

Definition 0.1. An incidence geometry is a set P , together with a set L of subsets called lines, satisfying I1, I2, I3.

An affine plane is an incidence geometry satisfying in addition:

(P') For every line l and every point $A \notin l$ there exists a unique line m with $A \in m$ and m parallel to l .

Later we will consider the weaker version:

(P) For every line l and every point $A \notin l$ there exists at most one line m with $A \in m$ and m parallel to l .

Exercise 0.2. Let P be an affine plane.

- (1) Given two intersecting lines l, l' , find a bijection between the points of l , and the lines parallel to l' . Find a third line l'' intersecting both l and l' . Find a bijection between the points of l and of l'' .
- (2) If one line has a finite number of points n , In this case show that the plane has n^2 points.

Axioms of order

" B is between A and C " is a new, undefined ternary relation on points; denoted: $A * B * C$.

B1: If $A * B * C$ then A, B, C are distinct and colinear; and $C * B * A$ holds.

B2: For any two distinct points A, B there exists a point C with $A * B * C$

B3: Given three distinct points on a line, one and only one of them is between the other two.

B4: Let A, B, C be three non-colinear points, and let l be a line not containing any of A, B, C . If l contains a point D lying between A and B , then it must also contain either a point lying between A and C , or a point lying between B and C , but not both.

Hilbert had an additional axiom:

(*) Any four points A, B, C, D of a straight line can always be so arranged that B shall lie between A and C and also between A and D , and, furthermore, that C shall lie between A and D and also between B and D .

It is not logically needed, but you may use it in the exercises below.

Exercise 0.3. (1) If we have given any finite number of points situated upon a straight line, we can always arrange them in a sequence A, B, C, D, E, \dots, K so that B shall lie between A and C, D, E, \dots, K ; C between A, B and D, E, \dots, K ; D between A, B, C and E, \dots, K , etc. Aside from this order of sequence, there exists but one other possessing this property namely, the reverse order K, \dots, E, D, C, B, A .

(2) Between any two points of a straight line, there always exists an unlimited number of points.

(3) Show that (2) does not follow from [B1-B3].

Using (*) or without it, show (Hilbert Theorem 5 / Hartshorne Proposition 7.1)

Exercise 0.4. Define a segment. Define a ray (half line) \overrightarrow{AB}

Exercise 0.5. Every straight line l divides the remaining points of the plane into two regions (the two "sides" of l) having the following properties: Every of the one region determines with each point B of the other region a segment containing a point of the straight line l . On the other hand, any two points A, A' of the same region determine a segment AA' containing no point of l .

(Hint: show that the relation " AB does not intersect l " is an equivalence relation on the set of points not on l . Suppose transitivity fails: AB, BC are equivalent, but AC are not. Draw

the picture, first in the "general" case that A,B,C are not colinear, and remember B4. Then think of the case that A,B,C are colinear.)

Exercise 0.6. Let l, l' be two lines. Either l' has points from both sides of l , or l, l' are parallel.

Definition 0.7. An angle $\angle BAC$ is the union of two rays $\overrightarrow{AB}, \overrightarrow{AC}$, not lying on the same line.

The interior of $\angle BAC$ is the set of points D that are on the same side of the line AB as C , and on the same side of the line AC as B .

Exercise 0.8. (1) Show using axioms I1-I3, B1-B4, P that a line cannot be contained in the interior of an angle.

(2) Before reading (3), try to show this without using P. Assume ("by way of contradiction") that it is, and find some strange consequences.

(3) Assuming I1-I3, B1-B4, and that a line is never contained in the interior of an angle, prove (P).

Axioms of congruence: segments

" $AB \cong CD$ " is a 4-place relation on points, viewed as a binary relation on segments: " $\overline{AB}, \overline{CD}$ are congruent".

C1: Given a line segment AB , and given a ray r originating at C , there exists a unique point D on r such that $AB \cong CD$.

C2: Congruence is an equivalence relation on line segments.

C3: (Addition) If $A * B * C$ and $D * E * F$, $AB \cong DE$ and $BC \cong EF$, then $AC \cong DF$.

Axioms of congruence: angles " $\angle ABC \cong \angle CDF$ " is an undefined binary relation on angles.

Definition 0.9. Two triangles ABC, DEF are congruent if $AB \cong DE, AC \cong DF, BC \cong EF, \angle ABC \cong \angle DEF, \angle CAB \cong \angle FDE, \angle BCA \cong \angle EFD$.

The axioms are:

C4: Given an angle $\angle BAC$ and a ray \overrightarrow{DF} there exists a unique ray \overrightarrow{DE} on a given side of the line DF , such that $\angle BAC \cong \angle EDF$.

C5: Congruence is an equivalence relation on line segments.

C6: If $AB \cong DE, AC \cong DF$ and $\angle BAC \cong \angle EDF$ then the triangles ABC, DEF are congruent.

Exercise 0.10. Discuss the correspondence of these axioms to postulates, basic notions and propositions in Euclid I.

Exercise 0.11. Fill out the proof of Euclid I Prop. V, justifying each step using the axioms. (In some cases, more than one line of argument will be needed.)

Exercise 0.12 (Extra credit). (1) Define the interior of a triangle.

(2) Show the interior of a triangle does not contain a line.

(3) Show the interior of a triangle is convex (the segment between two interior points is contained in the interior.)

(4) Show the exterior of a triangle is connected (given two points outside the triangle, connect them by two segments, lying outside the triangle.)