D')$ may be extended to $x \in \widehat{V}$.

For $\alpha \in \Gamma$, let $V_\alpha$ be the set of points $x$ of $V$ with $d(x, D) \leq \alpha$. Let $W_\alpha = W \cap \widehat{V_\alpha}$. Then $W_\alpha \cap D' = \emptyset$. So $d(x, D') \in \Gamma$ for $x \in W_\alpha$. By Lemma 4.2.25 there exists $\delta(\alpha) \in \Gamma$ such that $d(x, D') \leq \delta(\alpha)$ for $x \in W_\alpha$. By Lemma 9.1.7 we may take $\delta$ to be a continuous function. (In all uses of Lemma 9.1.7 within this proof, local boundedness is easily verified.) Since any continuous definable function $\Gamma \to \Gamma$ extends to a continuous function $\Gamma_\infty \to \Gamma_\infty$, we may extend $\delta$ to a continuous function $\delta : \Gamma_\infty \to \Gamma_\infty$. Also, since any such function is bounded by a continuous function with value $\infty$ at $\infty$ we may assume $\delta(\infty) = \infty$. Let

$$Z_1 = \{x \in V : d(x, D') \leq \delta(d(x, D))\}.$$

Let $c$ be a realization of $p \in W$. We have $c \in Z_1$ and $Z_1 \cap D' \subseteq D$. Since, by Lemma 4.2.9, $W$ is contained in $\widehat{Z_2}$ with $Z_2$ a bounded v+g-closed definable subset of $V$, we may set $Z = Z_1 \cap Z_2$. $\square$

**Lemma 9.3.2.** *Let $D$ be a closed subvariety of a projective variety $V$ over a valued field $F$, and assume there exists an étale map $e : V \smallsetminus D \to U$, $U$ an open subset of $\mathbb{A}^n$. Then there exists a $F$-definable homotopy $H : [0, \infty] \times \widehat{V} \to \widehat{V}$ fixing $\widehat{D}$ (that is, such that $H(t, d) = d$ for $t \in [0, \infty]$ and $d \in \widehat{D}$), with image $\mathbf{Z} = H(0, \widehat{V})$, such that for any subvariety $D'$ of $V$ of dimension $< \dim(V)$ we have $\mathbf{Z} \cap \widehat{D'} \subseteq \widehat{D}$. Moreover given a finite number of $F$-definable v-continuous functions $\xi_i : V \smallsetminus D \to \Gamma$, one can choose the homotopy such that the levels of the $\xi_i$ are preserved. If a finite group $G$ acts on $V$ over $U$, inducing a continuous action on $\widehat{V}$ and leaving $D$ and the fibers of $e$ invariant, then $H$ is $G$-equivariant.*

*Proof.* Let $I = [0, \infty]$ and let $h_0 : I \times \mathbb{A}^n \to \widehat{\mathbb{A}^n}$ be the standard homotopy sending $(t, x)$ to the generic type of the closed polydisc of polyradius $(t, \cdots, t)$ around $x$.

Denote by $H_0 : I \times \widehat{\mathbb{A}^n} \to \widehat{\mathbb{A}^n}$ its canonical extension (cf. Lemma 3.7.2). Note the following fundamental inflation property of $H_0$: if $W$ is closed subvariety of $\mathbb{A}^n$ of dimension $< n$, then, for any $(t, x)$ in $I \times \widehat{\mathbb{A}^n}$, if $t \neq \infty$, then $H_0(t, x) \notin \widehat{W}$.

By Lemma 7.4.1, Lemma 7.4.5 or Lemma 7.4.4, for each $u \in U$ there exists $\gamma_0(u) \in \Gamma$ such that $h_0(t, u)$ lifts uniquely to $V \smallsetminus D$ beginning with any $v \in e^{-1}(u)$, up to $\gamma_0(u)$. By Lemma 9.1.7 we can take $\gamma_0$ to be v+g-continuous. For $t \geq \gamma_0(u)$, let $h_1(t, v)$ be the unique continuous lift.

Since $\xi_i$ is v-continuous outside $D$, $\xi_i^{-1}\xi_i(v)$ contains a v-neighborhood of $v$. So for some $\gamma_1(u) \geq \gamma_0(u)$, for all $t \geq \gamma_1(u)$ we have $\xi_i(h_1(t, v)) = \xi_i(v)$. Again we may use Lemma 9.1.7 to replace $\gamma_1$ by a v+g-continuous function.

At this point we can cut off to $h_0[\gamma_1]$; this is continuous by Lemma 9.1.6, and by Lemma 9.1.1, $h_1[\gamma_1 \circ e]$ is continuous on $V \smallsetminus D$. However we would like to fix $D$ and have continuity on $D$.

Let $m$ be a metric on $V$, as provided by Lemma 3.8.1. Given $v \in V$ let $\rho(v) = \sup\{m(d, v) : d \in D\}$. By Lemma 4.2.25 we have $\rho : V \smallsetminus D \to \Gamma$. Let $\gamma_2 : \mathbb{A}^n \to \Gamma$, $\gamma_2 > \gamma_1$, such that for $t \geq \gamma_2(u)$ we have $d(h_1(t, v)), v) > \rho(v)$ for each $v$ with $e(v) = u$. (So $\rho(h_1(t, v)) = \rho(v)$.) By Lemma 9.1.7 we can take $\gamma_2$ to be v+g-continuous.

Let $H$ the canonical extension of $h_1[\gamma_2 \circ e]$ to $\widehat{V \smallsetminus D} \times I$ provided by Lemma 3.7.2. We extend $H$ to $\widehat{V} \times I$ by setting $H(t, x) = x$ for $x \in \widehat{D}$. We want to show that $H$ is continuous on $\widehat{V}$. Since we already know it is continuous at each point of the open set $(\widehat{V \smallsetminus D}) \times I$, it is enough to prove $H$ is continuous at each point of $\widehat{D} \times I$.

Let $d \in \widehat{D}$, $t \in I$. Then $H(t, d) = d$. Let $\mathbf{G}$ be an open neighborhood of $d$. $\mathbf{G}$ may be taken to have the form: $\{x \in G_0 : \mathrm{val} r(x) \in J\}$, with $J$ open in $\Gamma_\infty$, and $r$ a regular function on a Zariski open neighborhood $G_0$ of $d$. So $\mathbf{G} = \widehat{G}$ where $G$ is a v+g-open subset of $V$.

We have to find an open neighborhood $W$ of $(t, d)$ such that $H(W) \subseteq G$. We may take $W \subseteq G \times \Gamma_\infty$, so we have $H(W \cap \widehat{D}) \subseteq \mathbf{G}$. Since the simple points of $W \smallsetminus \widehat{D}$ are dense in $W \smallsetminus \widehat{D}$, it suffices to show that for some neighborhood $W$, the simple points are mapped to $\widehat{G}$.

View $d$ as a type (defined over $M_0$); if $z \models d|M_0$, then for some $\varepsilon \in \Gamma$, $H(B(z; m, \varepsilon)) \subseteq \mathbf{G}$. Fix $\varepsilon$, independently of $z$. Let $W_0 = \{v \in V : B(v; m, \varepsilon) \subseteq G$. Then $W_0$ is v+g-open. Indeed the complement is $\{v \in V : (\exists y) m(x, y) \leq \varepsilon \wedge y \in (V \smallsetminus G)\}$. Now the projection of a (bounded) v+g-closed set is also v+g-closed.

If there is no neighborhood $W$ as desired, there exist simple points $v_i \in V \smallsetminus D$, $v_i \to d$, $t_i \to t$ with $H(t_i, v_i) \notin G$. Now $\rho(v_i) \to \rho(d) = m(d, D) = \infty$, so $H(t_i, v_i) = h_1(\gamma_2(e(v_i)))$, and by the above $m(H(t_i, v_i), v_i) \to \infty$. So for large

$i$ we have $H(t_i, v_i) \in \widehat{B(v_i; m, \varepsilon)}$, and also $v_i \in W_0$. So $B(v_i, m, \varepsilon) \subseteq G$, hence $H(t_i, v_i) \in \widehat{G} = \mathbf{G}$, a contradiction. This shows that $H$ is continuous.

It remains to prove that if $\mathbf{Z} = H(0, \widehat{V})$, then, for any subvariety $D'$ of $V$ of dimension $< \dim(V)$, we have $\mathbf{Z} \cap \widehat{D'} \subseteq \widehat{D}$. This follows from the inflation property of $H_0$ stated at the beginning, applied to $e(D' \cap (V \smallsetminus D))$.

The statement on the group action follows from the uniqueness of the continuous lift. □

**Remark 9.3.3.** Lemma 9.3.2 remains true if one supposes only that $D$ contains the singular points of $V$. Indeed, one can find divisors $D_i$ with $D = \cap_i D_i$, and étale morphisms $h_i : D_i \to \mathbb{A}^n$, and iterate the lemma to obtain successively $Z \cap D' \subseteq D_1 \cap \ldots \cap D_i$. In particular, when $V$ is smooth, Lemma 9.3.2 is valid for $D = \emptyset$.

9.4. **Connectedness, and the Zariski topology.** Let $V$ be an algebraic variety over some valued field. We say a strict pro-definable subset $Z$ of $\widehat{V}$ is *definably connected* if it contains no clopen strict pro-definable subsets other than $\emptyset$ and $Z$. We say that $Z$ is *definably path connected* if for any two points $a$ and $b$ of $Z$ there exists a definable path in $Z$ connecting $a$ and $b$. Clearly definable path connectedness implies definable connectedness. When $V$ is quasi-projective and $Z = \widehat{X}$ with $X$ a definable subset of $V$, the reverse implication will eventually follow from Theorem 10.1.1.

If $X$ a definable subset of $V$, $\widehat{X}$ is definably connected if and only if $X$ contains no v+g-clopen definable subsets, other than $X$ and $\emptyset$. Indeed, if $U$ is a clopen strict pro-definable subset of $\widehat{X}$, the set $U \cap X$ of simple points of $U$ is a v+g-clopen definable subset of $X$, and $U$ is the closure of $U \cap X$. When $X$ is a definable subset of $V$, we shall say $\widehat{X}$ has a finite number of connected components if $X$ may be written as a finite disjoint union of v+g-clopen definable subsets $U_i$ with each $\widehat{U_i}$ definably connected. The $\widehat{U_i}$ are called connected components of $\widehat{X}$.

**Lemma 9.4.1.** *Let $V$ be a smooth algebraic variety over a valued field and let $Z$ be a nowhere dense Zariski closed subset of $V$. Then $\widehat{V}$ has a finite number of connected components if and only if $\widehat{V \smallsetminus Z}$ has a finite number of connected components. Furthermore, if $\widehat{V}$ is a finite disjoint union of connected components $\widehat{U_i}$ then the $\widehat{U_i} \smallsetminus \widehat{Z}$ are the connected components of $\widehat{V \smallsetminus Z}$.*

*Proof.* By Remark 9.3.3, there exists a homotopy $H : I \times \widehat{V} \to \widehat{V}$ such that its final image $\Sigma$ is contained in $\widehat{V \smallsetminus Z}$. Also, by construction of $H$, the simple points of $V \smallsetminus Z$ move within $\widehat{V \smallsetminus Z}$, and so $H$ leaves $\widehat{V \smallsetminus Z}$ invariant. Thus, we have a continuous morphism of strict pro-definable spaces $\varrho : \widehat{V} \to \Sigma$. If $V$ is a finite disjoint union of v+g-clopen definable subsets $U_i$ with each $\widehat{U_i}$ definably connected, note that each $\widehat{U_i}$ is invariant by the homotopy $H$. Thus, $\varrho(\widehat{U_i}) = \Sigma \cap \widehat{U_i}$

is definably connected. Since $\Sigma \cap \widehat{U_i} = \Sigma \cap (\widehat{U_i} \smallsetminus \widehat{Z})$ and any simple point of $U_i \smallsetminus Z$ is connected via $H$ within $\widehat{U_i \smallsetminus Z}$ to $\Sigma \cap \widehat{U_i}$ it follows that $\widehat{U_i} \smallsetminus \widehat{Z}$ is definably connected. For the reverse implication, assume $V \smallsetminus Z$ is a finite disjoint union of v+g-clopen definable subsets $V_i$ with each $\widehat{V_i}$ definably connected. Then $\varrho(\widehat{V_i}) = \Sigma \cap \widehat{V_i}$ is definably connected. Let $U_i$ denote the set of simple points in $\varrho^{-1}(\Sigma \cap \widehat{V_i})$. Then $\widehat{U_i}$ is definably connected. □

**Proposition 9.4.2.** *Let $V$ be a quasi-projective variety over a valued field which is connected for the Zariski topology. Then $\widehat{V}$ is definably connected.*

*Proof.* We may assume $V$ is irreducible. It follows from Bertini's Theorem, cf. [22] p. 56, that any two points of $V$ are contained in a irreducible curve $C$ on $X$. So the lemma reduces to the case of irreducible curves, and by normalization, to the case of smooth irreducible curves $C$. The case of genus 0 is clear using the standard homotopies of $\mathbb{P}^1$. So assume $C$ has genus $g > 0$. By Proposition 7.6.1 there is a retraction $\varrho : \widehat{C} \to \Upsilon$ with $\Upsilon$ a $\Gamma$-internal subset. It follows from Proposition 6.3.8 that $\Upsilon$ is a finite disjoint union of connected $\Gamma$-internal subsets $\Upsilon_i$. Denote by $C_i$ the set of simple points in $C$ mapping to $\Upsilon_i$. Each $C_i$ is a v+g-clopen definable subset of $C$ and $\widehat{C_i}$ is definably connected, thus $\widehat{C}$ has a finite number of connected components. Assume this number is $> 1$. Then $\widehat{C^g}/\mathrm{Sym}(g)$ has also a finite number $> 1$ of connected components, since $\widehat{C^g}$ may be written has a disjoint union of the definably connected sets $\widehat{C_{i_1} \times \cdots \times C_{i_g}}$.

Let $J$ be the Jacobian variety of $C$. There exist proper subvarieties $W$ of $C^g$ and $V$ of $J$, with $W$ invariant under $\mathrm{Sym}(g)$, and a biregular isomorphism of varieties $(C^g \smallsetminus W)/\mathrm{Sym}(g) \to J \smallsetminus V$. By Lemma 9.4.1 $(\widehat{C^g \smallsetminus W)/\mathrm{Sym}}(g)$ has a finite number $> 1$ of connected components, hence also $\widehat{J \smallsetminus V}$. By Lemma 9.4.1 again, $\widehat{J}$ would have a finite number $> 1$ of connected components. The group of simple points of $J$ acts by translation on $\widehat{J}$, homeomorphically, and so acts also on the set of connected components. Since it is a divisible group, the action must be trivial. On the other hand, it is transitive on simple points, which are dense, hence on connected components. This leads to a contradiction, hence $\widehat{C}$ is connected, which finishes the proof. □

It will be convenient to use the following terminology. Let $Y \subseteq \Gamma_\infty^w$ be a definable set. By a z-closed subset of $Y$ we mean one of the form $Y \cap [x_i = \infty]$, an intersection of such sets, or a finite union of such intersections. By a z-irreducible set we mean a z-closed subset which cannot be written as the union of two proper z-closed subsets. Any z-closed set can be written as a union of z-closed z-irreducible sets; these will be called components. A z-open set is the complement of a z-closed set $Z$. A z-open set is *dense* if its complement does not contain any component of $Y$.

