Thursday, May 31.

Definition of angle and symmetry. An angle is defined as the union of two rays with a common initial point. If congruence presented as a 6-place relation on points $\angle ABC \cong \angle DEF$, additional axioms are implied, saying that the angles ABC and CBA are congruent; also that ABC, ABC' are congruent, if $B * C * C'$.

Hartshorne Prop. 9.2 = Hilbert Theorem 12 (p. 11): Supplements to congruent angles are congruent. (proof on May 28).

Corollary (Euclid I-5): equilateral triangles have congruent angles opposite the congruent sides.

Converse to 9.2: Congruence to supplementary angles implies straightness of line.

Discussion of the term "addition of angles". (We will not formally define addition of angles, nor inequality of angles until we have the theory of rigid motions.)

Proof of Hartshorne Prop. 9.4= Hilbert Theorem 14 (where however the order condition is omitted.)

The following excercises are in a Hilbert plane. For the first one, you can either use Prop. 9.2 above, or else follow the hint below to prove it directly.

Excercise 0.1. *(=Hilbert theorem 13.) Assume $\angle ABC \cong \angle A'B'C'$, and let D lie in the interior of the angle $\angle ABC$. . Then, there exists a point D' in the interior of the angle $\angle A'B'C'$ such that $\angle ABD = \angle A'B'D'$ and $\angle CBD = \angle C'B'D'$.*

(Hint: we may assume $AB \cong A'B'$ and $BC \cong B'C'$. (explain why.) Show that the line AC intersects the line BD in some point; we may assume this point is D. Show that $C * D * A$, and find D' with $C' * D' * A'$ and such that $CD \cong C'D'$. Show that $DA \cong D'A'$, and that this point D' satisfies the conclusion.)

Excercise 0.2 ("subtraction of angles".)**.** *Let $\angle ABC \cong \angle A'B'C'$, and let D be an interior point of $\angle ABC$, and D' an interior point of $\angle A'B'C'$, such that $\angle ABD \cong \angle A'B'D'$. Show that $\angle CBD = \angle C'B'D'$.*

Excercise 0.3. *Let D be a point outside the triangle $\triangle BAC$, but in the interior of the angle $\angle BAC$. Show that B is in the interior of $\angle ACD$, and A is in the interior of $\angle CDB$.*

Excercise 0.4. *Go through Euclid, Book I, Proposition 7 and justify the proof.*

Excercise 0.5 (extra)**.** *Show that any two right angles are congruent.* This is an axiom in Euclid, but Hilbert points out it can be proved. See Theorem 15 there. There is also a proof in Hartshorne, but it uses the theory of ordering on angles which we have not yet developed.

0.1. **Rigid motions.** We defined rigid motions. (Hartshorne, p. 149.)

By an *oriented ray* we mean a ray r, together with a choice of one of the sides of the line through r.

Excercise 0.6. *Let ϕ be a rigid motion. Then $\phi(r)$ is a ray; and if S is a side of the line containing r, then $\phi(S)$ is a side of of the line containing $\phi(r)$.*

We discussed two properties:

ERM: For any two oriented rays r, r' there exists a rigid motion ϕ with $\phi(r) = r'$.

EURM: For any two oriented rays r, r' there exists a unique rigid motion ϕ with $\phi(r) = r'$.

Remark 0.2. The set of rigid motions forms a *group* G under composition. In the language of group theory, ERM says that G *acts transitively* on the set of oriented rays; EURM says that G acts *sharply transitively.*

Excercise 0.7. *An* affine transformation *is a nonconstant affine map $F \to F$, i.e. a map $x \mapsto ax + b$ with $a \neq 0$. Show that the affine transformations of a field F are sharply transitive on pairs of distinct points. In other words, if $a \neq b \in F$ and $c \neq d \in F$, show that there exists a unique affine transformation $\phi : F \to F$ with $\phi(a) = c, \phi(b) = d$. (Hint: first take $a = 0, b = 1$.)*