1. Group theory - geometry connection.

We showed three statements:

1.1. **Geometry-to-Group.** Let Π be a Hilbert plane (satisfying I.1-3, B.1-4, C.1-6.) Let G be the group of rigid motions. Then G is sharply transitive on directed rays.

This includes both an existence and a uniqueness statement. Here is a direct verification of the uniqueness, under weaker assumptions.

Lemma 1.2. *Let Π satisfy I 1-3, B 1-4, and the uniqueness part of C1 and C4. Let r be a ray, and let g be a rigid motion fixing r and fixing one side of r. Then $g = Id$, i.e. fixes every point (and line) of Π.*

Proof. Say $r = \overrightarrow{AA'}$, and let l be the line containing r.

Claim: Let B be any point of l. Then g fixes B.

If $B = A$ this is true by definition. If $B \neq A$ and B lies on r, then $g(B)$ lies on $g(r) = r$. Moreover $AB \cong g(A)g(B) = Ag(B)$ by definition of a rigid motion. By the uniqueness part of C1 we have $B = g(B)$. If $B \in l$ but $B \notin r$, we use the opposite ray.

Now let B be any point not on l. Then $g(B)$ lies on the same side of l as B, and $\angle BAA' \cong \angle g(B)AA'$ since g is a rigid motion. By the uniqueness part of C4, $g(B) = B$.

□

1.3. **Group to geometry: (=June 21 HW).** Let Π be a structure satisfying I.1-3, B.1-4. So Π has a notion of I and of *, but no notion of congruence is given. Let H be a group of automorphisms of Π, and assume H is sharply transitive on directed rays. DEFINE congruence on segments/angles by: $x \cong y$ if there exists $h \in H$ with $h(x) = y$. THEN with this notion, Π is a Hilbert plane (satisfying C.1-6.)

1.4. **Geometry+ Group axioms to geometric axioms.** Let Π be a satisfy I.1-3, B.1-4, and the uniqueness parts of C1 and C4. Also let G be the group of rigid motions of Π,and assume G is transitive on directed rays. Then Π satisfies C.1-6. Also two segments/angles are congruent if and only if there exists an element of G taking one to the other.

Proof. Let us prove this using (1) and (2). We are given the group G *and* a notion of congruence.

Claim 1. For two segments x, y, we have $x \cong y$ iff there exists $g \in G$ with $g(x) = y$.

Proof: Since G is a group of rigid motions, if $g(x) = y$ then $x \cong y$.

On the other hand suppose $x \cong y$. Say $x = AB, y = CD$; let $r = \overrightarrow{AB}, s = \overrightarrow{CD}$. Since G is transitive on directed rays, it is transitive on rays, i.e. there exists $g \in G$ with $g(r) = s$. So $g(A) = C$, and $g(B) \in CD$. Again since g is a rigid motion, $AB \cong Cg(B)$. But $AB \cong CD$ by assumption. By the uniqueness part of axiom C1, we have $g(B) = D$.

Claim 2. For two angles x, y we have $x \cong y$ iff there exists $g \in G$ with $g(x) = y$.

The proof is similar.

Now by Lemma 1.2, G is sharply transitive on directed rays. By (2), the plane Π with the congruence relations defined using G satisfies (C1-C6). But by the claim these are the same as the given congruence relations of Π. So (C1-C6) hold in Π.

□