**Lemma 9.4.3.** *Let $V$ be an algebraic variety over a valued field $F$ and let $f : V \to \Gamma_\infty$ be a v+g-continuous $F$-definable function. Then $f^{-1}(\infty)$ is a subvariety of $V$.*

*Proof.* Since $f^{-1}(\infty)$ is v-closed, it suffices to show that it is constructible. By Noetherian induction we may assume $f^{-1}(\infty) \cap W$ is a subvariety of $W$, for any proper subvariety $W$ of $V$. so it suffices to show that $f^{-1}(\infty) \cap V'$ is an algebraic variety, for some Zariski open $V' \subset V$. In particular we may assume $V$ is affine, smooth and irreducible. Since any definable set is v-open away from some proper subvariety, we may also assume that $f^{-1}(\infty)$ is v-open. On the other hand $f^{-1}(\infty)$ is v-closed. The point $\infty$ is an isolated point in the g-topology, so $f^{-1}(\infty)$ is g-closed and g-open. By Lemma 3.6.4 it follows that $\widehat{f^{-1}(\infty)}$ is a clopen subset of $\widehat{V}$. Since $\widehat{V}$ is definably connected by Proposition 9.4.2, one deduces that $f^{-1}(\infty) = V$ or $f^{-1}(\infty) = \emptyset$, proving the lemma. □

Let $Y$ be a definable subset of $\Gamma_\infty^n$. Define a *Zariski closed* subset of $Y$ to be a clopen subset of a z-closed subset of $Y$. By o-minimality, there are finitely many such clopen subsets, the unions of the definably connected components. A definable set $X$ thus has only finitely many Zariski closed subsets; if $X$ is connected and z-irreducible, there is a maximal proper one.
[2]

Lemma 9.4.3 can be strengthened as follows:

**Lemma 9.4.4.** *Let $V$ be an algebraic variety over a valued field $F$ and let $f : V \to Y \subset \Gamma_\infty^n$ be a v+g-continuous $F$-definable function. Then $f^{-1}(U)$ is Zariski open (closed) in $V$, whenever $U$ is Zariski open (closed) in $Y$.*

*Proof.* It suffices to prove this with "closed". So $U$ is a clopen subset of $U'$, with $U'$ z-closed. By Lemma 9.4.3, $f^{-1}(U')$ is Zariski closed; write $f^{-1}(U') = V_1 \cup \ldots \cup V_m$ with $V_i$ Zariski irreducbile. It suffices to prove the lemma for $f|V_i$, for each $i$; so we may assume $V_i = V$ is Zariski irreducible. By Lemma 9.4.2, $f^{-1}(U) = V$. □

Here is a converse:

**Lemma 9.4.5.** *Let $X \subset \Gamma_\infty^n$ and let $\beta : X \to \widehat{V}$ be a continuous, pro-definable map. Let $W$ be a Zariski closed subset of $\widehat{V}$. Then $\beta^{-1}(W)$ is Zariski closed in $X$.*

*Proof.* Let $F_1, \ldots, F_l$ be the nonempty, proper Zariski closed subsets of $X$. Removing from $X$ any $F_i$ with $F_i \subseteq \beta^{-1}(W)$, we may assume no such $F_i$ exist. By working separately in each component, we may assume $X$ is connected, and in fact z-irreducible. Moreover by induction on z-dimension, we can assume the

[2]This has nothing to do with the topology on $\Gamma^n$ generated by translates of subspaces defined by $\mathbb{Q}$-linear equations, for which the name Zariski would also be natural. We will use this latter topology little, and will refer to it as the linear Zariski topology on $\Gamma^n$, when required.

lemma holds for proper z-closed subsets of $X$. **Claim .** $\beta^{-1}(W) \cap F_i = \emptyset$ for each $i$.
Otherwise, let $P$ be a minimal $F_i$ with nonempty intersection with $\beta^{-1}(W)$. [3] Then, by induction, $\beta^{-1}(W) \cap P$ is Zariski closed in $P$. Any proper z-closed subset of $P$ meets $\beta^{-1}(W)$ trivially; it follows that $\beta^{-1}(W) \cap P$ is a component of $P$ itself; as $P$ is connected, $\beta^{-1}(W) = \emptyset$ or , $\beta^{-1}(W) = P$; in the latter case, $P$ is contained in $\beta^{-1}(W)$ and should have been removed from $X$. Thus $\beta^{-1}(W) \cap F_i = \emptyset$.

Say $\beta^{-1}(W) \subseteq \Gamma_\infty^m \times \{\infty\}^\ell$ with $m + l = n$ and $m$ maximal, even allowing for rearrangements of the coordinates. Then $\beta^{-1}(W) \cap (x_i = \infty) = \emptyset$ for $i = 1, \ldots, m$, i.e. $\beta^{-1}(W) \subseteq \Gamma^m \times \{\infty^l\}$. Projecting homeomorphically to $\Gamma^m$, we may assume $m = n$ and $X \subseteq \Gamma^n$. However, $W$ is g-clopen, so $\beta^{-1}(W)$ is g-clopen, i.e. clopen. This implies that it is after all Zariski closed in $X$. □

**Corollary 9.4.6.** *Let $\Upsilon$ be an iso-definable subset of $\widehat{V}$, $X$ a definable subset of $\Gamma_\infty^n$, and let $\alpha : \Upsilon \to X$ be a pro-definable homeomorphism. Then $\alpha$ takes the Zariski topology on $\Upsilon$ to the Zariski topology on $X$.*

*Proof.* Follows from Lemma 9.4.4 and Lemma 9.4.5. □

## 10. The main theorem

### 10.1. Statement.

**Theorem 10.1.1.** *Let $V$ be a quasi-projective variety, $X$ a definable subset of $V \times \Gamma_\infty^\ell$ over some base set $A \subset \mathrm{VF} \cup \Gamma$. Then there exists an $A$-definable deformation retraction $h : I \times \widehat{X} \to \widehat{X}$ to a pro-definable subset $\Upsilon$ definably homeomorphic to a definable subset of $\Gamma_\infty^w$, for some finite $A$-definable set $w$. One can furthermore require the following additional properties for $h$:*

1. *Given finitely many $A$-definable functions $\xi_i : V \to \Gamma_\infty$, one can choose $h$ to respect the $\xi_i$, i.e. $\xi_i(h(t,x)) = \xi_i(x)$ for all $t$. In particular, finitely many subvarieties or more generally definable subsets $U$ of $X$ can be preserved, in the sense that the homotopy restricts to one of $\widehat{U}$.*
2. *Assume given, in addition, a finite algebraic group action on $V$ preserving $X$. Then the homotopy retraction can be chosen to be equivariant.*
3. *When $X = V$ and $\ell = 0$, one may require that $\Upsilon$ is Zariski dense in $\widehat{V_i}$ in the sense of 3.9, for every irreducible component of maximal dimension $V_i$ of $V$.*
4. *The homotopy $h$ satisfies condition $(*)$ of 5.3, i.e.: $h(e_I, h(t,x)) = h(e_I, x)$ for every $t$ and $x$.*

[3] More precisely, let $Q$ be the z-closure of $P$; then $Q \neq X$. As Zariski-closed in $Q$ implies Zariski-closed in $X$, $Q \cap \beta^{-1}(W) = \emptyset$. Thanks to Zoé Chatzidakis for this comment.

(5) *Any element of $\Upsilon$, viewed as a stably dominated type, has equal transcendence degree and residual transcendence degree.*

**Remarks 10.1.2.** (1) Without parameters, one cannot expect $Z$ to be definably homeomorphic to a subset of $\Gamma_\infty^n$, since the Galois group may have a nontrivial action on the cohomology of $V$, even on the Berkovich part. (See the earlier observation regarding quotients.)

(2) Let $\pi : V' \to V$ be a finite morphism, and $\xi' : V' \to \Gamma_\infty^m$. Then, when $X = V$ one can find $h$ as in the Theorem lifting to $h' : I \times \widehat{V'} \to \widehat{V'}$ respecting $\xi'$. To see this, let $V'' \to V'$ be such that $V'' \to V$ admits a finite group action $H$, and $V'$ is the quotient variety of some subgroup. Find an equivariant homotopy of $\widehat{V''}$, then induce homotopies on $\widehat{V'}$ and on $\widehat{V}$. See Lemma 5.3.3 for the continuity of the induced homotopies, and Lemma 2.2.5 for the isodefinability of their image.

(3) By Lemma 6.3.13 (and Remark 6.3.10), in (3) we can also take a proper $\Gamma$-internal covering in place of a finite one.

(4) It is also possible to preserve an $A$-definable map $\xi : V \to \Gamma_\infty^w$. There exist definable sets $U_i$ such that $\xi|U_i$ is continuous, and such that $\xi(u)$ is a function from $w$ onto some $m_i$-element set. Moreover there exists a map $\xi' : V \to \Gamma_\infty^m$ (where $m = |w|$) such that for $v \in U_i$, $\xi'(v)$ is an $m_i$-tuple in non-decreasing order, enumerating the underlying set of the $w$-tuple $\xi(v)$. We can ask that $H$ preserve the $U_i$ and $\xi'$. Then along each path of $H$, $\xi$ is preserved up to a permutation of $w$, hence by continuity it is preserved.

(5) Item (5) means the following: let $M$ be a valued field containing $A \cap \mathrm{VF}$, with $A \cap \Gamma \subset \mathrm{val}(M)$. Let $p \in \Upsilon(M)$ and view $p$ as a stably dominated type. Let $c \models p|M$ and let $M' = M(c)$. Let $m$ be the residue field of $M$, and $m'$ of $M'$. Then $\mathrm{tr.deg}_m(m') = \mathrm{tr.deg.}_M(M') \leq \dim(V)$. We cannot ensure that the transcendence degrees equal $\dim(V)$ because of possible singularities of $V$; see Theorem 11.1.1 (4).

10.2. **Proof of Theorem 10.1.1: Preparation.** The theorem reduces easily to the case $\ell = 0$ (for instance, take the projection of $X$ to $V$, and add $\xi_i$ describing the fibers, as in the first paragraph of Lemma 6.3.13). We assume $\ell = 0$ from now on.

We may assume $V$ is a projective variety[4]. By adding the valuation of the characteristic function of $X$ to the functions $\xi_i$, we can assume that any homotopy respecting the functions $\xi_i$ must leave $X$ invariant. After replacing $V$ by an equidimensional projective variety of the same dimension containing $V$ and

[4]This uses in particular the existence of an equivariant projective completion for a finite group action on an algebraic variety. In fact if a finite group $H$ acts on an algebraic variety $V$ with projective completion $\bar{V}$, one can embed $V$ diagonally in $H \times V$ where $H$ acts on the left coordinate, and take the Zariski closure of the image in $H \times \bar{V}$.

adding the valuation of the characteristic function of the lower dimensional components of $V$ to the functions $\xi$, one may also assume $V$ is equidimensional.

Hence, we may assume $X = V$ is projective and equidimensional and we need to find a deformation retraction preserving certain functions $\xi_i$, and a finite algebraic group action $H$.

At this point we note that we can take the base $A$ to be a field. Let $F = \mathrm{VF}(A)$ be the field part. Then $V$ and $H$ are defined over $F$. Write $\xi = \xi_\gamma$ with $\gamma$ from $\Gamma$. Let $\xi'(x)$ be the function: $\gamma \mapsto \xi_\gamma(x)$. Clearly if the fibers of $\xi'$ is preserved then so is each $\xi_\gamma$. By stable embeddedness of $\Gamma$, $\xi'$ can be coded by a function into $\Gamma^k$ for some $k$. And this function is $F$-definable. Thus all the data can be taken to be defined over $F$, and the theorem over $F$ will imply the general case.

We may assume $F$ is perfect (and Henselian), since this does not change the notion of definability over $F$.

We use induction on $n = \dim(V)$. For $n = 0$, take the identity deformation $h(t, x) = x$, $w = V$, and map $a \in w$ to $(0, \dots, 0, \infty, 0, \dots, 0)$ with $\infty$ in the $a$'th place.

We start with a hypersurface $D_0$ of $V$ containing the singular locus $V_{\mathrm{sing}}$, and such that there exists an étale morphism $V \smallsetminus D_0 \to \mathbb{A}^n$, factoring through $V/H$[5].

Note that the $\xi_i$ factor through v+g-continuous functions into $\Gamma_\infty^n$. (If $f, g$ are homogeneous polynomials of the same degree, then $\mathrm{val}(f/g)$ is a function of $\max(0, \mathrm{val}(f) - \mathrm{val}(g))$ and $\max(0, \mathrm{val}(g) - \mathrm{val}(f))$. The characteristic function of a set defined by $\mathrm{val} f_i \geq \mathrm{val} f_j$ is the composition of the characteristic function of $x_i \geq x_j$ on $\Gamma_\infty^m$, with the function $(\mathrm{val} f_1, \dots, \mathrm{val} f_m)$.) Hence we may take the $\xi_i$ to be continuous. By enlarging $D_0$, we may assume $D_0$ contains $\xi_i^{-1}(\infty)$ (Lemma 9.4.3). Moreover, we can demand that $D_0$ is $H$-invariant, and that the *set* $\{\xi_i : i \in I\}$ is $H$-invariant, by increasing both if necessary. Note that there exists a continuous function $m = (m_1, \dots, m_n) : \Gamma_\infty^I \to \Gamma_\infty^n$ whose fibers are the orbits of the symmetric group acting on $I$, namely $m((x_i)_{i \in I}) = (y_1, \dots, y_n)$ if $(y_1, \dots, y_n)$ is a non-decreasing enumeration of $(x_i : i \in I)$, with appropriate multiplicites. Then $(m \circ \xi_i : i \in I)$ is $H$-invariant. It is clear that a homotopy preserving $m \circ \xi$ also preserves each $\xi_i$. Thus we may assume that each $\xi_i$ is $H$-invariant.

Let $E$ be the blowing-up of $\mathbb{P}^n$ at one point. Then $E$ admits a morphism $\pi : E \to \mathbb{P}^{n-1}$, whose fibers are $\mathbb{P}^1$. We now show one may assume $V$ admits a finite morphism to $E$, with composed morphism to $\mathbb{P}^{n-1}$ finite on $D_0$.

**Lemma 10.2.1.** *Let $V$ be a projective variety of dimension $n$. Then $V$ admits a finite morphism $\pi$ to $\mathbb{P}^n$ and there is a finite closed subset $Z$ of $V$ such that if $v : V_1 \to V$ denote the blow-up at $Z$, there exists a finite morphism $m : V_1 \to E$*

[5]Such a $D_0$ exists using generic smoothness, after choosing a separating transcendence basis at the generic point of $V/H$.

*making the diagram*

$$\begin{tikzcd} V_1 \arrow[r, "v"] \arrow[d, "m"] & V \arrow[d, "\pi"] \\ E \arrow[r] & \mathbb{P}^n \end{tikzcd}$$

*commutative. Moreover, if a divisor $D_0$ on $V$ is given in advance, we may arrange that $\pi \circ v$ is finite on $v^{-1}(D_0)$. If a finite group $H$ acts on $V$, we may take all these to be $H$-invariant.*

*Proof.* Let $m$ be minimal such that $V$ admits a finite morphism to $\mathbb{P}^m$. If $m > n$, choose a $\mathbb{P}^{m-1}$ inside $\mathbb{P}^m$, and a point neither on the $\mathbb{P}^{m-1}$ nor on the image of $V$; and project the image of $V$ to the $\mathbb{P}^{m-1}$ through this point. Hence $m = n$, i.e. there exists a finite morphism $V \to \mathbb{P}^n$.

Given a divisor $D_0$ on $V$, choose a point $z$ of $\mathbb{P}^n$ not on the image of this divisor. The projection through this point to a $\mathbb{P}^{n-1}$ contained in $\mathbb{P}^n$, and not containing $z$ determines a morphism $E \to \mathbb{P}^{n-1}$. If $V_1$ is the blow-up of $V$ at the pre-image $Z$ of $z$, we find a morphism $V_1 \to E$; composing with $E \to \mathbb{P}^{n-1}$ we obtain the required morphism. To arrange for $H$-invariance, apply the lemma to $V/H$. □

Hence we may find an $H$-equivariant birational morphism $v : V_1 \to V$, whose exceptional locus lies above a finite subset of $V$, such that $V_1$ admits a finite morphism to $E$, and moreover the composed morphism to $\mathbb{P}^{n-1}$ is finite over the full pullback of $D_0$.

Since only finitely many points of $V$ are blown up, their pre-images are finitely many subvarieties of $V_1$. Thus, by Lemma 5.3.3, any deformation retraction of $V_1$ leaving these invariant descends to a deformation retraction on $V$. Pulling back the data of Theorem 10.1.1 to $V_1$, and adding the above invariance requirement, we see that it suffices to prove the theorem for $V_1$. Hence we may assume that $V = V_1$, i.e. $V$ admits a finite morphism to $E$, with composed morphism to $\mathbb{P}^{n-1}$ finite on $D_0$.

10.3. **Construction of a relative homotopy $H_{curves}$.** We fix a non empty open affine $U_0$ subset of $U = \mathbb{P}^{n-1}$ over which the restriction $E_0$ of $E$ may be identified with $U_0 \times \mathbb{P}^1$. We also fix three points $0, 1, \infty$ in $\mathbb{P}^1$. We are now in the setting of §9.2 with $U = \mathbb{P}^{n-1}$. Recall that $D_{ver}$ denotes the vertical part of a divisor $D$. For any divisor $D$ on $E$ such that $D_{ver}$ contains $U \smallsetminus U_0$ we consider $\psi_D : [0, \infty] \times E \to \widehat{E/U}$ as in §9.2.

**Lemma 10.3.1.** *Let $F$ be an $A$-iso-definable subset of $\widehat{E_0/U_0}$ such that $F \to U_0$ has finite fibers. There exists a divisor $D'$ on $E_0$, generically finite over $U_0$, such that for every $u$ in $U_0$, for every $x$ in $F$ over $u$, the intersection of $D'$ with the ball in $\mathbb{P}^1_u$ corresponding to $x$ is non empty.*

*Proof.* Recall we are working over a field-base $A$. By splitting $F$ into two parts (then taking the union of the divisors $D'$ corresponding to each part), we may assume $F \subseteq \widehat{\mathcal{O}} \times U_0$ where $\mathcal{O}$ is the unit ball. Let $a$ be a point in $U_0$; so $F_a \subseteq \widehat{\mathcal{O}}$.

