1. Hilbert's axioms

In this section we will pay attention to some formal aspects of Hilbert's axioms. Let us begin with axioms (I1)-(I3).

Definition 1.1. *An* incidence geometry *consists of:*

(1) *a set* P *(called* the set of points.*)*
(2) *a set* L *(called* the set of lines.*)*
(3) *a set* $I \subseteq P \times L$*, called* incidence

satisfying axioms I1-I3.

We have here three undefined terms: point, line, incidence; and three axioms they must satisfy. Incidence is viewed as a relation between a point a and a line b; when $(a, b) \in I$ we say indifferently that *a lies on b*, or that *b passes through a*, or that *a, b are incident*.

Those who learned logic will recognize the framework of first order logic (=relational logic, predicate calculus, tahshiv hayehasim). We write $P(x)$ to mean: "x is a point", $I(x, y)$ to mean: "x is a point, y is a line, and x lies on y". We assume that P, L are disjoint and $I(x, y)$ implies $P(x)$ and $Q(y)$ (this can be built into the framework, or stated as an additional axiom.)

In addition to the symbols I, P, Q, we give ourselves logical symbols $\wedge$ (meaning "and"), $\vee$ (meaning "or"), $\implies$ (meaning: "implies"), $\neg$ ("not"), $(\exists)$ (there exists) $(\forall)$ (for all); and a symbol for equality, $=$. We abbreviate $\neg(x = y)$ by $x \neq y$, and use other symbols a little informally.

For instance the sentence: "through any two points there passes a line" can be expressed thus:

$$(\forall x)(\forall y)((P(x) \wedge P(y)) \implies (\exists z)(L(z) \wedge I(x, z) \wedge I(y, z))) \tag{1}$$

Even if you didn't learn logic, you should be able to make sense of the above formula (read slowly!)

The axiom (I1) says more than $I1 - E$, since it asserts the *uniqueness* of the line z passing through x, y. In other words, two lines through x, y must be equal:

(2)
$$(\forall x)(\forall y)((P(x) \wedge P(y) \wedge x \neq y) \implies (\forall z_1)(\forall z_2)((I(x, z_1) \wedge I(y, z_1)) \wedge (I(x, z_2) \wedge I(y, z_2)) \implies z_1 = z_2)$$

So (I1) is the conjunction of (1) and (2).

Excercise 1.1. *Write (I2) and (I3) in a similar way.*

Given a line b, let Λ_b be the set of points lying on b:

$$\Lambda_b := \{x : P(x) \wedge I(x, b)\}$$

Excercise 1.2. *Let* (P, L, I) *be an incidence geometry. If* $b, c \in L$ *and If* $\Lambda_b = \Lambda_c$ *show that* $b = c$.

Excercise 1.2 allows us to identify a line with a certain set of points, namely b with Λ_b. This is the point of view is taken in Hartshorne's book.

To explain the equivalence of the two points of view in a precise way, we use the notion of *isomorphism*. Let (P, L, I) and (P', L', I') be incidence geometries. An *isomorphism* between them is a pair of bijective functions $f : P \to P'$ and $g : L \to L'$, such that for $a \in P$ and $b \in L$ we have:

$$I(a, b) \iff I(f(a), g(b))$$

We say that two geometries are isomorphic if there exists an isomorphism between them. Intuitively, this means they are mathematically the same, only with a renaming of their elements.

Excercise 1.3. *Let (P, L, I) be an incidence geometry. Then there exists an isomorphic incidence geometry (P, L', I') such that:*

- *Each line is a subset of P.*
- *Incidence agrees with membership, i.e. for $a \in P, b \in L'$ we have $I'(a,b) \iff (a \in b)$.*

(hint: Let $L' = \{\Lambda_b : b \in L\}$.)

Excercise 1.4. *There is another isomorphic geometry where each point is a set of lines!*

The possibility of these two poitns of view is called *duality*; it can be very useful.

Hilbert's formalization of Euclidean geometry has three more relation symbols.

Betweenness, a 3-place relation symbol $B(x, y, z)$ (also written $x * y * z$). We assume (or add an axiom to say that) $B(x, y, z)$ implies $P(x), P(y), P(z)$.

Segment congruence, a 4-place relation symbol C_{seg} on points. $C_{seg}(x, y, z, w)$ is intended to mean: x, y, z, w are points; $x \neq y$, $z \neq w$, and $xy \cong zw$.

Angle congruence, a 6-place relation symbols C_{ang} on points.

