Let Π be an incidence geometry, together with a betweenness relation (written $x * y * z$), satisfying the incidence axioms and the betweenness axioms B1-B4.

We can then define a *ray*, and a *side of a line*. Let Y be the set of directed rays. Let H be a group of bijections $f : \Pi \to \Pi$ preserving $*$; i.e. $x * y * z \iff f(x) * f(y) * f(z)$.

We showed that if, in addition, we are given two relations $\cong$ (on segments and on angles) satisfying C1-C6, then EURM holds. Here we discuss the converse.

Assume: (*) For any $e, e' \in Y$ there exists a unique $f \in H$ with $f(e) = e'$.

Recall that a *segment* AB is the set of points between A and B, together with A and B. Also an *angle* is the union of two rays with a common initial point.

Excercise 0.1. *Show that if $h \in H$ and AB is a segment, then $h(AB) = CD$ where $h(A) = C, h(B) = D$. Show that h takes angles to angles. Define congruence, for segments and for angles, by:*

$X \cong X'$ iff there exists $h \in H$ with $h(X) = X'$.

Show that this is an equivalence relation.

Excercise 0.2. *Prove the existence part of C1.*

Hint: Given a ray r with initial point P, and given a segment AB, find $f \in H$ with $f(\overrightarrow{AB}) = r$ (why does f exist?). Consider $f(B)$.

Excercise 0.3. *Prove the uniqueness in C1.*

Hint: If uniqueness in C1 fails, show that there exist points A, B, C with $A * B * C$ and $AB \cong AC$. By definition there exists $f \in H$ with $f(AB) = AC$. Let f^2 be the composition $f \circ f$. Show that $A * C * f(C)$, and conclude that $f^2(B) \neq B$. Show that f^2 preserves the ray $\overrightarrow{AB}$, and also the sides of the line $\overleftrightarrow{AB}$. Conclude that $f^2 = Id$, and draw a contradiction.

Excercise 0.4. *Prove C3.*

Hint: we are given $A*B*C, D*E*F$, with $AB \cong DE, BC \cong EF$. Let $f \in H$, $f(BC) = AB$. Let $C' = f(F)$. Use the uniqueness in C1 to show that $C' = C$. Conclude that $AC \cong DF$.

Excercise 0.5. *Prove the existence in C4.*

Excercise 0.6. *Prove the uniqueness in C4.*

(This is more direct than the uniqueness in C1.)

Excercise 0.7. *prove C6.*

Hint: We are given $\triangle ABC$ and $\triangle DEF$ with $\angle ABC \cong \angle DEF$, and $AB \cong DE$, and $AC \cong DF$.

Find $f \in H$ with $f(\overrightarrow{EF}) = \overrightarrow{BC}$, and such that $f(F)$ lies on the same side as A of the line $\overleftrightarrow{BC}$. Show using 0.6 that $f(\overrightarrow{ED}) = \overrightarrow{AB}$. Show using 0.3 that $f(D) = B$ and $f(F) = C$. Conclude that $\triangle ABC \cong \triangle DEF$.

Excercise 0.8. *Define when $\angle ABC < \angle DEF$, and show that it is a linear ordering.*

Hint: we discussed in class asymmetry and transitivity; fill in the details. To show the trichotomy law, let $\angle ABC, \angle DEF$ be given. We have to show that $\angle ABC \cong \angle DEF$ or $\angle ABC < \angle DEF$ or $\angle DEF < \angle ABC$. Explain why we may assume that $AB = DE$, and that F lies on the same side of $\overleftrightarrow{AB}$ as C. With these assumptions, if F is in the interior of $\angle ABC$, then $\angle DEF < \angle ABC$. If F lies on the ray BC, show that $\angle ABC \cong \angle DEF$. Otherwise show that C lies in the interior of $\angle ABF$.