We claim that there exists a finite $A(a)$-definable subset $D'_a$ of $\mathcal{O}$ such that for every $x$ in $F_a$, the intersection of $D'_a$ with the ball in $\mathcal{O}$ corresponding to $x$ is non empty. If $F_a$ contains some simple points, let $D'_a$ be the union of these simple points. If it does not, and $A(a)$ is trivially valued, any $A$-definable closed sub-ball of $\mathcal{O}$ must have valuative radius 0, i.e. must equal $\mathcal{O}$. In this case one may set $D'_a = \{0\}$. Otherwise, $A$ is a nontrivially valued field, and so $\mathrm{acl}(A(a))$ is a model of ACVF. Hence, if we denote by $\tilde{F}_a$ the finite set of closed balls corresponding to the points in $F_a$, for every $b$ in $\tilde{F}_a$, $b \cap \mathrm{acl}(A(a)) \neq \emptyset$, thus there exists a finite $A(a)$-definable set such that $D'_a \cap b \neq \emptyset$ for every $b$ in $\tilde{F}_a$.

By compactness we get a constructible set $D''$ finite over $U_0$ with the required property. Taking the Zariski closure of $D''$ we get a Zariski closed set $D'$ generically finite over $U_0$ with the required property. □

**Lemma 10.3.2.** *There exists a divisor $D'$ such that, for any divisor $D$ containing $D'$ and such that $D_{ver}$ contains $U \smallsetminus U_0$, $\psi_D$ lifts to an $A$-definable map $h : [0, \infty] \times V \to \widehat{V/U}$.*

*Proof.* We proceed as in the proof of Proposition 7.6.1. Note that $V \to U$ is a relative curve so that $\widehat{V/U}$ is iso-definable over $A$ by Lemma 7.1.3. For every $u$ in $U$ write $f_u : V_u \to \mathbb{P}^1_u$ for the restriction of $V \to E$ over $u$. There is an iso-definable over $A$ subset $F_0$ of $\widehat{V_0/U_0}$ containing, for every point $u$ in $U_0$, all singular points of $C_u$, all ramification points of $f_u$ and all forward-branching points of $f_u$, and such that the fibers $F_0 \to U_0$ are all finite. Such an $F_0$ exists by Lemma 7.5.4 (uniform finiteness of the set of forward-branching points). Let $F$ be the image of $F_0$ in $E_0$. Then $D'$ provided by Lemma 10.3.1 does the job. □

Let $D$ be a divisor on $E$ as in Lemma 10.3.2, and such that $D$ contains the image of $D_0$ in $E$. Then $\psi_D$ lifts to an $A$-definable map $h_{curves} : [0, \infty] \times V \to \widehat{V/U}$. By Lemma 9.2.2, after enlarging $D$, one can arrange that $h_{curves}$ preserves the functions $\xi_i$. Note that $H$-invariance follows from uniqueness of the lift. Denote by $D_{vert}$ the preimage of $D_{ver}$ in $V$. By Lemma 9.1.1 and Lemma 9.2.1, $h_{curves}$ is g-continuous on $[0, \infty] \times V$ and v-continuous at each point of $[0, \infty] \times (V \smallsetminus D_{vert}) \cup D_0$. By Lemma 3.7.3 the restriction of $h_{curves}$ to $[0, \infty] \times (V \smallsetminus D_{vert}) \cup D_0$ extends to a deformation retraction $H_{curves} : [0, \infty] \times (\widehat{V \smallsetminus D_{vert}}) \cup D_0 \to (\widehat{V \smallsetminus D_{vert}}) \cup D_0$. Since $D_0$ is finite over $U$, the image $\Upsilon_{curves} = h_{curves}(0, (V \smallsetminus D_{vert}) \cup D_0)$ is iso-definable over $A$ in $\widehat{V/U}$ and relatively $\Gamma$-internal. Furthermore, the image $H_{curves}(0, (\widehat{V \smallsetminus D_{vert}}) \cup D_0)$ is contained in $\widehat{\Upsilon_{curves}}$ (we identify here $\widehat{\Upsilon_{curves}}$ with its image in $\widehat{V}$, as above Lemma 6.3.13). By construction $H_{curves}(\infty, x) = x$ for every $x$ and $H_{curves}$ satisfies $(*)$.

Let $x_v : U \to \Gamma_\infty$ be a v+g-continuous $A$-definable function measuring the valuative schematic distance to $D_{ver}$ in $U$, so that $x_v^{-1}(\infty) = D_{ver}$; let $x_h : V \to \Gamma_\infty$ be a similar distance function to $D_0$. Note that $x_v$ is $\Gamma$-valued on $\Upsilon_{curves} \smallsetminus x_h^{-1}(\infty)$, and that $\{a \in \Upsilon_{curves} : x_h(a) \in \Gamma, x_v(a) \leq \alpha\}$ is definably compact for any $\alpha \in \Gamma$, being the continuous image of a definably compact. Thus $\Upsilon_{curves}$ is $\sigma$-compact with respect to $(x_h, x_v)$.

10.4. **The base homotopy.** By Lemma 6.3.13 there exists a finite pseudo-Galois covering $U'$ of $U$ and a finite number of $A$-definable functions $\xi'_i : U' \to \Gamma_\infty$ such that, for $I$ a generalized interval, any $A$-definable deformation retraction $h : I \times U \to \widehat{U}$ lifting to a deformation retraction $h' : I \times U' \to \widehat{U'}$ respecting the functions $\xi'_i$, also lifts to an $A$-definable deformation retraction $I \times \widehat{\Upsilon_{curves}} \to \widehat{\Upsilon_{curves}}$ respecting the restrictions of the functions $\xi_i$ on $\Upsilon_{curves}$ and the $H$-action.

Now by the induction hypothesis applied to $U'$ and $\mathrm{Gal}(U'/U)$, such a pair $(h, h')$ does exist; we can also take it to preserve $x_v$, the distance from $D_{vert}$. Set $h_{base} = h$. Hence, $h_{base}$ lifts to a deformation retraction

$$H_{\widetilde{base}} : I \times \widehat{\Upsilon_{curves}} \to \widehat{\Upsilon_{curves}},$$

respecting the restrictions of the functions $\xi_i$ and $H$, using the "moreover" in Lemma 6.3.13. Recall the notion of Zariski density in $\widehat{U}$, 3.9. By induction $h_{base}$ has a $\Gamma$-internal $A$-iso-definable final image $\Upsilon_{base}$ and we may assume $\Upsilon_{base}$ is Zariski dense in $\widehat{U}$. By Lemma 6.3.13 $H_{\widetilde{base}}$ has a $\Gamma$-internal $A$-iso-definable final image, and by induction we may assume $H_{\widetilde{base}}$ satisfies $(*)$.

By composing the homotopies $H_{curves}$ and $H_{\widetilde{base}}$ one gets an $A$-definable deformation retraction

$$H_{bc} = H_{\widetilde{base}} \circ H_{curves} : I' \times (V \smallsetminus \widehat{D_{vert}}) \cup D_0 \longrightarrow \widehat{V},$$

where $I'$ denotes the generalized interval obtained by gluing $I$ and $[0, \infty]$. The image is contained in the image of $H_{\widetilde{base}}$, but contains $H_{\widetilde{base}}(I \times \widehat{\Upsilon_{curves}/U})$, the image over the simple points of $U$. As these sets are equal, the image is equal to both, and is iso-definable and $\Gamma$-internal; we denote it $\Upsilon_{bc}$. In general $\Upsilon_{bc}$ is not definably compact, but it is $\sigma$-compact via $(x_h, x_v)$, since $H_{\widetilde{base}}$ fixes $x_v$ and $\Upsilon_{curves}$ is $\sigma$-compact with respect to the same functions.

**Lemma 10.4.1.** *The subset $\Upsilon_{bc}$ is Zariski dense in $\widehat{V}$.*

*Proof.* Let $V_i$ denote the irreducible components of $V$, $\pi : \widehat{V/U} \to U$ and $\widehat{\pi} : \widehat{V} \to \widehat{U}$ denote the projections. By construction $H_{curves}$ respects the $\widehat{V_i}$ and there exists an open dense subset $U_1 \subseteq U$ such that, for every $x \in U_1$, $\pi^{-1}(x) \cap \Upsilon_{curves} \cap \widehat{V_i}$ is Zariski dense for every $i$. Thus, by the construction in the proof of Lemma 6.3.13, for every $x \in \widehat{U_1}$, $\widehat{\pi}^{-1}(x) \cap \widehat{\Upsilon_{curves}} \cap \widehat{V_i}$ is Zariski dense for every $i$. Pick $x \in \Upsilon_{base}$ which is Zariski dense in $\widehat{U}$, then $\widehat{\pi}^{-1}(x) \cap \Upsilon_{bc}$ is Zariski dense in $\widehat{V}$. $\square$

10.5. **The homotopy in $\Gamma_\infty^w$.** By Corollary 6.3.7, there exists an $A$-definable, continuous, injective map $\alpha : \Upsilon_{bc} \to \Gamma_\infty^w$, with image $W \subseteq [0,\infty]^w$, where $w$ is a finite $A$-definable set. We also have continuous $A$-definable maps $\underline{v}$ and $\underline{h} : W \to \Gamma_\infty$, such that $\underline{v} \circ \alpha$ measures distance to $D_{ver}$, and $\underline{h} \circ \alpha$ measures distance to $D_0$ (these functions, defined on $V$ and hence on $\Upsilon_{bc}$, may be taken to factor through $\Gamma_\infty^w$).

For $a \in \Gamma_\infty^w$, and $i \in w$, we write $x_i(a)$ for the $i$'th coordinate $a_i$. By adding two points $h, v$ to $w$ (fixed by the group actions), we can assume that $\underline{v} = x_v, \underline{h} = x_h$ for some $h, v \in w$. We write $[x_i = x_j]$ for $\{a \in [0,\infty]^w : x_i(a) = x_j(a)\}$, and similarly $[x_i = 0]$, etc.

Since $\Upsilon_{bc}$ is $\sigma$-compact via $(x_h, x_v)$, $W$ is $\sigma$-compact with respect to $(\underline{h}, \underline{v})$. In particular, $W \smallsetminus [\underline{v} = \infty]$ is $\sigma$-compact via $\underline{v}$, and hence closed in $\Gamma_\infty^w \smallsetminus [\underline{v} = \infty]$; so $W \cap \Gamma^w$ is closed in $\Gamma^w$.

We let $H$ act on $W$, so that $\alpha : \Upsilon_{bc} \to \Gamma_\infty^w$ is equivariant. By re-embedding $W$ in $\Gamma_\infty^{w \times H}$, via $w \mapsto (\sigma(w) : \sigma \in H)$, we may assume $H$ acts on the coordinate set $w$, and the induced action of $H$ on $\Gamma_\infty^w$ extends the action of $H$ on $W$.

Entirely within $\Gamma_\infty^w$, we show the existence of deformations from a $\sigma$-compact such as $(W \smallsetminus [\underline{v} = \infty]) \cup [\underline{h} = \infty]$ to a definably compact set. We begin with $W \cap \Gamma^w$.

**Lemma 10.5.1.** *Let*
$$W' = (W \cap \Gamma^w) \cup [\underline{h} = \infty].$$
*There exists an $A$-definable deformation retraction $H_\Gamma : [0,\infty] \times W' \to W'$ whose image is a definably compact subset $W_0$ of $W'$ and such that $H_\Gamma$ leaves the $\xi_i$ invariant, fixes $[\underline{h} = \infty]$, and is $H$-equivariant.*

In this lemma, we take 0 to be the initial point, $\infty$ the final point. On $\Gamma_\infty$, we view $\infty$ as the unique simple point. In this sense the flow will is still "away from the simple points", as for the other homotopies. Moreover, starting at any given point, the flow will terminate at a finite time. The homotopy we obtain will in fact be a semigroup action, i.e. $H_\Gamma(s, H_\Gamma(t,x)) = H_\Gamma(s+t, x)$, in particular it will satisfy $(*)$ (in the form: $H_\Gamma(\infty, H_\Gamma(t,x)) = H_\Gamma(\infty, x)$).

*Proof.* Find an $A$-definable cellular decomposition $\mathcal{D}$ of $\Gamma^w$, compatible with $W \cap \Gamma^w$ and with $[x_a = 0]$ and $[x_a = x_b]$ where $a, b \in w$, and such that each $\xi_i$ is linear on each cell of $\mathcal{D}$. We also assume $\mathcal{D}$ is invariant under both the Galois action and the $H$-action on $w$. This can be achieved as follows. Begin with a finite set of pairs $(\alpha_j, c_j) \in \mathbb{Q}^w \times \Gamma^w$, such that each of the subsets of $\Gamma^w$ referred to above is defined by inequalities of the form $\alpha_j v - c_j \odot_j 0$, where $\odot_j$ is $<$ or $>$ or $=$. Take the closure of this set under the Galois action and the $H$-action. A cell of $\mathcal{D}$ is any nonempty set defined by conditions $\alpha_j v - c_j \odot_j 0$, where $\odot_j$ is any function from the set of indices to $\{<, >, =\}$. Such a cell is an open convex subset of the affine hyperplane that it spans.

Any bijection $b : w \to \{1, \ldots, |w|\}$ yields a bijection $b_* : \Gamma^w \to \Gamma^{|w|}$; the image of $c_j$ under these various bijections depends on the choice of $b$ only up to reordering. Thus $b_*(c_j)$ gives a well-defined subset of $\Gamma$, which belongs to $\Gamma(A)$. Let $\mathbf{A}$ be the convex subgroup of $\Gamma = \Gamma(\mathbb{U})$ generated by $\Gamma(A)$, and let $B = \Gamma(\mathbb{U})/\mathbf{A}$. For each cell $C$ of $\mathcal{D}$, let $\beta C$ be the image of $C$ in $B^w$. Note that $\beta C$ may have smaller dimension than $C$; notably, $\beta C = (0)$ iff $C$ is bounded. At all events $\beta C$ is a cell defined by homogeneous linear equalities and inequalities. When $\Gamma(A) \neq (0)$, $\beta C$ is always a closed cell, i.e. defined by weak inequalities.

For any $C \in \mathcal{D}$, let $C_\infty$ be the closure of $C$ in $\Gamma^w_\infty$. Let $\mathcal{D}_0$ be the set of cells $C \in \mathcal{D}$ such that $C_\infty \smallsetminus \Gamma^w \subseteq [\underline{h} = \infty]$. Equivalently, $C \in \mathcal{D}_0$ if and only if for each $i \in w$, an inequality of the form $x_i \leq m\underline{h} + c$ holds on $C$, for some $m \in \mathbb{N}$ and $c \in \Gamma$. Other equivalent conditions are that $x_i \leq m\underline{h}$ on $\beta C$ for some $i$, or that there exists no $e \in C$ with $\underline{h}(e) = 0$ but $x_i(e) \neq 0$. Let

$$W_0 = \cup_{C \in \mathcal{D}_0} C \cup [\underline{h} = \infty].$$

It is clear that $W_0$ is a definably compact subset of $\Gamma^w_\infty$, contained in $W' = \Gamma^w \cup [\underline{h} = \infty]$.

More generally, define a quasi-ordering $\leq_C$ on $w$ by: $i \leq_C j$ if for some $m \in \mathbb{N}$, $x_i(c) \leq m x_j(c)$ for all $c \in \beta C$. Since the decomposition respects the hyperplanes $x_i = x_j$, we have $i \leq_C j$ or $j \leq_C i$ or both. Thus $\leq_C$ is a linear quasi-order. Let $\beta' C = \beta C \cap [\underline{h} = 0]$. We have $\beta' C = 0$ iff $\underline{h}$ is $\leq_C$-maximal iff $C \in \mathcal{D}_0$. If $C \in \mathcal{D}_0$, let $e_C = 0$. Otherwise, $\beta' C$ is a nonzero rational linear cone, in the positive quadrant. Let $e_C$ be the barycenter of $\beta' C \cap [\sum x_i = 1]$. The choice is thus $H$ and Galois invariant.

For $t \in \Gamma$, we have $te_C := e_C t \in \Gamma^w$. If $e_C \neq 0$ then $\Gamma e_C$ is unbounded in $\Gamma^w$, so for any $x \in V$ there exists $t \in \Gamma$ such that $x - te_c \notin C$. Let $\tau(x)$ be the unique smallest such $t$.

We will now define $H_\Gamma$ on each cell $C \in \mathcal{D}$ separately. For cells $C \in \mathcal{D}_0$, let $H_\Gamma(t, x) = x$ be the constant homotopy.

