Let \( \Pi \) be an incidence geometry, together with a betweenness relation (written \( x \ast y \ast z \)), satisfying the incidence axioms and the betweenness axioms B1-B4.

We can then define a ray, and a side of a line. Let \( Y \) be the set of directed rays. Let \( H \) be a group of bijections \( f : \Pi \to \Pi \) preserving \( \ast \); i.e. \( x \ast y \ast z \iff f(x) \ast f(y) \ast f(z) \).

We showed that if, in addition, we are given two relations \( \equiv \) (on segments and on angles) satisfying C1-C6, then EURM holds. Here we discuss the converse.

Assume: (*) For any \( e, e' \in Y \) there exists a unique \( f \in H \) with \( f(e) = e' \).

Recall that a segment \( AB \) is the set of points between \( A \) and \( B \), together with \( A \) and \( B \). Also an angle is the union of two rays with a common initial point.

**Exercise 0.1.** Show that if \( h \in H \) and \( AB \) is a segment, then \( h(AB) = CD \) where \( h(A) = C, h(B) = D \). Show that \( h \) takes angles to angles. Define congruence, for segments and for angles, by:

\[ X \cong X' \text{ iff there exists } h \in H \text{ with } h(X) = X'. \]

Show that this is an equivalence relation.

**Exercise 0.2.** Prove the existence part of C1.

Hint: Given a ray \( r \) with initial point \( P \), and given a segment \( AB \), find \( f \in H \) with \( f(AB) = r \) (why does \( f \) exist?). Consider \( f(B) \).

**Exercise 0.3.** Prove the uniqueness in C1.

Hint: If uniqueness in C1 fails, show that there exist points \( A, B, C \) with \( A \ast B \ast C \) and \( AB \equiv AC \). By definition there exists \( f \in H \) with \( f(AB) = AC \). Let \( f^2 \) be the composition \( f \circ f \). Show that \( A \ast C \ast f(C) \), and conclude that \( f^2(B) \neq B \). Show that \( f^2 \) preserves the ray \( AB \), and also the sides of the line \( AB \). Conclude that \( f^2 = Id \), and draw a contradiction.

**Exercise 0.4.** Prove C3.

Hint: we are given \( A \ast B \ast C, D \ast E \ast F \), with \( AB \equiv DE, BC \equiv EF \). Let \( f \in H \), \( f(BC) = AB \). Let \( C' = f(F) \). Use the uniqueness in C1 to show that \( C' = C \). Conclude that \( AC \equiv DF \).

**Exercise 0.5.** Prove the existence in C4.

**Exercise 0.6.** Prove the uniqueness in C4.

(This is more direct than the uniqueness in C1.)

**Exercise 0.7.** prove C6.

Hint: We are given \( \triangle ABC \) and \( \triangle DEF \) with \( \angle ABC \cong \angle DEF \), and \( AB \equiv DE \) , and \( AC \cong DF \).

Find \( f \in H \) with \( f(\overrightarrow{BC}) = \overrightarrow{EF} \), and such that \( f(F) \) lies on the same side as \( A \) of the line \( \overrightarrow{BC} \). Show using 0.6 that \( f(\overrightarrow{ED}) = \overrightarrow{AB} \). Show using 0.3 that \( f(D) = B \) and \( f(F) = C \). Conclude that \( \triangle ABC \cong \triangle DEF \).

**Exercise 0.8.** Define when \( \angle ABC \prec \angle DEF \), and show that it is a linear ordering.

Hint: we discussed in class asymmetry and transitivity; fill in the details. To show the trichotomy law, let \( \angle ABC, \angle DEF \) be given. We have to show that \( \angle ABC \cong \angle DEF \) or \( \angle ABC \prec \angle DEF \) or \( \angle DEF \prec \angle ABC \). Explain why we may assume that \( AB = DE \), and that \( F \) lies on the same side of \( AB \) as \( C \). With those assumptions, if \( F \) is in the interior of \( \angle ABC \), then \( \angle DEF \prec \angle ABC \). If \( F \) lies on the ray \( BC \), show that \( \angle ABC \cong \angle DEF \). Otherwise show that \( C \) lies in the interior of \( \angle ABF \).