## 1. The hyperbolic plane

Let $\Pi$ be a Hilbert plane, $\Lambda$ the set of lines of $\Pi$; let $I$ be the incidence relation, i.e. the set of incident pairs $(a, b)$ with $a \in \Pi, b \in \Lambda$. Let $I^+$ be the set of directed rays of the plane.

**Exercise 1.1.** *Let $G$ be a group of rigid motions of a plane $\Pi$. Explain the meaning of $g(e)$, where $e \in I^+$. We say that $G$ acts sharply transitively on $I^+$ if for any $e, e' \in I^+$ there exists a unique $g \in G$ with $g(e) = e'$. Show that this is equivalent to the conjunction of:*

1. *$G$ acts transitively on $I$. In other words for any $i, i' \in I$ there exists $g \in G$ with $g(i) = i'$.*
2. *For some line $l$, and $p \in l$, there exists $g \in G$ fixing $p$ and exchanging the two rays of $l$; and another $g' \in G$ fixing each point of $l$, and exchanging the sides of $l$.*
3. *For some line $l$, the only $g \in G$ fixing the points of $l$ and the sides of $l$ is the identity.*

**Remark** Let $G_l = \{g \in G : g(l) = l\}$. In (3) we are only given information about $G_l$ for ONE line $l$. However, from (1) we can deduce that for any other line $l$ there exists $h \in G$ with $h(l') = l$. Now we have a bijection $G_l \to G_{l'}$:

$$g \mapsto h^{-1}gh$$

(check!). Also, $g$ fixes the sides of $l$ iff $h^{-1}gh$ fixes the sides of $l'$ (check!.) Using this we see that (3) is true for ANY line $l'$. A similar argument applies to (2).

Let $V$ be a finite dimensional vector space over the field $\mathbb{R}$. A *nondegenerate symmetric bilinear form* is a map $b : V \times V \to \mathbb{R}$ such that (writing simply $(u, v)$ for $b(u, v)$)):

1) $(u, v) = (v, u)$
2) $(\alpha u + \beta u', v) = \alpha(u, v) + \beta(u', v)$ for $\alpha, \beta \in \mathbb{R}, u, u', v \in V$.
3) If $v \in V$ and $(\forall u \in V)((v, u) = 0)$ then $v = 0$.

Any bilinear form on $\mathbb{R}^n$ can be expressed as $(u, v) =< u, Mv >$, where $<, >$ is the standard inner product, and $M$ is a symmetric, invertible matrix.

We say that $u \perp v$ (in the sense of $b$) if $(u, v) = 0$.

An *isometry* of $(V, b)$ is an invertible linear transformation $T : V \to V$ such that $b(Tu, Tv) = b(u, v)$.

Recall from linear algebra the following theorem (Sylvester-Witt).

**Theorem 1.2.** *There exists a basis $v_1, \dots, v_n$ of $V$ such that:*
*(i) $v_i \perp v_j$ if $i \neq j$ (i.e. $(v_i, v_j) = 0$.)*
*(ii) $(v_i, v_i) = \pm 1$.*
*In fact any $v_1, \dots, v_m$ satisfying (i),(ii) can be completed to a basis satisfying (i),(ii).*

If we re-arrange the basis, we can assume the elements with $(v_i, v_i) = -1$ come before the elements with $(v_i, v_i) = 1$. Such a basis is called orthonormal of type $- - \dots - + \dots +$.

The most familiar case is of a (Euclidean) *inner product*, or type $+ \dots +$.

When $\dim(V) = 3$ we have two possibilities of interest: (The other two differ by a sign.)

(Type +++) $V_{euclid}$: $\mathbb{R}^3$ with the usual inner product: $< (x, y, t), (x', y', t') >= xx' + yy' + tt'$.
(Type - -+) $V_{hyperbolic}$: $\mathbb{R}^3$ with the bilinear map $((x, y, t), (x', y', t'))_{hyp} = -xx' - yy' + tt'$

We will write $(u, v)$ for $(u, v)_{hyp}$, for short.