Define $H_\Gamma : [0, \infty] \times C \to \Gamma^w$ (separately on each cell $C \in \mathcal{D}$) by induction on the dimension of $C$, as follows. If $C \in \mathcal{D}_0$, $H_\Gamma(t, x) = x$. Assume $C \in \mathcal{D} \smallsetminus \mathcal{D}_0$. If $x \in C$ and $t \leq \tau(x)$, let $H_\Gamma(t, x) = x - te_C$. So $H_\Gamma(\tau(x), x)$ lies in a lower-dimensional cell $C'$. For $t \geq \tau(x)$ let $H_\Gamma(t, x) = H_\Gamma(t - \tau(x), \tau(x))$. So $H_\Gamma(t', x) \in C'$ for $t \geq \tau(x)$. For fixed $a$, $H_\Gamma(t, a)$ thus traverses finitely many cells as $t \to \infty$, with strictly decreasing dimensions.

We claim that $H_\Gamma$ is continuous on $[0, \infty] \times \Gamma^w$. To see this fix $a \in C \in \mathcal{D}$ and let $(t', a') \to (t, a)$. We need to show that $H_\Gamma(t', a') \to H_\Gamma(t, a)$. By curve selection it suffices to consider $(t', a')$ varying along some line $\lambda$ approaching $(t, a)$. For some cell $C'$ we have $a' \in C'$ eventually along this line.

If $a' \in W_0$ then $a \in W_0$ since $W_0$ is closed. In this case we have $H_\Gamma(a', t') = a', H_\Gamma(a, t) = a$, and $a' \to a$ tautologically. Assume therefore that $a' \notin W_0$, so $C' \notin \mathcal{D}_0$, $e' \neq 0$ (where $e' = e_{C'}$), and $\tau(a') \neq \infty$.

Consider first the case: $t' \leq \tau(a')$ (cofinally along $\lambda$). Then by definition we have $H_\Gamma(t', a') = a' - t'e'$. Now $C$ must be a boundary face of $C'$, cut out from the closure of $C'$ by certain hyperplanes $\alpha_j v - c_j = 0$ ($j \in J(C, C')$). We have $\alpha_j v = c_j$ for $v \in C$, and (we may assume) $\alpha_j v \geq c_j$ for $v \in C'$.

If $\gamma_j = \alpha_j e' > 0$ for some $j$, fix such a $j$. As $t' \leq \tau(a')$, we have $\alpha_j(a' - t'e') = \alpha_j a' - \gamma_j t' \geq c_j$, so $t' \leq \gamma_j^{-1}(\alpha_j a' - c_j)$. Now $a' \to a$ so $\alpha_j a' - c_j \to 0$. Thus $t' \to 0$, i.e. $t = 0$. So $H_\Gamma(t, a) = a$, and $H_\Gamma(t, a) - H_\Gamma(t', a') = a - (a' - t'e') = (a - a') + t'e \to 0$ (as $(t', a') \to (t, a)$ along $\lambda$).

The remaining possibility is that $\alpha_j e' = 0$ for each $j \in J(C, C')$. So $\alpha_j v = 0$ for each $v \in \beta' C'$. Hence $\beta' C' \subseteq \beta C$. Since $\beta' C \subseteq \beta' C'$, it follows that $\beta' C = \beta' C'$ and so $e_C = e_{C'}$. Now $(t, x) \mapsto x - te'$ is continuous on all of $\Gamma \times \Gamma^w$ so on $C \cup C'$, and hence again $H_\Gamma(t', a') \to H_\Gamma(t, a)$.

This finishes the case $t' \leq \tau(a')$ (including $\tau(a') = \infty$). In particular, letting $a'' = H_\Gamma(a', \tau(a'))$, we find that $a'' \to a$; and $\tau(a') \to t^*$ for some $t^*$. Now by induction on the dimension of the cell $C'$, we have $H_\Gamma(t' - \tau(a'), a'') \to H_\Gamma(t - t^*, a)$; it follows that $H_\Gamma(t', a') \to H_\Gamma(t, a)$. This shows continuity on $[0, \infty] \times \Gamma^w$.

Note that if $C \in \mathcal{D} \smallsetminus \mathcal{D}_0$, then $\xi_i$ depends only on coordinates $x_i$ with $x_i \leq_C \underline{h}$. This follows from the fact that $\xi_i$ is bounded on any part of $C$ where $\underline{h}$ is bounded; so $\xi_i \leq m\underline{h}$ for some $m$. Since $x_i(e_C) = 0$ for $i \leq_C \underline{h}$, it follows that $\xi_i$ is left unchanged by the homotopy on $C$. So along a path in the homotopy, $\xi_i$ takes only finitely many values (one on each cell); being continuous, it must be constant. In other words the $\xi_i$ are preserved. The closures of the cells are also preserved, hence, as $W \cap \Gamma^w$ is closed, $W \cap \Gamma^w$ is preserved by the homotopy.

Extend $H_\Gamma$ to $W'$ by letting $H_\Gamma(t, x) = x$ for $x \in W' \smallsetminus \Gamma^w$. We argue that $H_\Gamma$ is continuous at $(t, a)$ for $a \in W' \smallsetminus \Gamma^w$, i.e. $\underline{h}(a) = \infty$. We have to show that for $a'$ close to $a$, for all $t$, $H_\Gamma(t, a')$ is also close to $a$. If $a' \notin \Gamma^w$ we have $H_\Gamma(t, a') = a'$. Assume $a' \in \Gamma^w$; so $a' \in C$ for some $C \in \mathcal{D}$. If $C \in \mathcal{D}_0$, again we have $H_\Gamma(t, a') = a'$. Otherwise, there will be a time $t'$ such that $H_\Gamma(t', a') = a'' \notin C$. So $a''$ will fall into a another cell, with lower $\underline{v}$. We will show that $H_\Gamma(t, a')$ remains close to $a$ for $t \leq \tau(a')$. In particular, $a''$ is close to $a$; so (inductively) $H_\Gamma(t, a'') = H_\Gamma(t' + t, a)$ is close to $a$. Thus it suffices to show for each coordinate $i \in w$ that $x_i(a')$ remains close to $x_i(a)$. If $i \leq_{C'} \underline{h}$ then the homotopy does not change $x_i(a')$ so (as $a$ is fixed) we have $x_i(H_\Gamma(t, a')) = x_i(a') \to x_i(a) = x_i(H_\Gamma(t, a))$. So assume $h <_C i$. Since $\underline{h}(a) = \infty$ we have $\underline{h}(a') \to \infty$ and hence $x_i(a') \to \infty$. So $x_i(a) = \infty = x_i(H_\Gamma(t, a))$. For any $c = H_\Gamma(t, a')$, $t \leq \tau(a')$, we have $x_i(c) \geq \underline{h}(c)/m = \underline{h}(a')/m$. Since $a' \to a$, $\underline{h}(a')$ is large, so $x_i(c)$ is large, i.e. close to $x_i(a)$. This proves the continuity of $H_\Gamma$ on $W'$. This ends the proof of Lemma 10.5.1. $\square$

**Lemma 10.5.2.** *There exists a z-dense open subset $W^o$ in $W$ such that with*

$$W' = (W^o \smallsetminus [\underline{v} = \infty]) \cup [\underline{h} = \infty],$$

*there exists an A-definable deformation retraction* $H_\Gamma : [0,\infty] \times W' \to W'$ *whose image is a definably compact set* $W_0$ *of* $W'$ *and such that* $H_\Gamma$ *leaves the* $\xi_i$ *invariant, fixes* $[\underline{h} = \infty]$*, and is* $H$*-equivariant.*

*Proof.* First assume $W$ is z-irreducible and is not contained in $[\underline{v} = \infty]$. Let $w^0$ be the set of all $i \in w$ such that the $i$'th projection $\pi_i : W \to \Gamma_\infty$ does not take the constant value $\infty$ on $W$. Clearly $\pi^o = \Pi_{i \in w^o} \pi_i$ is a homeomorphism on $W$ onto the image. Note that $\pi^o(W) \cap \Gamma^{w^o}$ is z-open and z-dense in $\pi^o(W)$, and disjoint from $[\underline{v} = \infty]$. Applying Lemma 10.5.1 to $\pi^o(W) \cap \Gamma^{w^o}$ and pulling back by $\pi^o$ we obtain the required homotopy $H_\Gamma = H_{\Gamma,W}$.

In general let $W = \cup_\nu W_\nu$ be the decomposition of $W$ into components. For each $\nu$, let $H_\nu = H_{\Gamma,W_\nu}$ be as above. Note that the intersection of two components is a proper subset of each. Let $W_\nu^o = W_\nu \smallsetminus \cup_{\nu' \neq \nu} W_{\nu'}$. Then $W_\nu^o$ is z-open and z-dense in $W_\nu^o$. Let $H_{\Gamma,W}$ be the union of all $H_\nu | W_\nu^o$, over all components $W_\nu$ that are not contained in $[\underline{v} = \infty]$ (equivalently, $W_\nu^o$ has empty intersection with $[\nu = \infty]$). The process in Lemma 10.5.1 and in the first paragraph of the present lemma is entirely canonical, the retraction $H_{\Gamma,W}$ obtained this way is $A$-definable and $H$-invariant. □

Note that since $W_0$ is definably compact and contained in $\Gamma^w \cup [\underline{h} = \infty]$, for each $i \in w$ there exists some $m_i \in \mathbb{N}$ and $c_i \in \Gamma(A)$ such that $x_i \leq m_i x_h + c_i$ on $W_0 \cap \Gamma^w$. We will use these $m_i, c_i$ below.

**Lemma 10.5.3.** *Let* $\Upsilon$ *be a* $\Gamma$*-internal iso-definable subset of* $\widehat{V}$ *which is the image of* $\widehat{V}$ *under an A-definable strong homotopy retraction. Assume* $\Upsilon$ *is Zariski dense in* $\widehat{V}$ *in the sense of 3.9. Then there exists a continuous A-pro-definable map* $\beta : \widehat{V} \to \Gamma_\infty^{w'}$*, injective on* $\Upsilon$*, such that if* $Z \subset \beta(\Upsilon)$ *is a z-open dense subset of* $\Gamma_\infty^{w'}$*, then* $\beta^{-1}(Z)$ *is a Zariski open dense subset of* $\widehat{V}$*.*

*Proof.* Let $\alpha : \widehat{V} \to \Gamma_\infty^w$ be the composition of a retraction to $\Upsilon$ with an appropriate $A$-definable injective continuous map $\Upsilon \to \Gamma_\infty^w$ as provided by Corollary 6.3.7. Let $C_1, \ldots C_r$, be the irreducible components of $V$. For each $C_j$, let $x_j$ be a valuative schematic distance function to $C_j$ and let $\beta(x) = (\alpha(x), x_1, \ldots, x_r)$. By Lemma 9.4.3, $\beta^{-1}(Z)$ is Zariski closed if $Z$ is z-closed. Hence the same holds for z-open. If $Z \subset Y$ is z-closed in $Y = \beta(\Upsilon)$ and contains no z-component of $Y$, suppose $\beta^{-1}(Z)$ contains some $\widehat{C_{j_0}}$. Then $\beta^{-1}(Z) \cup \cup_{j \neq j_0} \widehat{C_j}$ contains $\Upsilon$, so $Z \cup \cup_{j \neq j_0} [x_j = \infty]$ contains $Y$. It follows that $\cup_{j \neq j_0} [x_j = \infty]$ contains $Y$ already. But then as $\widehat{C_j} = \beta^{-1}([x_j = \infty])$ we have $\Upsilon \subseteq \cup_{j \neq j_0} \widehat{C_j}$, contradicting the hypothesis on $\Upsilon$. □

10.6. **The inflation homotopy.** By Lemma 9.3.2 there exists an $A$-definable homotopy $H_{inf} : [0,\infty] \times \widehat{V} \to \widehat{V}$ respecting the functions $\xi_i$ and the group action $H$ with image contained in $(V \smallsetminus \widehat{D_{vert}}) \cup D_0$. (In fact, by Lemma 9.3.1

the image is contained in $\widehat{Z}$ with $Z$ a v+g-closed bounded definable subset of $V$ with $Z \cap D_{vert} \subseteq D_0$.) We also require that the functions $\phi_i = \min(x_i, m_i x_h + c_i)$ be preserved, for each $i \in w$. This is possible by the same lemma, since these functions into $\Gamma$ are continuous off $D_0$. Now the restriction of $\phi_i$ to $W_0$ is just the $i$'th coordinate function; so $\alpha^{-1}(W_0)$ is fixed pointwise by $H_{inf}$. By construction $H_{inf}$ satisfies $(*)$.

We will define $H$ as the composition (or concatenation) of homotopies

$$H = H_\Gamma^\alpha \circ ((H_{\widetilde{base}} \circ H_{curves}) \circ H_{inf}) : I'' \times \widehat{V} \to \widehat{V}$$

where $H_\Gamma^\alpha$ is to be constructed, and $I''$ denotes the generalized interval obtained by gluing $[\infty, 0]$, $I'$ and $[0, \infty]$. Being the composition of homotopies satisfying $(*)$, $H$ satisfies $(*)$.

Since the image of $H_{inf}$ is contained in the domain of $H_{bc}$, the first composition makes sense.

By construction, we have a continuous $A$-pro-definable retraction $\beta_{bc}$ from $(\widehat{V \smallsetminus D_{vert}}) \to \Upsilon_{bc}$, sending $b$ to the final value of $t \mapsto H_{bc}(t, v)$. Furthermore, by Lemma 10.4.1, $\Upsilon_{bc}$ is Zariski dense in $\widehat{V}$. Applying Lemma 10.5.3 to $\Upsilon_{bc}$ let $\beta : \widehat{V \smallsetminus D_{vert}} \to \Gamma_\infty^w$ be a continuous $A$-pro-definable map, injective on $\Upsilon_{bc}$, such that if $Z \subset \beta(\Upsilon_{bc})$ is a z-open dense subset of $\Gamma_\infty^w$, then $\beta^{-1}(Z)$ is a Zariski open dense subset of $\widehat{V}$. Denote by $\alpha$ the restriction of $\beta$ to $\Upsilon_{bc}$. Let $W^o$ be as in Lemma 10.5.2. Then $\beta^{-1}(W^o)$ is Zariski open and dense. Now for any Zariski dense open $O$, the image $I_{inf}$ of $H_{inf}$ is contained in $\widehat{O \cup D_0}$. Thus $\beta_{bc}(I_{inf})$ is a definably compact subset of $\beta^{-1}(W^o) \cap \Upsilon_{bc}$. Note that $\beta$ restricts to a homeomorphism $\alpha_1$ between this set and a definably compact subset $W_1$ of $W$. One sets $H_\Gamma^\alpha(t, x) = \alpha_1^{-1} H_\Gamma(t, \alpha_1(x))$: in short, $H_\Gamma^\alpha$ is $H_\Gamma$ conjugated by $\alpha$, restricted to an appropriate definably compact set. So $H$ is well-defined by the above quadruple composition.

Since $H_{inf}$ fixes $\alpha^{-1}(W_0)$, and $W_0$ is the image of $H_\Gamma$, $H_{inf}$ fixes the image of $H$. One the other hand $H_{bc}$ fixes $\Upsilon_{bc}$ and hence the subset $\alpha^{-1}(W_0) \subseteq \Upsilon_{bc}$. Thus $H$ fixes its own image $\Upsilon = \alpha^{-1}(W_0)$.

Any limit point $q$ of $\Upsilon$ in $D_{vert}$ is necessarily in $D_0$, as one sees by applying $\alpha$. Hence $\Upsilon$ is definably compact and $\alpha$ is a homeomorphism from $\Upsilon$ to the definably compact subset $W_0$ of $\Gamma_\infty^w$.

We now check that $\Upsilon$ is Zariski dense in $\widehat{V}$. Otherwise there would exist a definable continuous function $\eta : W' \to \Gamma_\infty$ such that $W_0 \subseteq \eta^{-1}(\infty)$ and $W' \not\subseteq \eta^{-1}(\infty)$. Pick a point $x$ in $W'$ with $\eta(x)$ finite. By construction of $H_\Gamma$, for some finite $t_0$, $H_\Gamma(t_0, x)$ lies in $W_0$. This is a contradiction, since the function $t \mapsto \eta(H_\Gamma(t, x))$ can only take finite values for finite $t$.

This ends the proof of Theorem 10.1.1, except for the verification of (5). If $p, q$ are stably dominated types satisfying (5), where $p$ is $M$-definable, $c \models p|M$,

$q$ is $M(c)$-definable, $a \models q|M(c)$, and $r$ is the unique stably dominated type over $M$ with $\mathrm{tp}(ca/M) = q|M$ (given by Lemma 2.5.5), then it is clear from the definitions that $r$ has property (5) too. Now (5) is clear for the homotopy on a curve. Inductively it holds for the skeleton of the base homotopy. Hence by transitivity it holds for each element of f $\Upsilon_{bc}$, away from $D_{ver}$, or on $D_0$. Since any element of $\Upsilon$ is in fact such an element of $\Upsilon_{bc}$, one deduces (5). □

10.7. **Variation in families.** When $h : X \to Y$ is a pro-definable map, forming a commutative diagram with maps $X \to T$, $Y \to T$, the family of maps $h_t : X_t \to Y_t$ obtained by restriction to fibers above $t \in T$ is referred to as *uniformly pro-definable.* We will be interested in the case where $T$ is a definable set.