Definition 1.2. *A* Hilbert plane *is a structure $(P, L; I, B, C_{seq}, C_{ang})$ satisfying axioms (I1)-(I3), (B1)-(B4), (C1)-(C6).*

2. Ordered fields

An *ordered field* is a field F together with a linear ordering $<$ on F, such that $1 > 0$, if $x > y$ then $x + u > y + u$, and if in addition $u > 0$ then $xu > yu$.

Given an ordered field F, we can try constructing a Hilbert plane, as follows.

Let $P_F = F^2$. (The set of *points* of the Hilbert plane will be the set of ordered pairs (a, b), with $a, b \in F$.

Define a *line* to be the solution set of an equation $ax + by + c = 0$, where $(a, b, c) \in F^3 \setminus \{(0, 0, 0)\}$. In other words a line is a set of the form

$$\Lambda_{(a,b,c)} = \{(x, y) \in F^2 : ax + by + c = 0\}$$

This does *not* mean that L can be defined to be $F^3 \setminus \{(0, 0, 0)\}$, since it can happen that $\Lambda_{(a,b,c)} = \Lambda_{(a',b',c')}$ yet $(a, b, c) \neq (a', b', c')$.

Excercise 2.1. *1) Describe explicitly when $\Lambda_{(a,b,c)} = \Lambda_{(a',b',c')}$ yet $(a, b, c) \neq (a', b', c')$.*

2) Show that the set of lines is in 1-1 correspondence with the the set S of triples (a, b, c) such that:

- *$a = 1$, or*
- *$a = 0, b = 1$*

(Show that any line has the form λ_s for some $s = (a, b, c) \in S$, and that if $\lambda_s = \lambda_{s'}$ with $s, s' \in S$ then $s = s'$).

Define $I_F = \{((x, y), (a, b, c))) : ax+by+c = 0\}$. We saw (March 26, Ex. 3) that (P_F, L_F, I_F) is an incidence geometry.

Let us now define the ternary relation B. We first define a notion of "betweenness" on the field F itself: for $x, y, z \in F$, say that $x * y * z$ iff $x < y < z$ *or* $z < y < x$.

Excercise 2.2. *Let F be an ordered field. Recall that a function $f : F^m \to F^n$ is* linear *if for any $x, y, z \in F^m$ and any $\alpha, \beta, \gamma \in F$ we have*

$$f(\alpha x + \beta y + \gamma z) = \alpha f(x) + \beta f(y) + \gamma f(z) \tag{3}$$

We say that f is affine *if* (3) *holds* when $\alpha + \beta + \gamma = 1$.

(1) *Show that a function* $f : F \to F$ *is affine iff it has the form* $f(x) = ax + b$ *for some* a, b. *(*Hint: let $b = f(0)$, $a = f(1) - b$.*)*.
(2) *Let* $f : F \to F$ *be a non-constant affine function. Show that* f *is bijective and preserves betweenness, i.e.* $x * y * z$ *iff* $f(x) * f(y) * f(z)$.
(3) *Show that the image of a non-constant affine function* $f : F \to F^2$ *is a line, and any line is the image of an affine function.*
(4) *Let* $f : F \to F^2$ *an affine function, such that* $f(F) = g(F) = l$ *is a line. Assume* $f(x) = g(x')$, $g(y) = g(y')$, $f(z) = g(z')$. *Show that* $x * y * z \iff x' * y' * z'$. (Hint: consider gf^{-1}.)
(5) Define betweenness $*$ on a line l, and show that any non-constant affine function $f : F \to l$ preserves betweenness.

Excercise 2.3. *Let* F *be an ordered field. Show that axioms B1-B4 of Hilbert planes are valid in the plane* (P_F, L_F, I_F)

An ordered field is called *Pythagorean* if the sum of two squares is a square. Over such a field, we can define the *length* of a segment $[a, b]$ (where$a = ((a_1, a_2), b = (b_1, b_2)))$ in P_F to be the positive square root of $(b_1 - a_1)^2 + (b_2 - a_2)^2$.

Even without this definition of length, in any field, we can define the *length squared* of the segment $[a, b]$ to be $(b_1 - a_1)^2 + (b_2 - a_2)^2$. We can define C_{seg} in the following way. $C_{seg}(a, b, c, d)$ holds iff $[a, b]$ and $[c, d]$ have the same length-squared. (If the field is pythagorean, this is equivalent to saying they have the same length.)

Excercise 2.4. *If* F *is pythagorean, show that C1-C4 hold. Where do you need the pythagorean property?*

2.1. **The circle-circle existence property.** Recall that Euclid's treatment missed axioms guaranteeing the existence of intersections of circles with circles, or of lines with circles. These are not part of the axioms of Hilbert planes either, but will be considered as additional properties. The *circle-circle existence property* is the following sentence:
E_2 Given two circles Γ, Δ, if Δ contains at least one point inside Γ and at least one point outside Γ, then Γ, Δ intersect.