The group of isometries is denoted $O(3)$ in the case $+ + +$, and $O(2, 1)$ in the case $- - +$. The group of all invertible linear transformations of $\mathbb{R}^3$ is denoted $GL(3)$.

**Exercise 1.3.** *Show:*

1. *$GL(3)$ is sharply transitive on the set of bases of $V$.*
2. *$O(2, 1)$ is sharply transitive on the set of orthonormal $-, -, +$- bases of $V_{hyperbolic}$.*

Hint for (2): Let $(v_1, v_2, v_3)$ and $(u_1, u_2, u_3)$ be bases of this type. Using (1), find the linear transformation $T$ with $T(u_i) = v_i$. Show that $T$ is an isometry. It is unique even as a linear transformation.

Let

$$H_\pm = \{v \in V_{hyperbolic} : (v, v) = 1\} = \{(x, y, t) \in \mathbb{R}^3 : t^2 = 1 + x^2 + y^2\}$$
$$H = \{(x, y, t) \in H_\pm : t > 0\}$$

Define a *line* on $H$ to be a nonempty set of the form $H \cap U$, where $U$ is a a linear subspace $U$ of $V$ of dimension 2.

**Exercise 1.4.** *Show that $H$ is a an incidence plane.*

**Exercise 1.5.** *Let $\pi_1(x, y, t) = (x/t, y/t)$. Show that $\pi_1$ defines a bijection between $H$ and the open disk $\bigcirc = \{(x, y) : x^2 + y^2 < 1\}$. Describe the sets $\pi_1(l)$ where $l$ is a line.*

**Description of the image of the lines .** These are precisely the nonempty intersections of the usual Euclidean lines on the $x, y$-plane with the open disk $\bigcirc$. Indeed a line $l$ of $H$ has the form $l \cap U$, where $U$ is a 2-dimensional subspace of $\mathbb{R}^3$. Let $s = s(U) := \{(x, y) : (x, y, 1) \in U\}$. Then $s(U)$ is a line in the $(x, y)$-plane. (It is essentially the intersection of the plane $U$ with the plane $t = 1$.) If $(x, y, t) \in U$ then $(x/t, y/t, 1) = t^{-1}(x, y, t) \in U$, so $\pi_1(x, y, t) \in s(U)$. Conversely if $(x, y) \in s(U)$ then $(tx, ty, t) \in U$ for any $t$. If also $(x, y) \in \bigcirc$, taking $t = (1 - x^2 - y^2)^{-1/2}$ we see that $(tx, ty, t) \in H$ also, so $(tx, ty, t) \in l$ and $(x, y) \in \pi_1(l)$. Thus $\pi_1(U \cap H) = s \cap \bigcirc$. Conversely, if $s$ is a given line intersecting $\bigcirc$, let $U$ be the subspace of $\mathbb{R}^3$ spanned by $s \times \{1\}$, and check that $\pi_1(U \cap H) = s \cap \bigcirc$.

Define a betweenness relation on $H$ by: $A * B * C$ if $\pi_1(A) * \pi_1(B) * \pi_1(C)$.

**Exercise 1.6.** *Show that the betweenness axioms B1-B4 hold.*

**Exercise 1.7.** *If $g \in O(2, 1)$ then $g(H_\pm) = H_\pm$. We have $g(H) = H$ or $g(H) = H_-$, where $H_- = \{(x, y, t) \in H_\pm : t < 0\}$.*

*Proof.* $g$ preserves the bilinear map $(u, v)$; so if $(u, u) = 1$ then $(g(u), g(u)) = 1$. This shows that $H_\pm$ is preserved by $g$.

We now show that $g(H) \subseteq H$ or $g(H) \subseteq H_-$.

This can be done either algebraically or using topology; we give both proofs.