Consider a situation where $(V, X) = (V_t, X_t)$ are given uniformly in a parameter $t$, varying in a definable set $T$. For each $t$, Theorem 10.1.1 guarantees the existence of a strong homotopy retraction $h_t : I \times \widehat{X_t} \to \widehat{X_t}$, and a definable homeomorphism $j_t : W_t \to h_t(e_I, \widehat{X_t})$, with $W_t$ a definable subset of $\Gamma_\infty^{w(t)}$. Such statements are often automatically uniform in the parameter $t$. However here the pro-definable $h_t$ is given by an infinite collection of definable maps, so compactness does not directly apply. Nevertheless the proof is uniform in the parameter $t$. We state this as a separate proposition.

**Proposition 10.7.1.** *Let $V_t$ be a quasi-projective variety, $X_t$ a definable subset of $V_t \times \Gamma_\infty^\ell$, definable uniformly in $t \in T$ over some base set $A$. Then there exists a uniformly pro-definable family $h_t : I \times \widehat{X_t} \to \widehat{X_t}$, a finite set $w(t)$ a definable set $W_t \subseteq \Gamma_\infty^{w(t)}$, and $j_t : W_t \to h_t(0, \widehat{X_t})$, pro-definable uniformly in $t$, such that for each $t \in T$, $h_t$ is a deformation retraction, and $j_t : W_t \to h_t(0, \widehat{X_t})$ is a pro-definable homeomorphism.*

*Moreover, (1), (2) and (3) of Theorem 10.1.1 can be gotten to hold, if the $\xi_i$ and the group action are given uniformly.*

*Proof.* The homotopy in the conclusion of Theorem 10.1.1 is a composition of four homotopies; these in turn are obtained by composing a number of constructions. The lemmas in these constructions have the following general form:

($*$) Let $E_1, E_2, \ldots, E_k$ be pro-definable sets over $A$; assume property $P$ holds of $E = (E_1, \ldots, E_k)$; then there exists a pro-definable $Y$ such that $Q$ holds of $(E, Y)$.

We have to check, in each case, the following:

($*$u) If $A = A_0(a)$, and if $E = E_a$ is given uniformly in $a$ and $P(E_a)$ holds for all $a$ in some $A_0$-definable set $D$, then $Y$ can be taken to be uniformly definable in $a$, and $Q(E_a, Y_a)$ holds for all $a \in D$.

Here if the property $P$ involves itself an existential quantifier over a pro-definable object, e.g. a bijection between a subset of $\Gamma^n$ and $E$, then this bijection should be taken as part of the data; similarly for the conclusion.

There are two cases in which (∗) (proved for all $A$) automatically implies (∗u). We can view $E$ as a sequence $(E_n)$ of definable sets, and similarly $Y$ as a sequence $(Y_n)$. Assume the property $Q(E, Y)$ is a (usually infinite) collection of first-order sentences, in a language enriched with predicates for $E_n, Y_n$; and similarly $P$.

Case (i): When $Y$ in (∗) is definable rather than prodefinable, the uniformity (∗u) follows from compactness.

This is the case in the lemmas on relative curves, since $\widehat{V/U}$ is definable. It is also the case for the homotopy within $\Gamma$, since again it lives entirely on a definable set; and also for Lemma 10.3.1 and Lemma 10.3.2.

Case (ii): Assume in (∗) that $Y$ not only exists but is unique, in the strong sense that for any model of the theory, and any $E$ with $P(E)$, there exists at most one $Y = (Y_n)_n$ with $Q(E, Y)$. Then Beth's theorem implies that $Y$ is prodefinable (as we are already assuming); but furthermore, since Beth's theorem applies to the incomplete theory with a constant for $t$, it implies that if the data is uniformly definable then so is $Y$.

Examples where (ii) applies are Lemma 5.3.3 and Lemma 3.7.3.

For some lemmas, however, we do not know an a posteriori proof of automatic uniformity, and have no better way than repeating the proof, dragging along an additional parameter $t$. Let us consider the case of Lemmas 6.3.9 and 6.3.13 which are good examples, leaving the verification of the remaining lemmas to the reader. The hypothesis of Lemma 6.3.9 includes the hypothesis that each fiber $X_u$ is $\Gamma$-internal; in the uniform version, we assume that this internality is uniform in $t$, i.e. that there are uniformly $t$-definable bijections $g_u : Z_u \to X_u$ with $Z_u$ a definable subset of $\Gamma^n$. Hence the second sentence of the proof of Lemma 6.3.9 goes through, i.e. compactness yields $(m, d)$ such that $\tau_h \circ g_u$ is injective for all $u, t$, and hence $\tau_h$ is injective. The rest of the proofs of these lemmas goes through even more routinely. □

## 11. The nonsingular case

**11.1.** For definable sets avoiding the singular locus it is possible to prove the following variant of Theorem 10.1.1. The proof uses the same ingredients but is considerably simpler in that only birational versions of most parts of the construction are required.

Given an algebraic variety $V$ one denotes by $V_{\mathrm{sing}}$ its singular locus.

**Proposition 11.1.1.** *Let $V$ be a quasi-projective variety over a valued field $F$ and let $X$ be a v-open $F$-definable subset of $V$, with empty intersection with $V_{\mathrm{sing}}$. Then there exists an $F$-definable homotopy $h : I \times \widehat{X} \to \widehat{X}$ between the identity and a continuous map to a pro-$F$-definable subset definably homeomorphic to a definable subset of $w' \times \Gamma^w$, for some finite $F$-definable sets $w$ and $w'$.*

*Moreover,*

(1) *Given finitely many continuous $F$-definable functions $\xi_i : X \to \Gamma$, one can choose $h$ to respect the $\xi_i$, i.e. $\xi_i(h(t,x)) = \xi_i(x)$ for all $t$.*
(2) *Assume given, in addition, a finite algebraic group action on $V$ preserving $X$. Then the homotopy retraction can be chosen to be equivariant.*
(3) *If $X$ is definably compact, the interval $I$ can be taken to be the standard interval $[\infty, 0]$, and $h$ can be taken to be a deformation retraction.*
(4) *If $V$ has dimension $d$ at each point $x \in X$, then each point of the image of $h$, viewed as a stably dominated type, has transcendence degree $d$.*

*In particular this holds for $X = V$ when $V$ is nonsingular.*

Note that the conclusion mentions $\Gamma$ rather than $\Gamma_\infty$. The finite set $w'$ can be dispensed with if $\Gamma(F) \neq (0)$, or if $\widehat{X}$ is connected, but not otherwise, as can be seen by considering the case when $X$ is finite.

The proof depends on two lemmas. The first recaps the proof of Theorem 10.1.1, but on a Zariski dense open set only. The second is a stronger form of inflation, using smoothness, moving into the Zariski open.

**Lemma 11.1.2.** *Let $V$ be a quasi-projective variety defined over $F$. Then there exists a Zariski open dense subset $V_0$ of $V$, and an $F$-definable deformation retraction $h : I \times \widehat{V_0} \to \widehat{V_0}$ whose image is a pro-definable subset, definably homeomorphic to an $F$-definable subset of $w' \times \Gamma^w$, for some finite $F$-definable sets $w'$ and $w$.*

*Moreover:*

(1) *Given finitely many $F$-definable functions $\xi_i : V \to \Gamma$, one can choose $h$ to respect the $\xi_i$, i.e. $\xi_i(h(t,x)) = \xi_i(x)$ for all $t$.*
(2) *Assume given, in addition, a finite algebraic group action on $V$ preserving $X$. Then $V_0$ and the homotopy retraction can be chosen to be equivariant.*

*Proof.* Find a Zariski open $V_1$ with $\dim(V \setminus V_1) < \dim(V)$, and a morphism $\pi : V_1 \to U$, whose fibers are curves. Let $H_{curves}$ be the homotopy described in §10.3. It is continuous outside some subvariety $U'$ of $U$ with $\dim(U') < \dim(U)$; replace $V_1$ by $V_1 = \pi^{-1}(U')$. So $H_{curves}$ is continuous on $V_1$; the image $S_1$ is relatively $\Gamma$-internal over $U$. By a (greatly simplified version of) the results of §6, over some étale neighborhood $U'$ of $U$, $S_1$ becomes isomorphic to a subset of $U' \times \Gamma_\infty^n$.

**Claim.** On a Zariski dense open subset of $V_1$, $S_1$ is isomorphic to a subset of $U' \times \{1, \dots, N\} \times \Gamma^n$.

*Proof of the Claim.* By removing a proper subvariety, we may assume $V_1$ is a disjoint union of irreducible components, and work within each component $W$ separately. The part of $S_1$ mapping to $U' \times \Gamma^n$ is Zariski open in $S_1$; if it is not empty, by irreducibility of $V_1$ it must be dense, and so we can move to this dense open set and obtain the lemma with $N = 1$. Otherwise $S_1$ is isomorphic to a subset of $U' \times \partial\Gamma_\infty^n$, where $\partial\Gamma_\infty^n = \Gamma_\infty^n \setminus \Gamma^n$. In this case we can remove a proper subvariety and decompose the rest into finitely many algebraic pieces, each mapping into one hyperplane at $\infty$ of $\Gamma_\infty^n$. $\square$

We may thus assume $S_1$ is isomorphic to a subset of $U' \times \{1, \dots, N\} \times \Gamma^n$. Inductively, the lemma holds for $U'$, so there exists a homotopy $H_{base}$ defined outside some proper subvariety $U''$. Let $V_0 = V_1 \setminus \pi^{-1}(U'')$. As in Theorem 10.1.1, lift to a homotopy $H_{\widetilde{base}}$ defined on $S_1 \cap \widehat{V_0}$. The homotopies can be taken to meet conditions (1) and (2). Composing, we obtain a deformation retraction of $V_0$ to a subset $S$, and a homeomorphism $\alpha : S \to Z \subset \{1, \dots, M\} \times \Gamma^m$, defined over $\mathrm{acl}(A)$. We may assume $M > 1$. As in Lemma 6.3.7 we can obtain an $A$-definable homeomorphism into $(\{1, \dots, M\} \times \Gamma^m)^w$. $\square$

**Lemma 11.1.3.** *Let $V$ be a subvariety of $\mathbb{P}^n$, and let $a \in V$ be a nonsingular point. Then the standard metric on $\mathbb{P}^n$ restricts to a good metric on $V$ on some v-open neighborhood of $a$.*

*Proof.* For sufficiently large $\alpha$, the set of points of distance $\geq \alpha$ from $a$ may be represented as the $\mathcal{O}$-points of a scheme over $\mathcal{O}$ with good reduction, whose special fiber is irreducible, in fact a linear variety. $\square$

*Proof of Proposition 11.1.1.* Let $H_c$ be a homotopy as in Lemma 11.1.2, defined on $V_0$. In particular we obtain a continuous map $f(0) : V_0 \to S_0$, where $S_0$ is the skeleton. Now $S_0$ admits an continuous, 1-1 map $g$ into $\Gamma^w$ for some $w$. For large $t$, let $H_{inf}(v, t)$ be the generic type of the ball of valuative radius $t$ around $v$. Since $X$ is v-open, $H_{inf}(v, t)$ stays within $X$. As in the proof of Theorem 10.1.1, find a continuous cutoff. Note that the image of $H_{inf}$ is contained in $V_0$. Let $H$ be the composition of $H_c$ and $H_{inf}$.

Now assume $X$ is definably compact. Then the image $I_{inf}$ of $X$ under $H_{inf}$ is bounded away from $Z_0 = V \setminus V_0$, say at distance $\geq \alpha_0$. In particular $g \circ f \circ h_0$ is a continuous map into $\Gamma$ (where $h_0$ is the final map of $H_{inf}$). Now modify $H_{inf}$ so as to stop as soon as distance $\geq \alpha_0$ from $Z_0$ is reached. Then the image of $X$ under $H_{inf}$ is still bounded away from $Z_0$ at distance $\alpha_0$, but now all points at such distance are fixed by $H_{inf}$. It follows that $H_{inf}$ is a deformation retraction. To ensure that the composition is also a deformation retraction we proceed as in Theorem 10.1.1, composing with an additional homotopy internal to $\Gamma$.

Given any chain of composed homotopies $h_1 \circ \cdots h_m$, by precomposing with $H_{inf}$ we obtain $h_1 \circ \cdots \circ h_m \circ h$, such that each $h_i$, restricted to the image of $h_{i+1} \circ \cdots \circ h$, is constant on some semi-infinite interval $[a, \infty]$. Thus as in §11.2 the intervals can be glued to a single interval. The homotopy internal to $\Gamma$ is only needed on a compact, where $\underline{v}$ is bounded, and hence requires a finite interval too. $\square$

**Remark 11.1.4** (A birational invariant)**.** It follows from the proof of Proposition 11.1.1 that the definable homotopy type of $\widehat{V \smallsetminus V_{\mathrm{sing}}}$ (or more generally of $\widehat{X \smallsetminus V_{\mathrm{sing}}}$ when $X$ is a v-open definable subset of $V$) is a birational invariant of $V$ (of the pair $(V, X)$). This rather curiously complements a theorem of Thuillier [27].

**11.2. On the number of intervals.** Our proof of Theorem 10.1.1 uses the induction hypothesis for the base $U$, lifted to a certain o-minimal cover (using the same generalized interval.) This is composed with three additional homotopies: inflation, and the relative curve homotopy, and the homotopy internal to $\Gamma$. Each of these use the standard interval from $\infty$ to 0. The number $h(n)$ of basic intervals needed for an $n$-dimensional variety thus satisfies $h(1) = 1$, $h(n+1) \leq h(n) + 3$, so $h(n) \leq 3n - 2$.

Observe that if $I$ is a glueing of $n$ intervals $[-\infty, \infty]$, and $f : I \to \widehat{V}$ is a path which is constant on some $[-\infty, a]$ in each copy of $[-\infty, \infty]$, and constant on some $[b, \infty]$ in all but the right-most interval, then one can collapse the generalized interval to an ordinary interval, and the path $f$ is equivalent to one defined on an interval $[0, \infty] \subset \Gamma_\infty$. Similar considerations apply to homotopies.

For a homotopy whose interval cannot be simplified in this way, consider $\mathbb{P}^1 \times \mathbb{P}^1$. With the natural choice of fibering in curves, the proof of Theorem 10.1.1 will lead to an iterated homotopy to a point: first collapse to $\{\text{point}\} \times \mathbb{P}^1$, then to $\{\text{point}\} \times \{\text{point}\}$.

Nevertheless one may ask if Theorem 10.1.1 remains true with homotopies using a single standard interval $[0, \infty] \subset \Gamma_\infty$. This is not important for our purposes but may become so in future work involving higher resolution. At least on a smooth projective variety $V$, the answer is likely to be positive; see Proposition 11.1.1.

## 12. An equivalence of categories

**12.1.** A semi-algebraic subset of $\widehat{V}$ is by definition a subset of the form $\widehat{X}$, where $X$ is a definable subset of $V$.

Let $C_{VF}$ be the category of semi-algebraic subsets of $\widehat{V}$, $V$ an algebraic variety; the morphisms are pro-definable continuous maps. We could also say that the objects are definable subsets of $V$, but the morphisms $U \to U'$ are still pro-definable continuous maps $\widehat{U} \to \widehat{U'}$.

Let $C_\Gamma$ be the category of definable subsets $X$ of $\Gamma_\infty^w$ (for various definable finite sets $w$), with definable continuous maps. Any such map is piecewise given by an element of $\mathrm{GL}_n(\mathbb{Q})$ composed with a translation, and with coordinate projections and inclusions $x \mapsto (x, \infty)$ and $x \mapsto (x, 0)$. (If $X$ is a definable subset of $\Gamma_\infty^n$ with no irreducible components of dimension $< n$, and $Y \subseteq \Gamma_\infty^w$, then any definable continuous map is piecewise given by an element of $\mathrm{GL}_n(\mathbb{Q})$ composed with a translation.)

Let $C^i_\Gamma$ be the category of separated $\Gamma$-internal definable subsets $X$ of $\widehat{V}$ (for various varieties $V$) with definable continuous maps.

These categories can be viewed as ind-pro definable: more precisely $\mathrm{Ob}_C$ is an ind-definable set, and for $X, Y \in \mathrm{Ob}_C$, $\mathrm{Mor}(X,Y)$ is a pro-ind definable set. But usually we will be interested only in the subcategory consisting of $A$-definable objects and morphisms. It can be defined in the same way in the first place, only replacing "definable" by "$A$-definable".

The three categories admit natural functors to the category TOP of topological spaces with continuous maps.

