Ex. 1.1, (I2).

$$(\forall z)L(z) \implies (\exists x)(\exists y)(I(x,z) \wedge I(y,z) \wedge x \neq y)$$

Ex. 2.2 (1) It is easy to verify that a function of the form $g(x) = ax + b$ is affine, by computing the value of both sides of (3). In the other direction, let $f : F \to F$ be affine. Let $b = f(0), a = f(1) - b$. Let $g(x) = ax + b$. We have to show that $f = g$. In equation (3), substitute $x = 1, y = 0, \beta = 1 - \alpha, \gamma = 0$ to obtain:

$$f(\alpha) = \alpha f(1) + \beta f(0) = \alpha(a + b) + (1 - \alpha)b = a\alpha + b = g(\alpha)$$

Another proof.
(a) If h is affine and $h(0) = 0$ then h is linear.
(b) If f, g are affine then $f - g$ is linear.
(c) If h is linear and $h(1) = 0$, then $h = 0$.
Using (b) and (c) it follows that above, $h = f - g$ is 0, so $f = g$.

(3) Let $f : F \to F^2$ be a nonconstant affine function. Write $f(x) = (f_1(x), f_2(x))$. Then f_1, f_2 are also affine. Indeed f_i is the composition of f with the projectin $F^2 \to F$, $(x_1, x_2) \mapsto x_i$, which is affine. Hence by (1), for $i = 1, 2$ we have: $f_i(x) = a_i x + b_i$. So $a_2 f_1(x) - a_1 f_2(x) - d = 0$, where $d = a_2 b_1 - a_1 b_2$. Let l be the line with equation: $a_2 x_1 - a_1 x_2 - d = 0$ (note that a_1, a_2 can't both be zero.) Then we have shown that $f(x) \in l$. Conversely, given $(c_1, c_2) \in l$, we have to find x with $f(x) = (c_1, c_2)$. Say $a_1 \neq 0$. Find a such that $f_1(a) = c_1$. Check that $f_2(a) = c_2$. This shows that the image of f is the line l.

Conversely, let l be a line, say with equation $ax_1 + bx_2 + c = 0$. Then not both a, b are zero. Say $b \neq 0$; then we can assume $b = 1$. Let $f(x) = (x, -c - ax)$; check that this works.

(4) Let f^{-1} denote the inverse function to f (viewed as a function $F \to l$.) Then $f^{-1} : l \to F$ is affine. Hence so is $h := f^{-1} \circ g$. We have $h(x') = x, h(y') = y, h(z') = z$ so (2) applies.

(5) Fix any non-constant affine function $f_0 : F \to l$; it exists by (3). Let $f_0^{-1} : l \to F$ be the inverse function. Define, for $u, v, w \in l$: $u * v * w$ iff $f_0^{-1}(u) * f_0^{-1}(v) * f_0^{-1}(w)$. Then f_0 is an isomorphism between $(F, *)$ and $(l, *)$, so all the axioms B1-B3 are valid for $(l, *)$. Let $f : F \to l$ be any nonconstant affine function. Then by (4) (or (2)) we have $f(x) * f(y) * f(z)$ iff $f_0^{-1} f(x) * f_0^{-1} f(y) * f_0^{-1} f(z)$ iff $x * y * z$.

Ex. 2.3 See Hartshorne, p. 138.

Note that the incidence geometry, with notion of congruence, over the rational field should not be called a Hilbert plane, since C1 does not hold.

Proof of B4 First consider a

Special case. Let l be a line through the origin $(0, 0)$, but not one of the axes. Let $a < 0 < b$. Then l intersects the segment $[(0, a), (1, 0)]$ or the segment $[(0, b), (1, 0)]$, but not both.

Proof of special case The line l must contain some $(c, d) \neq (0, 0)$. Since l is not one of the axes, we have $c \neq 0$ and $d \neq 0$. Replacing (c, d) by $(-c, -d)$ if necessary, we may assume $c > 0$. If $d > 0$, then l intersects the segment $[(0, b), (1, 0)]$ in the point (tc, td), where $t = b/(cb + d)$. Also any point $(x, y) \neq (0, 0)$ of l has the form $t(c, d)$ and hence if $x > 0$ then $y > 0$, so (x, y) cannot lie on the segment $[(0, a), (1, 0)]$. The case $d < 0$ is similar.

The general case can be done by computation, but instead we will use 2.2. Using 2.2, any invertible affine transformation $T : F^2 \to F^2$ preserves $*$. Thus (B4) is true of A, B, C, l iff it is true of $T(A), T(B), T(C), T(l)$. We will use this to simplify the problem.