*Topological proof:* The map $\pi_1^{-1} : \bigcirc \to H$ is a homeomorphism, so $H$ is connected. Thus $g(H)$ is connected. Since $H, H_-$ are disjoint and open, $g(H)$ must be contained entirely in one of them.

*Algebraic proof:* First check that $(a, b) > 0$ if $a, b$ are both in $H$ or both in $H_-$, but $(a, b) < 0$ if they are in opposite parts. (If $a = (0, 0, 1)$ this is very easy. But by transitivity on points, this case suffices.) Now suppose $a \in H$ and $g(a) \in H$. Let $b \in H$. Then $(a, b) > 0$, so $(g(a), g(b)) > 0$ since the form is preserved; hence $g(b) \in H$. This shows that if $g(a) \in H$ for one $a$, then $g(b) \in H$ for all $b$. Hence $g(H) \subseteq H$ or $g(H) \subseteq H_-$.

Similarly, $g^{-1}(H) \subseteq H$ or $g^{-1}(H) \subseteq H_-$. Applying $g$, we have $H \subseteq g(H)$ or $H \subseteq g(H_-)$. Since $H, H_-$ are disjoint it follows that $g(H) = H$ or $g(H) = H_-$. □

Let $O^+(2, 1)$ denote the group of elements $g \in O(2, 1)$ with $g(H) = H$. So if $g \in O(2, 1)$ and $g(p) = p'$ for some $p, p\ inH$ then $g \in O^+(2, 1)$.

**Exercise 1.8.** (1) *For any $p, p' \in H$ there exists $g \in O(2, 1)$ with $g(p) = p'$. (So $O^+(2, 1)$ is transitive on $H$.)*

(2) *For* $0 \neq u \in H$*, let* $u^{\perp} = \{v \in V_{hyperbolic} : u \perp v\}$*. Show that* $-(,)$ *is an inner product on* $u^{\perp}$*, i.e. if* $0 \neq v \in u^{\perp}$ *then* $(v, v) < 0$.

(3) $O^{+}(2, 1)$ *is transitive on the set of pairs* $(p, l)$ *with* $p$ *a point of* $H$*,* $l$ *a line of* $H$*, and* $p \in l$.

Hints: (1) follows from Theorem 1.2 and Exercise 1.3(2).

(2) Check this for the point $(0, 0, 1)$. Then explain why this suffices (using (1)).

(3) Any line has the form $l = H \cap U$ with $U$ a subspace of $V$ of dimension 2. Given $l$ and a point $p \in l$, find $u \in U$ with $u \neq 0$ and $p \perp v$. Show that one can take $u$ with $(u, u) = -1$. Find $w$ with $u, w, p$ a $-, -, +$ - orthonormal basis. Use Exercise 1.3.

**Exercise 1.9.** *Show that (2,3) of Exercise 1.1 hold.*

(Hint: take $p = (0, 0, 1)$, and use reflections.)

**Corollary 1.10.** *Define congruence of segments and of angles by:* $x \cong y$ *iff there exists* $g \in O^{+}(2, 1)$ *with* $g(x) = y$*. Then the congruence axioms C1-C6 hold.*

**Exercise 1.11.** *Show that given a line* $l$ *and a point* $p$ *not on* $l$*, there exist infinitely many lines* $l'$ *through* $p$*, parallel to* $l$*.*

Hint: use Exercise 1.5.

**Exercise 1.12.** *Show that the matrix* $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & \sqrt{3} \\ 0 & \sqrt{3} & 2 \end{pmatrix}$ *is in* $O^{+}(2, 1)$*. Let* $p_0 = (0, 0, 1)$, $p_1 = Tp_0, \ldots, p_{n+1} = Tp_n$*. Show that the the coordinates of* $p_n$ *grow exponentially. But* $[p_n, p_{n+1}] \cong [p_0, p_1]$*.*