There is a natural functor $\iota : C_\Gamma \to C^i_\Gamma$, commuting with the natural functors to TOP; namely, given $X \subseteq \Gamma^n_\infty$, let $\iota(X) = \{p_\gamma : \gamma \in X\}$, where $p_\gamma$ is as defined above Lemma 3.4.2. By this lemma and Lemma 3.4.3, the map $\gamma \mapsto p_\gamma$ induces a homeomorphism $X \to \iota(X)$.

**Lemma 12.1.1.** *The functor $\iota$ is an equivalence of categories.*

*Proof.* It is clear that the functor is fully faithful. The essential surjectivity follows from Corollary 6.3.8. □

We now consider the corresponding homotopy categories $HC_{VF}$, $HC_\Gamma$ and $HC^i_\Gamma$. These categories have the same objects as the original ones, but the morphisms are factored out by (strong) homotopy equivalence. Namely two morphisms $f$ and $g$ from $X$ to $Y$ are identified if there exists a generalized interval $I = [0,1]$ and a continuous pro-definable map $h : X \times I \to Y$ with $h_0 = f$ and $h_1 = g$. One may verify that composition preserves equivalence; the image of $\mathrm{Id}_X$ is the identity morphism in the category.

The equivalence $\iota$ above induces an equivalence $HC_\Gamma \to HC^i_\Gamma$.

**Lemma 12.1.2.** *For a definable $X \subseteq \Gamma^w_\infty$, let $C(X) = \{x \in \mathbb{A}^w : \mathrm{val}(x) \in X\}$. Then the inclusion $\iota(X) \subseteq \widehat{C(X)}$ is a homotopy equivalence.*

*Proof.* For $t \in [0,\infty]$ one sets $H_0 = G_m(\mathcal{O})$, $H_\infty = \{1\}$, and for $t > 0$, with $t = \mathrm{val}(a)$, $H_t$ denotes the subgroup $1 + a\mathcal{O}$ of $G_m(\mathcal{O})$. For $x$ in $C(X)$ one denotes by $p(H_t x)$ the the unique $H_t$-translation invariant stably dominated type on $H_t x$. In this way one defines a homotopy $C(X) \times [0,\infty] \to \widehat{C(X)}$ by sending $(x,t)$ to $p(H_t x)$, whose canonical extension $\widehat{C(X)} \times [0,\infty] \to \widehat{C(X)}$ is a deformation retraction with image $\iota(X)$. □

**Theorem 12.1.3.** *The categories $HC_\Gamma$ and $HC_{VF}$ are equivalent by an equivalence respecting the subcategories of definably compact objects.*

To prove Theorem 12.1.3, we introduce a third category $C_2$ whose objects are pairs $(X,\pi)$, with $\pi : X \to X$ a retraction (strongly homotopy equivalent to identity) with $\Gamma$-internal image. A morphism $f : (X,\pi) \to (X',\pi')$ is a continuous definable map $f : X \to X'$ such that $f \circ \pi = \pi' \circ f$. We define a homotopy equivalence relation $\sim_2$ on $\mathrm{Mor}_{C_2}((X,\pi),(X',\pi'))$: $f \sim_2 g$ if there exists a continuous pro-definable $h : X \times I \to X'$, $h_0 = f, h_1 = g$, such that $h_t \circ \pi = \pi' \circ h_t$ for all $t$. Again one checks that this is a congruence and that one can define a quotient category, $HC_2$.

There is an obvious functor $C_2 \to C_{VF}$ forgetting $\pi$, and also a functor $C_2 \to C^i_\Gamma$, mapping $(X,\pi)$ to $\pi(X)$. One checks that the natural maps on morphisms are well-defined and that they induce functors $HC_2 \to HC_{VF}$ and $HC_2 \to HC^i_\Gamma$. To prove the theorem, it suffices therefore to prove, keeping in mind Lemma 12.1.1, that each of these two functors is essentially surjective and fully faithful, and to observe that they restrict to functors on the definably compact objects, essentially surjective on definably compact objects.

(If the categories are viewed as ind-pro-definable, these functors are morphisms of ind-pro-definable objects, but we do not claim that a direct definable equivalence exists.)

**Lemma 12.1.4.** *The functor $HC_2 \to HC_{VF}$ is surjective on objects, and fully faithful.*

*Proof.* Surjectivity on objects is given by Theorem 10.1.1. Let $(X,\pi),(X',\pi') \in \mathrm{Ob}HC_2 = \mathrm{Ob}C_2$. Let $f : X \to X'$ be a morphism of $C_{VF}$. Then the composition $\pi' \circ f \circ \pi$ is homotopy equivalent to $f$, since $\pi \sim \mathrm{Id}_X$ and $\pi' \sim \mathrm{Id}_{X'}$, and is a morphism of $C_2$. This proves surjectivity of $\mathrm{Mor}_{HC_2}((X,\pi),(X',\pi')) \to \mathrm{Mor}_{HC_{VF}}(X,X')$. Injectivity of this map is clear. □

**Lemma 12.1.5.** *The functor $HC_2 \to HC^i_\Gamma$ is essentially surjective and fully faithful.*

*Proof.* To prove essential surjectivity it suffices to consider objects of the form $\iota(X)$, with $X \in \mathrm{Ob}C_\Gamma$. For these, Lemma 12.1.2 does the job.

Let $(X,\pi),(X',\pi') \in \mathrm{Ob}HC_2 = \mathrm{Ob}C_2$. Let $g : \pi(X) \to \pi'(X')$ be a morphism of $C_\Gamma$. Then $\pi' \circ g \circ \pi : X \to X'$ is a morphism of $C_2$. So even $\mathrm{Mor}_{C_2}((X,\pi),(X',\pi')) \to \mathrm{Mor}_{C_\Gamma}(X,X')$ is surjective.

To prove injectivity, suppose $g_, g_2 : X \to X'$ are $C_2$-morphisms, and $h : \pi(X) \times I \to \pi'(X')$ is a homotopy between $g_1|\pi(X)$ and $g_2|\pi'(X')$. We wish to show that $g_1$ and $g_2$ are $C_2$-homotopic. Now for $i = 1,2$, $g_i$ and $\pi' g_i \pi$ have the same image in $\mathrm{Mor}(\pi X, \pi' X')$, and there is a homotopy between $\pi' g_1 \pi$ and $\pi' g_2 \pi$ as remarked before. So we may assume $g_i = \pi' g_i \pi$ for $i = 1,2$. Define $H : X \times I \to X'$ by $H(x,t) = \pi' h(\pi(x),t)$. This is a $C_2$-homotopy between $g_1$ and $g_2$ showing that $g_1$ and $g_2$ have the same class as morphisms in $HC_2$. □

**12.2. Questions on homotopies over imaginary base sets.** Is Theorem 10.1.1 true over an arbitrary base?

Assume $(V,X)$ are given as in Theorem 10.1.1, but over a base $A$ including imaginary elements. A homotopy $h_c$ is definable over additional field parameters

$c$, satisfying the conclusion of Theorem 10.1.1 over $A(c)$. By the uniformity results of §10.7, there exists a $A$-definable set $Q$ such that any parameter $c \in Q$ will do. One can find a definable type $q$, over a finite extension $A'$ of $A$ (i.e. $A' = A(a'), a' \in \mathrm{acl}(A)$). We know that $q = \int_r f$, with $r$ an $A$-definable type on $\Gamma^n$, and $f$ an $A$-definable $r$-germ of a function into $\widehat{Q}$. Define $h(t,v) = \lim_{u \in r} \int_{c \models f(u)} f_c(t,v)$. Then $h(t,v)$ is an $A'$-definable homotopy. It seems that the final image of $h$ is $\Gamma$-parameterized, and has property (5) of Theorem 10.1.1; isotriviality is likely to follow, and separatedness follows since one can take $V$ to be complete.

Similar issues arise when one tries to find a homotopy fixing a prescribed 0-definable element of $\widehat{V}$.

Moreover it is not clear if $h$ can be found over $A$ instead of $\mathrm{acl}(A)$.

## 13. Applications to the topology of Berkovich spaces

**13.1. Berkovich spaces.** Set $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$. Let $F$ be a valued field with $\mathrm{val}(F) \subseteq \mathbb{R}_\infty$, and let $\mathbf{F} = (F, \mathbb{R})$ be viewed as a substructure of a model of ACVF (in the VF and $\Gamma$-sorts). Here $\mathbb{R} = (\mathbb{R}, +)$ is viewed as an ordered abelian group.

Let $V$ be an algebraic variety over $F$, and let $X$ be an $F$-definable subset of the variety $V$. We define the Berkovich space $B_F(X)$ to be the space of types over $\mathbf{F}$ that are almost orthogonal to $\Gamma$. Thus for any $F$-definable function $f$ with values in the $\Gamma$-sort and $a \models p$, we have $f(a) \in \Gamma_\infty(\mathbf{F}) = \mathbb{R}_\infty$. So $f(a)$ does not depend on $a$, and we denote it by $f(p)$. We endow $B_F(X)$ with a topology by defining a pre-basic open set to have the form $\{p \in X \cap U : \mathrm{val}(f)(p) \in W\}$, where $U$ is an affine open subset of $V$, $f$ is regular on $U$, and $W$ is an open subset of $\mathbb{R}_\infty$. A basic open set is a finite intersection of pre-basic ones.

When we wish to consider $q \in B_F(X)$ as a type, rather than a point, we will write it as $q|\mathbf{F}$.

When $V$ is an algebraic variety over $F$, $B_F(V)$ can be identified with the underlying topological space of the Berkovich analytification of $V$; see [9]. When $X$ is a definable subset of $V$, $B_F(X)$ is a semi-algebraic subset of $B_F(V)$ in the sense of [8]; conversely any semi-algebraic subset has this form.

An element of $B_F(X)$ has the form $\mathrm{tp}(a/\mathbf{F})$, where $\mathbf{F}(a)$ is an extension whose value group remains $\mathbb{R}$. To see the relation to stably dominated types, note that if there exists a $\mathbf{F}$-definable stably dominated type $p$ with $p|\mathbf{F} = \mathrm{tp}(a/\mathbf{F})$, then $p$ is unique; in this case the Berkovich point can be directly identified with this element of $\widehat{X}$. If there exists a stably dominated type $p$ defined over a finite Galois extension $F'$ of $F$, $\mathbf{F}' = (F', \mathbb{R})$, with $p|\mathbf{F} = \mathrm{tp}(a/\mathbf{F})$, then the Galois orbit of $p$ is unique; in this case the relation between the Berkovich point and the point of $\widehat{X}$ is similar to the relation between closed points of $\mathrm{Spec}(V)$ and points of $V(F^{alg})$. In general the Berkovich point of view relates to ours in rather the same way that

Grothendieck's schematic points relates to Weil's points of the universal domain. We proceed to make this more explicit.

Let $K$ be a maximally complete algebraically closed field, containing $F$, with value group $\mathbb{R}$, and residue field equal to the algebraic closure of the residue field of $F$. Such a $K$ is unique up to isomorphism over $\mathbf{F}$ by Kaplansky's theorem, and it will be convenient to pick a copy of this field $K$ and denote it $F^{max}$.

We have a restriction map from types over $K$ to types over $\mathbf{F}$. On the other hand we have an injective restriction map from stably dominated types defined over $K$, to types defined over $K$. Composing, we obtain a continuous map from the set of stably dominated types in $X$ defined over $K$ to the set of types over $\mathbf{F}$ on $X$:

$$\pi_X : \widehat{X}(K) \to B_F(X).$$

We shall sometimes omit the subscript when there is no ambiguity.

**Lemma 13.1.1.** *Let $X$ be an $F$-definable subset of an algebraic variety over $F$. The mapping $\pi : \widehat{X}(K) \to B_F(X)$ is surjective.*

*Proof.* If $q$ lies in the image of $\pi$, then $q = \mathrm{tp}(c/\mathbf{F})$ for some $c$ with $\mathrm{tp}(c/K)$ orthogonal to $\Gamma$; hence $\Gamma(\mathbf{F}(c)) \subseteq \Gamma(K(c)) = \Gamma(K) = \Gamma(\mathbf{F})$.

Conversely, suppose $q = \mathrm{tp}(c/\mathbf{F})$ is almost orthogonal to $\Gamma$. Let $L = F(c)^{max}$. Then $\Gamma(\mathbf{F}) = \Gamma(\mathbf{F}(c)) = \Gamma(L)$. The field $K = F^{max}$ embeds into $L$ over $\mathbf{F}$; taking it so embedded, let $p = \mathrm{tp}(c/K)$. Then $p$ is almost orthogonal to $\Gamma$, and $q = p|\mathbf{F}$. Since $K$ is maximally complete, $p$ is orthogonal to $\Gamma$, cf. Theorem 2.8.2. □

The Berkovich points can thus be viewed as a certain quotient of the stably dominated points: Berkovich points of $V$ over $F$ are types over $F$ of elements of $\widehat{V}$. Conversely, provided we restrict attention to fields $F$ with $\Gamma(F)$ archimedean, the stably dominated points can be described in terms of the Berkovich points: let $C_F$ be the category of valued field extensions $F'$ of $F$ with value group $\mathbb{R}$. A point of $\widehat{V}$ is a (proper class) function choosing a point $a_{F'} \in B_{F'}(V)$, for any $F' \in C_F$; such that $a_{F'}$ is functorial in $F'$, i.e. for any valued field embedding $j : F' \to F''$, with induced map $j_* : V(F') \to V(F'')$, we have $j_*(a_{F'}) = a_{F''}$. If $F$ is not maximally complete, not every point of $B_K(V)$ extends to such a functor.

Recall § 3.2, and the remarks on definable topologies there.

**Proposition 13.1.2.** *Let $X$ be an $F$-definable subset of an algebraic variety $V$ over $F$. Let $\pi : \widehat{V}(K) \to B_F(V)$ be the natural map. Then $\pi^{-1}(B_F(X)) = \widehat{X}(K)$, and $\pi : \widehat{X}(K) \to B_F(X)$ is a closed map. Moreover, the following conditions are equivalent:*

1. *$\widehat{X}$ is definably compact*
2. *$X$ is bounded and v+g closed*
3. *$\widehat{X}(K)$ is compact*
4. *$B_F(X)$ is compact*

(5) *$B_F(X)$ is closed in $B_F(V')$, where $V'$ is any complete $F$-variety containing $V$.*

*The natural map $B_{F'}(X) \to B_F(X)$ is also closed, if $F \leq F'$ and $\Gamma(F') \leq \mathbb{R}$. In particular, $B_F(X)$ is closed in $B_F(V)$ iff $B_{F'}(X)$ is closed in $B_{F'}(V)$.*

*Proof.* We first consider the five conditions. The equivalence of (1) and (2) is already known by Proposition 4.2.17. Assume (2). The topology on $\widehat{X}$ is induced from a family of maps into $\Gamma_\infty$. These maps are all bounded on $X$, so they embed $\widehat{X}$ into a product of spaces $[a, \infty] \subset \Gamma_\infty$, hence $\widehat{X}(K)$ into $[a, \infty] \subset \mathbb{R}_\infty$. By Tychonoff's theorem, $\widehat{X}(K)$ is compact, (3). In this case, $\pi$ is a closed map, and we saw it is surjective, so $B_F(X)$ is compact, (4). Now the inclusion $B_F(X) \to B_F(V')$ is continuous, and $B_F(V')$ is Hausdorff, so (4) implies (5).

On the other hand if (1) fails, let $V'$ be some complete variety containing $V$, and let $K = F^{max}$. There exists a $K$-definable type on $\widehat{X}$ with limit point $q$ in $\widehat{V'} \smallsetminus \widehat{X}$. So $\pi(q)$ is in $B_F(V')$ and in the closure of $B_F(X)$, but not in $B_F(X)$. This proves the equivalence of (1-5).