First let T_1 be the transformation subtracting the vector D. So $T_1(D) = (0, 0)$. Transforming the problem with T_1, we may assume $D = (0, 0)$. By assumption, A, B, C are not colinear, but $A * D * B$ are, so D, B, C are not colinear; hence B, C are a basis for F^2.

Next let T_2 be an invertible linear transformation, with $T_2(B) = (0, 1)$ and $T_2(C) = (1, 0)$. Transforming using T_2, we now have $B = (0, 1), C = (1, 0)$. Since $A * D * B$, the point A must have the form $(0, a)$ for some $a < 0$. Let $b = 1$, and apply the special case.

Ex. 2.4 (C3). Let $A * B * C$ and $D * E * F$. Assume that $AB \cong DE$ and $BC \cong EF$. We have to show that $AC \cong DF$.

Write $ls(AB)$ for the length-squared of AB. We know that $ls(AB) = ls(DE)$ and also $ls(BC) = ls(EF)$. We have to show that $ls(AC) = ls(DF)$.

View $A, B, \ldots$ as vectors in F^2. Since A, B, C lie on a line, we have $C - B = \alpha(B - A)$ for some $\alpha \in F$; since $A * B * C$ we have $\alpha > 0$. Similarly $F - E = \beta(E - D)$ for some $\beta > 0$. It follows that $ls(DE) = \alpha^2 ls(AB)$ and $ls(EF) = \beta^2 ls(DE)$. So $\alpha^2 = \beta^2$, hence $(\alpha - \beta)(\alpha + \beta) = 0$, so $\alpha = \pm\beta$. Since $\alpha, \beta > 0$ we have $\alpha = \beta$.

Say $B - A = (u, v)$. Then $C - B = (\alpha u, \alpha v)$, and $C - A = ((1 + \alpha)u, (1 + \alpha)v)$. Hence $ls(AC) = (1 + \alpha)^2(u^2 + v^2) = (1 + \alpha)^2 ls(AB)$.

Similarly $ls(DF) = (1 + \beta)^2 ls(DE)$.

Since $\alpha = \beta$ and $ls(AB) = ls(DE)$, we have $ls(AC) = ls(DF)$.

Note We did not define congruence of angles here (though it is possible to do so); the question was intended to ask about congruence of segments only, (C1-C3).

Ex. 2.6: Assume F is an ordered field such that every positive element has a square root. Show that (E2) holds.

This involves some algebraic manipulations, and it helps to show first:

2.6(1): The line $Ax + By = C$ intersects the circle $x^2 + y^2 = 1$ in F^2, provided that $C^2 \leq A^2 + B^2$.

[Proof: We may assume here that either $B = 0$ or, dividing through by B to obtain $(A/B)x + y = (C/B)$, that $B = 1$. In each case compute the point of intersection using the quadratic formula, and check that you are taking the square root of a non-negative element.]

If $a, b \in F^2$, write $d(a, b)$ for the "length" of a segment $[ab]$, defined near Ex. 2.3. It is easy to see that $d(a, c) \leq d(a, b) + d(b, c)$ ("triangle inequailty".)

Returning to 2.6:

Let C_1 be a circle with center a_1, radius r_1, and similarly C_2. Assume C_1 has a point b_1 in the interior of C_2; thus $d(b, a_1) = r_1$ and $d(b, a_2) < r_2$. By the "triangle inequality" above, it follows that $d(a_1, a_2) < r_1 + r_2$, and also that $r_1 < r_2 + d(a_1, a_2)$. Similarly using a point b_2 on C_2 and in the interior of C_1, we obtain from the hypothesis of (E2):

$$|r_1 - r_2| < d(a_1, a_2) < r_1 + r_2 \tag{1}$$

Thus we have to show that if Equation (1) holds, then the circles C_1, C_2 intersect. Say $r_2 \leq r_1$. We can simplify the computation a little by assuming $a_1 = (0, 0)$; this is legitimate since we may apply the transformation $(x, y) \mapsto (x, y) - a_1$ without changing the problem. We can then rescale (apply the map $(x, y) \mapsto (x/r_1, y/r_1)$) so we may assume $r_1 = 1$. Say $a_2 = (b, c)$, and let $d = d(a_1, a_2) = b^2 + c^2$. We have to solve:

$$x^2 + y^2 = 1 \tag{2}$$

$$(x - b)^2 + (y - c)^2 = r_2^2 \tag{3}$$

Subtracting, we can replace (3) by :

$$2bx + 2cy = 1 - r_2^2 + d^2$$

Using (2.6(1)) above, this has a solution provided $4d^2 \geq (1 - r_2^2 + d^2)^2$, or equivalently $2d \geq |1 - r_2^2 + d^2|$, or (taking both possibilities of the absolute value)

$$(d + 1)^2 \geq r_2^2 \geq (d - 1)^2$$

This follows from Equation (1).