Excercise 2.5.
- *Let* (P, L, I) *be a Hilbert plane. Define the notions* circle*, a point being* inside *or* outside *a circle, and when two circles* intersect*.*
- *Let* $(P_{\mathbb{Q}}, L_{\mathbb{Q}}, I)$ *be the Hilbert plane formed from the ordered field of rationals. Show that the circle with center* $(0, 0)$ *and through the point* $(1, 1)$ *does not intersect the* x*-axis. Find two circles that do not intersect.*
- *Assume* F *is an ordered field such that any positive element has a square root. Show that* (E_2) *is true.*

3. Logical aspects of Hilbert's axioms

Discussion. Hilbert had an additional axiom, the Archimedean axiom. He stated it as follows:

Let A_1 *be any point upon a straight line between the arbitrarily chosen points* A *and* B. *Take the points* $A_2, A_3, A_4, \ldots$ *so that* A_1 *lies between* A *and* A_2, A_2 *between* A_1 *and* A_3, A_3 *between* A_2 *and* A_4 *etc., and such that* $A_1A_2 \cong A_2A_3 \cong A_3A_4, \ldots$. *Then, among this series of points, there always exists a certain point* A_n *such that* B *lies between* A *and* A_n.

If we try to express this axiom in the logical formalism as above, we see that it concerns not only points and lines, but also *natural numbers* and *sequences*. It is thus not purely geometric in the sense of the axioms of a Hilbert plane.

Hilbert also considered the possibility a *completeness axiom*, which goes out of the system and compares it to other possible Hilbert planes: it states that the given plane is Archimedean, and maximal in the sense that no Archimedean Hilbert plane extends it. Here Hilbert himself stated that he will make no use of it. It can be stated in terms of arbitrary *subsets* of a line in the plane; only later in the 20th century was it appreciated that this too involves another structure, set theory, with its own aspects going far out of geometry.

Following Hartshorne, we do not include either of these axioms in our definition of a Hilbert plane. We take the axioms to be those in Definition 1.2. (If we wish we can always consider Hilbert planes with special properties, e.g. *archimedean* Hilbert planes or *complete* Hilbert planes.)

3.1. **Consistency.** In the class Logic I, one defines a *proof system*, giving a formal way to obtain consequences of the axioms. A set of axioms is called *consistent* if it does not imply a contradiction, i.e. does not lead to a sentence and its negation. It is called *complete* if for any sentence ϕ in the language, either ϕ is a formal theorem (i.e. there exists a proof of ϕ from the axioms), or else ϕ can be contradicted, i.e. there exists a proof of $\neg\phi$ from the axioms. [1]

If we have in mind a particular model M of the axioms, *consistency* is automatically true. Conversely, Gödels *completeness theorem* asserts that any consistent system has a model. Given a model M, *completeness* can also be stated as follows: any sentence true in M can be proved from the axioms.

Are Hilbert's axioms consistent?

Yes. We will see that any ordered field F provides a model (P_F, L_F, I_F).

Are Hilbert's axioms complete?

No. We will see that the parallel axiom P can neither be proved, nor disproved from Hilbert's axioms.

OK, but if we add the parallel axiom as Euclid did, do we obtain a complete system?

No. It is possible to formulate analogs E_n of (E_2) for odd n, relating to curves of higher degree than circles. In terms of ordered fields, E_n is the statement that any nonconstant polynomial of degree n has a root. These statements are true in the ordered field $\mathbb{R}$. But if one begins with the rational numbers, and successively adds square roots of positive elements, it can be shown that we will not find a cube root of 2. Thus E_3 neither follows from, nor contradicts, the axioms of the Hilbert plane with the parallel axiom added.

Is this an example Gödel's incompleteness theorem?

No. Gödel's incompleteness theorem applies to theories that are able to discuss *natural numbers*. It does not apply to Hilbert's geometric system. [2]

Tarski proved that any sentence true in the plane $(P_{\mathbb{R}}, L_{\mathbb{R}}, I_{\mathbb{R}})$ follows from Hlibert's axioms along with the parallel axiom and some of the axioms E_n. Tarski's system, including P and the axioms E_n, is complete and consistent.

[1] Not to be confused with the quite different notion of completeness of a Hilbert plane, encountered above.

[2] Gödel's incompleteness theorem is sometimes described as applying to "any sufficiently rich formal system". This description is misleading; Hilbert's axioms and geometry are surely "rich", but Gödel's theorem does not apply s and in fact Tarski showed that the opposite is true