The equality $\pi^{-1}(B_F(X)) = \widehat{X}(K)$ is clear from the definitions. Now the restriction of a closed map $\pi$ to a set of the form $\pi^{-1}(W)$ is always closed, as a map onto $W$. So to prove the closedness property of $\pi$, we may take $X = V$, and moreover by embedding $V$ in a complete variety we may assume $V$ is complete. In this case $X = V$ is v+g-closed and bounded, so $\widehat{X}(K)$ is compact by condition (3). As $B_F(X)$ is Hausdorff, $\pi$ is closed. The proof that $B_{F'}(X) \to B_F(X)$ is also closed is identical, and taking $X = V$ we obtain the statement on the base invariance of the closedness of $X$. We could alternatively use the proof of Lemma 3.4.4. □

**Proposition 13.1.3.** *Assume $X$ and $W$ are $\mathbf{F}$-definable subsets of some algebraic variety over $F$.*

(1) *Let $h_0 : X \to \widehat{W}$ be an $F$-definable function. Then $h_0$ induces functorially a function $\widetilde{h} : B_F(X) \to B_F(W)$ such that $\pi_W \circ h = \widetilde{h} \circ \pi_X \circ i$, with $i : X \to \widehat{X}$ the canonical inclusion.*
(2) *Any continuous $F$-definable function $h : \widehat{X} \to \widehat{W}$ induces a continuous function $\widetilde{h} : B_F(X) \to B_F(W)$ such that $\pi_W \circ h = \widetilde{h} \circ \pi_X$.*
(3) *The same applies if either $X$ or $W$ is a definable subset of $\Gamma_\infty^n$ and we read $B_F(X) = X(\mathbf{F})$, respectively $B_F(W) = W(\mathbf{F})$.*

*Proof.* Define $\widetilde{h} : B_F(X) \to B_F(W)$ as in Lemma 3.7.1 (or in the canonical extension just above it). Namely, let $p \in B_F(X)$. We view $p$ as a type over $\mathbf{F}$, almost orthogonal to $\Gamma$. Say $p|\mathbf{F} = \mathrm{tp}(c/\mathbf{F})$. Let $d \models h(c)|\mathbf{F}(c)$. Since $h(c)$ is stably dominated, $\mathrm{tp}(d/\mathbf{F}(c))$ is almost orthogonal to $\Gamma$, hence so is $\mathrm{tp}(cd/\mathbf{F})$, and thus also $\mathrm{tp}(d/\mathbf{F})$. Let $\widetilde{h}(c) = \mathrm{tp}(d/\mathbf{F}) \in B_F(W)$. Then $\widetilde{h}(c)$ depends only on $\mathrm{tp}(c/M)$, so we can let $\widetilde{h}(p) = F(c)$.

For the second part, let $h_0 = h|X$ be the restriction of $h$ to the simple points. By Lemma 3.7.2, $h$ is the unique continuous extension of $h_0$. Define $\widetilde{h}$ as in (1). Let $\pi_X : \widehat{X}(K) \to B_F(X)$ and $\pi_W : \widehat{W}(K) \to B_W(X)$ be the restriction maps as above. It is clear from the definition that $\widetilde{h}(\pi_X(p)) = \pi_W(h(p))$. (In case $K$ is nontrivially valued, this is also clear from the density of simple points, since $\widetilde{h} \circ \pi_X$ and $\pi_W \circ h$ agree on the simple points of $\widehat{X}(K)$.)

It remains to prove continuity. By the discussion above, $\pi$ is a surjective, closed map. Hence since $\widetilde{h}^{-1}(U) = \pi_X(\pi_X^{-1}(\widetilde{h}^{-1}(U)))$, the continuity of $\widetilde{h}$ follows from that of $\widetilde{h} \circ \pi_X$.

(3) The proof goes through in both cases. □

**Lemma 13.1.4.** *Let $X$ be a $\mathbf{F}$-definable subset of $V \times \Gamma_\infty^n$ with $V$ a variety over $F$.*

(1) *Let $f : X \to Y$ be an $\mathbf{F}$-definable map, $q \in B_F(Y)$, and assume $U$ is an $\mathbf{F}$-definable subset of $X$, and $\widehat{U_b}$ is closed in $\widehat{X_b}$ for any $b \models q|\mathbf{F}$. Then $B_F(U)_q$ is closed in $B_F(X)_q$.*
(2) *Similarly if $g : X \to \Gamma_\infty$ is an $\mathbf{F}$-definable function, and $\widehat{g}|\widehat{X_b}$ is continuous for any $b \models q|\mathbf{F}$, then $B_F(g)$ induces a continuous map on $B_F(X)_q \to \Gamma_\infty$.*
(3) *More generally, if $g : X \to V'$ is an $\mathbf{F}$-definable map into some variety $V'$, and $\widehat{g}|X_b$ is v+g-continuous for any $b \models q|\mathbf{F}$, then $B_F(g)$, by which we mean $B_F$ of the graph of $g$, induces a continuous map $B_F(X)_q : B_F(X)_q \to B_F(Z)$.*

*Proof.* Indeed if $r \in B_F(X)_q \smallsetminus B_F(U)_q$, let $c \models r|\mathbf{F}$, $b = f(c)$. We have $c \in X_b \smallsetminus U_b$, so there exists a definable function $\alpha_b : X_b \to \Gamma_\infty$ and an open neighborhood $E_c$ of $\alpha_b(c)$ such that $\alpha_b^{-1}(E_b) \subset X_b \smallsetminus U_b$. By Lemma 3.4.4, $\alpha_b$ can be taken to be $\mathbf{F}(b)$-definable, and in fact to be a continuous function of the valuations of some $F$-definable regular functions, and elements of $\Gamma(\mathbf{F})$. There exists a $\mathbf{F}$-definable function $\alpha$ on $X$ with $\alpha_b = \alpha|X_b$. Now $\alpha$ separates $r$ from $B_F(U)_q$ on $B_F(X)_q$, showing that $U$ is closed in $B_F(X)_q$.

The statement on continuity (2) follows immediately: if $Z$ is a closed subset of $\Gamma_\infty$, then $g^{-1}(Z) \cap \widehat{X_b}$ is closed in each $\widehat{X_b}$, hence $g^{-1}(Z) \cap B_F(U)_q$ is closed.

The more general statement (3) follows since to show that a map into $B_F(Z)$ is continuous, it suffices to show that the composition with $B_F(s)$ is continuous for any definable, continuous $s : Z' \to \Gamma_\infty$, $Z'$ Zariski open in $Z$. □

We have the induced map $\tilde{f} : B_F(X) \to B_F(Y)$. Let $B_F(X)_q = \tilde{f}^{-1}(q)$, a subspace of $B_F(X)$. Here is a version of Proposition 13.1.3 relative to the base $Y$.

**Lemma 13.1.5.** *Let $X$, $Y$ and $W$ be $\mathbf{F}$-definable subsets of some algebraic variety over $F$. Let $f_X : X \to Y$ and $f_W : W \to Y$ be given v+g-continuous, $\mathbf{F}$-definable maps, and $h : X \to \widehat{W/Y}$ an $\mathbf{F}$-definable map inducing $H : \widehat{X/Y} \to \widehat{W/Y}$.*

*Assume $H|\widehat{X_b}$ is continuous for every $b \in Y$. Then for any $q \in B_F(Y)$, $h$ induces a continuous function $\widetilde{h}_q : B_F(X)_q \to B_F(W)_q$.*

*Proof.* The topology on $B_F(W)_q$ is induced from $B_F(W)$, and this in turn is the coarsest topology such that $B_F(g)$ is continuous for any $v+g$-continuous definable $g : W \to \Gamma_\infty$. Composing with $B_F(g)$, we see that we may assume $W = Y \times \Gamma_\infty$. We have $h : X \to \Gamma_\infty$, inducing $H : \widehat{X/Y} \to \Gamma_\infty$, and assume $H|\widehat{X_b}$ is continuous for $b \in Y$. We have to show that a continuous $\widetilde{h}_q : B_F(X)_q \to \Gamma_\infty$ is induced.

In case the map $\widehat{X} \to \Gamma_\infty$ induced from $h$ is continuous, by Lemma 13.1.3 $\widetilde{h}$ is continuous, and hence the restriction to each fiber $B_F(X)_q$ is continuous.

In general, let $X'$ be the graph of $h$, viewed as an isodefinable subspace of $X \times \Gamma_\infty$. The projection $X' \to W$ is continuous, so a natural, continuous function $B_F(X')_q \to \Gamma_\infty$ is induced, by the above special case. It remains to prove that the projection map $B_F(X')_q \to B_F(X)_q$ is a homeomorphism (with inverse induced by $(x \mapsto (x, f(x)))$. When $q = b \in Y$ is a simple point, this follows from the continuity of $H|\widehat{X_b}$. Hence by Lemma 13.1.4, it is true in general. □

In the Berkovich category, as in §5.3 and throughout the paper, by deformation retraction we mean a strong deformation retraction.

**Corollary 13.1.6.** (1) *Let $X$ be an $\mathbf{F}$-definable subset of some algebraic variety over $F$. Let $h : I \times \widehat{X} \to \widehat{X}$ be an $\mathbf{F}$-definable deformation retraction, with image $h(e_I, \widehat{X}) = Z$. Let $\mathbf{I} = I(\mathbb{R}_\infty)$ and $\mathbf{Z} = Z(\mathbf{F})$. Then $h$ induces a deformation retraction $\widetilde{h} : \mathbf{I} \times B_F(X) \to B_F(X)$ with image $\mathbf{Z}$.*

(2) *Let $X \to Y$ be an $\mathbf{F}$-definable morphism between $\mathbf{F}$-definable subsets of some algebraic variety over $F$. Let $h : I \times \widehat{X/Y} \to \widehat{X/Y}$ be an $\mathbf{F}$-definable deformation retraction satisfying $(*)$, with fibers $h_y$ having image $Z_y$. Let $q \in B_F(Y)$. Then $h$ induces a deformation retraction $\widetilde{h}_q : \mathbf{I} \times B_F(X)_q \to B_F(X)_q$, with image $\mathbf{Z}_q$.*

(3) *Assume in addition there exists a definable $\Upsilon \subseteq \Gamma_\infty^n$ and definable homeomorphisms $\alpha_y : Z_y \to \Upsilon$, given uniformly in $y$. Then $\mathbf{Z}_q \cong \Upsilon$.*

*Proof.* (1) follows from Lemma 13.1.3; the statement on the image is easy to verify. (2) follows similarly from Lemma 13.1.5. For (3), define $\beta : X \to \Upsilon$ by $\beta(x) = \alpha_y(h(0_I, x))$ for $x \in X_y$, $0_I$ being the final point of $I$. Then $\alpha_y \circ \beta(x) = h(0_I, x)$, $\beta(h(t,x)) = \beta(x)$, $\beta(\alpha_y^{-1}(x)) = x$. Applying $B_F$ and restricting to the fiber over $q$ we obtain continuous maps $\beta, \alpha_y^{-1}$ by Lemma 13.1.4; the identities survive, and give the result. □

For our purposes, a $\mathbb{Q}$-tropical structure on a topological space $X$ is a homeomorphism of $X$ with a subspace $S$ of $[0,\infty]^n$ defined as a finite Boolean combination of equalities or inequalities between terms $\sum \alpha_i x_i + c$ with $\alpha_i \in \mathbb{Q}, \alpha_i \geq$

$0, c \in \mathbb{R}$. Since $S$ is definable in $(\mathbb{R}, +, \cdot)$, $X$ is homeomorphic to a finite simplicial complex.

From Theorem 10.1.1 and Corollary 13.1.6 we obtain:

**Theorem 13.1.7.** *Let $X$ be an $F$-definable subset of a quasi-projective algebraic variety $V$ over a valued field $F$ with $\mathrm{val}(F) \subseteq \mathbb{R}_\infty$. There exists a deformation retraction $H : \mathbf{I} \times B_F(X) \to B_F(X)$, whose image $\mathbf{Z}$ has a $\mathbb{Q}$-tropical structure; in particular it is homeomorphic to a finite simplicial complex.*

We next state some functorial properties of the deformation retraction above. Like Theorem 13.1.7, these were proved by Berkovich assuming the base field $F$ is nontrivially valued, and that $X$ and $Y$ can be embedded in proper varieties which admit a pluri-stable model over the ring of integers of $F$. We thank Vladimir Berkovich for suggesting these statements to us.

Whenever we write $B_F(V)$, we assume $\mathrm{val}(F) \leq \mathbb{R}$, allowing the case that $\mathrm{val}(F) = 0$.

**Theorem 13.1.8.** *Let $X$ and $Y$ be quasi-projective algebraic varieties over a valued field $F$ with value group contained in $\mathbb{R}$.*

(1) *There exists a finite separable extension $F'$ of $F$ such that, for any non-Archimedean field $F''$ over $F'$, the canonical map $B_{F''}(X \otimes F'') \to B_{F'}(X \otimes F')$ is a homotopy equivalence. In fact, there exists a deformation retraction of $B_{F'}(X)$ to $S'$ as in Theorem 13.1.7 that extends to a deformation retraction of $B_{F''}(X)$ to $S''$, for which the canonical map $S'' \to S'$ is a homeomorphism.*
(2) *There exists a finite separable extension $F'$ of $F$ such that, for any non-Archimedean field $F''$ over $F'$, the canonical map $B_{F''}(X \times Y) \to B_{F''}(X) \times B_{F''}(Y)$ is a homotopy equivalence.*
(3) *For smooth $X$ and a dense open subset $U$ in $X$, the canonical embedding $B_F(U) \to B_F(X)$ is a homotopy equivalence.*

*Proof.* Let $S$ be the skeleton given in Theorem 10.1.1. According to this theorem, there exists an $F$-definable embedding $S \to \Gamma_\infty^w$, where $w$ is a finite set. Let $F'$ be a finite Galois extension of $F$, such that $\mathrm{Aut}(F^{alg}/F')$ fixes each point of $w$. Then there exists an $F'$-definable bijection $S \to \Gamma_\infty^n$, $n = |w|$. It follows that $S(\mathbf{F}'') = S(\mathbf{F}')$ whenever $\mathbf{F}'' \geq \mathbf{F}$ is a valued field extension with $\Gamma(\mathbf{F}'') = \mathbb{R}$. The image of $S$ in $B_F(X)$ is thus homeomorphic to $S(F')/G$ where $G = \mathrm{Aut}(F'/F)$. The image $S_{F''}$ of $S$ in $B_{F''}(X)$ is homeomorphic to $S = S(\mathbf{F}')$.

Moreover, there exists a finite separable extension $F'$ of $F$, such that $S(F') = S(F^{alg})$. Both of these statements are immediate from the $F$-definable bijection $S \to \Gamma_\infty^w$, where $w$ is a finite set; it suffices to choose $F'$ such that $\mathrm{Aut}(F^{alg}/F')$ fixes $w$ pointwise. Note that the canonical map $\widehat{V}(K) \to B_{F'}(V)$ restricts to an injective map on $S$, since $S(K) \subset S(F')$.

(1) The homotopy of Theorem 10.1.1 is $F$-definable, and so functorial on $F''$-points for any $F'' \geq F$. In particular for any $F \leq F' \leq F''$, the homotopy of $B_{F''}(X)$ is compatible with the homotopy of $B_{F'}(X)$ via the natural map $B_{F''}(X) \to B_{F'}(X)$ (restriction of types). The final image of the homotopies is respectively $S_{F''}$ and $S_{F'}$; we noted that these are homeomorphic images of $S$ and hence homeomorphic via the natural map.

(2) Follows in the same way from Corollary 8.7.4 (which was proved precisely with the present motivation). The deformation retraction $\widehat{X \times Y} \to (S \otimes T)$ induces, over any $F'' \geq F$, a deformation retraction on $B_{F'}(X \times Y)$ whose image is $(S \otimes T)/\mathrm{Aut}(F^{alg}/F'')$. If $F'' \geq F'$, the Galois action is trivial, so the image is canonically homeomorphic to $S \otimes T \cong S \times T$. The canonical map $B_{F''}(X \times Y) \to B_{F''}(X) \times B_{F''}(Y)$ is thus part of a commuting triangle where the other two maps are homotopy equivalences, as in the proof of Corollary 8.7.4, so it is itself a homotopy equivalence.

(3) The third item follows from Remark 11.1.4. □

The following result was previously known when $X$ is a smooth projective curve [2]:

**Theorem 13.1.9.** *Let $X$ be an $F$-definable subset of a quasi-projective algebraic variety $V$ over a valued field $F$ with $\mathrm{val}(F) \subseteq \mathbb{R}_\infty$ and assume $B_F(X)$ is compact. Then there exists a family $(X_i : i \in I)$ of finite simplicial complexes embedded in $B_F(X)$, where $I$ is a directed partially ordered set, such that $X_i$ is a subcomplex of $X_j$ for $i < j$, with deformation retractions $\pi_{i,j} : X_j \to X_i$ for $i < j$, and deformation retractions $\pi_i : B_F(X) \to X_i$ for $i \in I$, satisfying $\pi_{i,j} \circ \pi_j = \pi_i$ for $i < j$, such that the canonical map from $B_F(X)$ to the projective limit of the spaces $X_i$ is a homeomorphism.*

*Proof.* Let the index set $J$ consist of all $F$-definable continuous maps $j : \widehat{X} \to \widehat{X}$, such that for some $F$-definable deformation retraction $H$ as in Theorem 10.1.1, we have $j(x) = H(e_I, x)$. For $j \in J$, let $S_j = j(\widehat{X})$, and $X_j = S_j(\mathbf{F})$. Say that $j_1 \leq j_2$ if $S_{j_1} \subseteq S_{j_2}$. In this case, $j_1|S_{j_2}$ is a deformation retract $S_{j_2} \to S_{j_1}$; let $\pi_{j_1,j_2}$ be the induced map $X_{j_2} \to X_{j_1}$. It is a deformation retraction. This system is directed, i.e. given $j$ and $j'$ there exists $j''$ with $j, j' \leq j''$. To see this, given $j$ and $j'$, let $\alpha_j : S_j \to \Gamma_\infty^w$ be a definable injective map, and let $j''$ belong to a homotopy respecting $\alpha_1, \alpha_2$, cf. Remark 10.1.2 (4). We have a natural surjective map $\pi_j : B_F(X) \to X_j$ for each $j$, induced by the mapping $j$; it satisfies $\pi_{i,j} \circ \pi_j = \pi_i$ for $i < j$ and it is a deformation retraction. This yields a continuous and surjective map from $\theta : B_F(X) \to \varprojlim X_j$. We now show that $\theta$ is injective. Let $p \neq q \in B_F(X)$; view them as types almost orthogonal to $\Gamma$. For any open affine $U$ and regular $f$ on $U$, for some $\alpha$, either $x \notin U$ is in $p$ or $\mathrm{val}(f) = \alpha$ is in $p$; this is because $p$ is almost orthogonal to $\Gamma$. Thus as $p \neq q$, for some open affine $U$ and some regular $f$ on $U$, either $p \in U$ and $q \notin U$, or vice versa, or $p, q \in U$

and for some regular $f$ on $U$, $f(x) = \alpha \in p$, $f(x) = \beta \in q$, with $\alpha \neq \beta$. Let $H$ be as in Theorem 10.1.1 respecting $U$ and $\mathrm{val}(f)$, and let $j$ be a corresponding retract. Then clearly $\pi_j(p) \neq \pi_j(q)$. Thus, $\theta$ is a continuous bijection and by compactness it is a homeomorphism. □

**Remark 13.1.10.** Let $\Sigma$ be (image of) the direct limit of the $X_i$'s in $B_F(X)$. Note that $\Sigma$ contains all rigid points of $B_F(X)$ (that is, images of simple points under the mapping $\pi$ in Lemma 13.1.1): this follows from Theorem 10.1.1, by finding a homotopy to a skeleton $S_x$ fixing a given simple point $x$ of $\widehat{X}$. We are not certain whether $\Sigma$ can be taken to be the whole of $B_F(X)$. But given a stably dominated type $p$ on $X$, letting $S_p = S_x$ for $x \models p$ and averaging the homotopies with image $S_x$ over $x \models p$, we obtain a definable homotopy whose final image is a continuous, definable image of $S_p$. In this way we can express $B_F(X)$ as a direct limit of a system of finite simplicial complexes, with continuous transition maps.

## 13.2. Finitely many homotopy types.

**Theorem 13.2.1.** *Let $X$ and $Y$ be $\mathbf{F}$-definable subsets of algebraic varieties defined over a valued field $F$. Let $f : X \to Y$ be an $\mathbf{F}$-definable morphism that may be factored through a definable injection of $X$ in $Y \times \mathbb{P}^m$ for some $m$ followed by projection to $Y$.*

1. *For $b \in Y$, let $X_b = f^{-1}(b)$. Then there are finitely many possibilities for the homotopy type of $B_{F(b)}(X_b)$, as $b$ runs through $Y$. More generally, let $U \subset X$ be $\mathbf{F}$-definable. Then as $b$ runs through $Y$ there are finitely many possibilities for the homotopy type of the pair $(B_{F(b)}(X_b), B_{F(b)}(X_b \cap U))$. Similarly for other data, such as definable functions into $\Gamma$.*
2. *For any valued field extension $F \leq F'$ with $\Gamma(F') \leq \mathbb{R}$, let $\tilde{f} : B_{F'}(X) \to B_{F'}(Y)$ be the induced map, and $B_{F'}(X)_q = \tilde{f}^{-1}(q)$ for $q \in B_{F'}(Y)$. Then there are only finitely many possibilities for the homotopy type of $B_{F'}(X)_q$ as $q$ runs over $B_{F'}(Y)$ and $F'$ over extensions of $F$. More generally, let $U \subset X$ be $\mathbf{F}$-definable. Then as $q$ runs over $B_{F'}(Y)$ and $F'$ over extensions of $F$ there are finitely many possibilities for the homotopy type of the pair $(B_{F'}(X)_q, B_{F'}(X)_q \cap B_{F'}(U))$. Similarly for other data, such as definable functions into $\Gamma$.*

*Proof.* In the more general statement, we may take $X$ to be a complete variety. We thus assume $X$ is complete.

According to the uniform version of Theorem 10.1.1, Proposition 10.7.1, there exists an $\mathbf{F}$-definable map $W \to Y$ with finite fibers $W(b)$ over $b \in Y$, and uniformly in $b \in Y$ an $\mathbf{F}(b)$-definable homotopy retraction $h_b$ on $X_b$ preserving the given data, with final image $Z_b$, and an $\mathbf{F}(b)$-definable homeomorphism $\phi_b : Z_b \to S_b \subseteq \Gamma_\infty^{W(b)}$. We may find, definably uniformly in $b$, an $\mathbf{F}(b)$-definable subset $T_b \subseteq \Gamma^n_\infty$, an $\mathbf{F}(b)$-definable set $W_!(b)$, and for $w \in W_!(b)$, a definable homeomorphism $\psi_w : Z_b \to T_b$, such that $H_b = \{\psi_{w'}^{-1} \circ \psi_w : w, w' \in W_!(b)\}$

is a group of homeomorphisms of $Z_b$, and $H'_b = \{\psi_{w'} \circ \psi_w^{-1} : w, w' \in W_!(b)\}$ is a group of homeomorphisms of $T_b$. In fact for a fixed $b$, one can pick some $W(b)$-definable homeomorphism $\psi_b$ of $Z_b$ onto a definable subspace of $\Gamma_\infty^n$; let $\Xi_b = \{\psi_w : w \in W_!(b)\}$ be the set of automorphic conjugates of $\psi_b$ over $\mathbf{F}(b)$; and verify that $H_b$ is a finite group, $\Xi_b$ is a principal torsor for $H_b$, and so $H'_b$ is also a finite group (isomorphic to $H_b$). Thus, for a fixed $b$, one can do the construction as stated, obtaining the stated properties. To achieve uniform definability in $b$ we must renounce the fact that $\Xi_b$ are automorphic conjugates, but the other properties are uniformly definable in $b$, hence by compactness and "glueing" we may find $W_!(b)$ and $\Xi_b$ uniformly in $b$, with the required properties. In particular, there exists an $\mathbf{F}$-definable map $W_! \to Y$ with fibers $W_!(b)$ over $b \in Y$.

By stable embeddedness of $\Gamma$, and elimination of imaginaries in $\Gamma$, we may write $T_b = T_{\rho(b)}$ where $\rho : Y \to \Gamma^m$ is a definable function. Let $\Gamma^*$ be an expansion of $\Gamma$ to RCF. Then by Remark 13.2.2 (1), $T_b$ runs through finitely many $\Gamma^*$-definable homeomorphism types as $b$ runs through $Y$. Similarly, the pair $(T_b, H'_b)$ runs through finitely many $\Gamma^*$-definable equivariant homeomorphism types (e.g. we may find an $H'_b$-invariant cellular decomposition of $T_b$ and describe the action combinatorially). In particular, for $b \in Y$, $(T_b(\mathbb{R}), H'_b)$ runs through finitely many homeomorphism types (i.e. isomorphism types of pairs $(U, H)$ with $U$ a topological space, $H$ a finite group acting on $U$ by auto-homeomorphisms).

By Corollary 13.1.6 we have, for $b \in Y$, a deformation retraction of $B_{F(b)}(X_b)$ to $B_{F(b)}(Z_b)$. Pick $w \in W_!(b)$, and let $W^*(b)$ be the set of realizations of $\mathrm{tp}(w/\mathbf{F}(b))$. If $w, w' \in W^*(b)$ then $w' = \sigma(w)$ for some automorphism $\sigma$ fixing $\mathbf{F}(b)$; we may take it to fix $\Gamma$ too. It follows that $\psi_w^{-1} \circ \psi_{w'} = \sigma | Z_b$. Conversely, if $\sigma$ is any automorphism of $W_!(b)$, it may be extended by the identity on $\Gamma$, and it follows that $\psi_{\sigma(w)} = \psi_w \circ \sigma$; so $W^*(b)$ is a torsor of $H^*(b) = \{\psi_w^{-1} \circ \psi_{w'} : w, w' \in W^*(b)\}$, which is a group. Let $H_*(b) = \{\psi_w \circ \psi_{w'}^{-1} : w, w' \in W^*(b)\}$. It follows that $H_*(b)$ is a group, and for any $w \in W^*(b)$, $\psi_w$ induces a bijection $Z_b/H^*(b) \to T_b/H_*(b)$; moreover it is the same bijection, i.e. it does not depend on the choice of $w \in W^*(b)$.

We are interested in the case: $\Gamma(\mathbf{F}(b)) = \Gamma(\mathbf{F}) = \mathbb{R}$. In this case, since $H^*(b)$ acts by automorphisms over $\mathbf{F}(b)$, two $H^*(b)$-conjugate elements of $Z_b$ have the same image in $B_{F(b)}(X_b)$. On the other hand two non-conjugate elements have distinct images in $T_b/H_*(b)$, and so cannot have the same image in $B_{F(b)}(X_b)$. It follows that $B_{F(b)}(Z_b)$, $Z_b(\mathbf{F}(b))/H^*(b)$ and $T_b(\mathbb{R})/H_*(b)$ are canonically isomorphic. By compactness and definable compactness considerations one deduces that these isomorphisms between $B_{F(b)}(Z_b)$, $Z_b(\mathbf{F}(b))/H^*(b)$ and $T_b(\mathbb{R})/H_*(b)$ are in fact homeomorphisms. It is only for this reason that we made $X$ to be complete in the beginning of the proof.

The number of possibilities for $H_*(b)$ is finite and bounded, since $H'(b)$ is a group of finite size, bounded independently of $b$, and $H_*(b)$ is a subgroup of

$H'(b)$. Since the number of equivariant homeomorphism types of $(T_b(\mathbb{R}), H'(b))$ is bounded, we are done with the first statement in (1).

With the help of Corollary 13.1.6, this proof goes through for non-simple Berkovich points too. Let $q \in B_F(Y)$, and view it as a type over $\mathbf{F}$. By Corollary 13.1.6 (2), $B_F(X)_q$ has the homotopy type of $\mathbf{Z}_q$. Let $b \models q$, pick $w \in W_!(b)$ and let notation be as above. Let $b' = (b, w)$ and let $q' = \mathrm{tp}(b, w/\mathbf{F})$. Let $X' = X \times_Y W_!$. By Corollary 13.1.6 (2) applied to the pullback of the retraction $I \times \widehat{X/Y} \to \widehat{X/Y}$ to $\widehat{X'/W_!}$, $B_F(X')_{q'}$ retracts to a space $\mathbf{Z}_{q'}$ which is homeomorphic to $T_b(\mathbb{R})$. By the same reasoning as above, it follows that $\mathbf{Z}_q$ is homeomorphic to $\mathbf{Z}_{q'}$ *modulo* a certain subgroup $H^*(b)$ of $H(b)$, and also homeomorphic to $T_b$ modulo $H_*(b)$ for a certain subgroup of $H'(b)$, so again the number of possibilities is bounded. This holds uniformly when $F$ is replaced by any valued field extension, and the first statement in (2) follows.

The proof goes through directly to provide the generalization to pairs and $\Gamma$-valued functions of (1) and (2). $\square$

**Remarks 13.2.2.** (1) In the expansion of $\Gamma$ to a real closed field, definable subsets of $\Gamma_\infty^n$ are locally contractible and definably compact subsets of $\Gamma_\infty^n$ admit a definable triangulation, compatible with any given definable partition into finitely many subsets. By taking the closure in case the sets are not compact, it follows that given a definable family of semi-algebraic subsets of $\mathbb{R}_\infty^n$, there exist a finite number of rational polytopes (with some faces missing), such that each member of the family is homeomorphic to at least one such polytope. In particular the number of definable homotopy types is finite. In fact it is known that the number of definable homeomorphism types is finite. See [7], [29].

(2) Eleftheriou has shown [10] that there exist abelian groups interpretable in $\mathrm{Th}(\mathbb{Q}, +, <)$ that cannot be definably, homeomorphically embedded in affine space within DOAG. By Proposition 6.3.6, the skeleta of abelian varieties can be so embedded. It would be good to bring out the additional structure they have that ensures this embedding.

### 13.3. Tame topological properties for $B_F(X)$.

**Theorem 13.3.1** (Local contractibility)**.** *Let $X$ be an $F$-definable subset of an algebraic variety $V$ over a valued field $F$ with* $\mathrm{val}(F) \subseteq \mathbb{R}_\infty$*. The space $B_F(X)$ is locally contractible.*

*Proof.* We may assume $V$ is affine. Since the topology of $B_F(X)$ is generated by open subsets of the form $B_F(X')$ with $X'$ definable in $X$, it is enough to prove that every point $x$ of $B_F(X)$ admits a contractible neighborhood. By Theorem 10.1.1 and Corollary 13.1.6, there exists a strong homotopy retraction $H : I \times B_F(X) \to B_F(X)$ with image a subset $\Upsilon$ which is homeomorphic to a semi-algebraic subset of some $\mathbb{R}^n$. Denote by $\varrho$ the retraction $B_F(X) \to \Upsilon$. By

(4) in Theorem 10.1.1 one may assume that $\varrho(H(t,x)) = \varrho(x)$ for every $t$ and $x$. Recall that any semi-algebraic subset $Z$ of $\mathbb{R}^n$ is locally contractible: one may assume $Z$ is bounded, then its closure $\overline{Z}$ is compact and semi-algebraic and the statement follows from the existence of triangulations of $\overline{Z}$ compatible with the inclusion $Z \hookrightarrow \overline{Z}$ and having any given point of $Z$ as vertex. Is is thus possible to pick a contractible neighborhood $U$ of $\varrho(x)$ in $\Upsilon$. Since the set $\varrho^{-1}(U)$ is invariant by the homotopy $H$, it retracts to $U$, hence is contractible. □

**Remark 13.3.2.** Berkovich proved in [4] and [5] local contractibility of smooth non-archimedean analytic spaces. His proof uses de Jong's results on alterations.

Let us give another application of our results, in the spirit of a result of Poineau, [25] Théorème 2, cf. also Abbes and Saito [1] 5.1.

**Theorem 13.3.3.** *Let $X$ be an $F$-definable subset of a quasi-projective algebraic variety over a valued field $F$ with $\mathrm{val}(F) \subseteq \mathbb{R}_\infty$ and let $G : X \to \Gamma_\infty$ be a definable map. Consider the corresponding map $\mathbf{G} : B_F(X) \to \mathbb{R}_\infty$. Then there is a finite partition of $\mathbb{R}_\infty$ into intervals such that the fibers of $\mathbf{G}$ over each interval have the same homotopy type. Also, if one sets $B_F(X)_{\geq\varepsilon}$ to be the preimage of $[\varepsilon, \infty]$, there exists a finite partition of $\mathbb{R}_\infty$ into intervals such that for each interval $I$ the inclusion $B_F(X)_{\geq\varepsilon} \to B_F(X)_{\geq\varepsilon'}$, for $\varepsilon > \varepsilon'$ both in $I$, is a homotopy equivalence.*

*Proof.* Consider a strong homotopy retraction of $\widehat{X}$ leaving the fibers of $G$ invariant, as provided by Theorem 10.1.1. By Corollary 13.1.6 it induces a retraction $\varrho$ of $B_F(X)$ onto a subset $\Upsilon$ such that there exists a homeomorphism $h : \Upsilon \to S$ with $S$ a semi-algebraic subset of some $\mathbb{R}^n$. By construction $\mathbf{G}$ factors as $\mathbf{G} = g \circ \varrho$ with $g$ a function $S \to \mathbb{R}_\infty$. Furthermore, we may assume that $g' := h^{-1} \circ g$ is a semi-algebraic function $S$. Thus, it is enough to prove that there is a finite partition of $\mathbb{R}_\infty$ into intervals such that the fibers of $g'$ over each interval have the same homotopy type and that if $S_{\geq\varepsilon}$ is the locus of $g' \geq \varepsilon$, there exists a finite partition of $\mathbb{R}_\infty$ into intervals such that for each interval $I$ the inclusion $S_{\geq\varepsilon} \to S_{\geq\varepsilon'}$, for $\varepsilon > \varepsilon'$ both in $I$, is a homotopy equivalence. But such statements are well-known in o-minimal geometry, cf., e.g., [7] Theorem 5.22. □

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Department of Mathematics, The Hebrew University, Jerusalem, Israel
*E-mail address*: ehud@math.huji.ac.il

Institut de Mathématiques de Jussieu, UMR 7586 du CNRS,Université Pierre et Marie Curie, Paris, France
*E-mail address*: loeser@math.jussieu.fr