MODEL THEORETIC PROOF OF SZEMEREDI'S THEOREM

Theorem. *Szemeredi's Theorem*

$(\forall\delta > 0)(\forall k \in \mathbb{N})\, \exists n \in \mathbb{N} \forall N > n$ *for every* $A \subset [1, N]$ *if* $|A| \geq \delta N$ *there exists an arithmetic progression of length k:* $a, d \in \mathbb{N}$ *s.t* $\{a, a + d, a + 2d, ..., a + (k - 1)\, d\} \subset A$.

Suppose Szemeredi's Theorem fails. Fix $\delta > 0$, $k \in \mathbb{N}$ and $\{A_n\}_{n=1}^{\infty}$ $A_n \subset [1, n]$ $|A_n| > \delta \cdot n$ s.t A_n contains no arithmetic progressions of length k.

We'll view A_n as a subset of $[1, 2n + 1]$ and view $[1, 2n + 1]$ as an addetive group.

Claim. For $A_n \subset [1, n]$, there is a k-arithmetic progression in $A_n \subset \mathbb{N}$ iff there is a k-arithmetic progression in $A_n \subset [1, 2n + 1]$ (as a group).

So by our assumption there is no k-arithmetic progressions in the groups $([1, 2n + 1], A_n)$.

Definition 1. The structures $G_n = ([1, 2n + 1], A, +)$. The interpretation of A is A_n.

We also add symbols $m\phi$ for each formula (with parameters) $\phi(x_1, ..., x_d, \bar{y})$ and interpret them as $\frac{\left|\left\{\bar{x} \in [1,2n+1]^d;\, \phi(\bar{x},\bar{y})\right\}\right|}{\left|[1,2n+1]^d\right|}$

We take the ultraproduct (along a suitable ultrafilter) of the structures G_n s.t $2n + 1$ is prime: $(G, A, +, Symbols)$.

Let μ^d $d = 1, ..., k$ be the loeb measure generated from $\mu^d(B) = \frac{|B|}{\left|[1,2n+1]^d\right|}$ for $B \subset [1, 2n + 1]^d$.

μs satisfy fubini.

For any defineable set $\phi(x_1, ..., x_d, \bar{y})$ μ^d and $m\phi$ agree (since they are both defined as the limit along the ultraproduct of the finite measures)

By our assumption $\mu(A) > \delta$ and A contains no k-arithmetic progressions.

Definition 2. $\bar{x} = (x_1, ..., x_k)$, $\bar{x}_{\bar{i}} = (x_1, ..., x_{i-1}, x_{i+1}, ..., x_k)$

$A_i \subset G^{k-1}$ $i = 1, ..., k$ (note "they are subset of G^{k-1} with different parameters")

$$x_{\bar{i}} \in A_i \iff \sum_{i\neq j<k} j \cdot x_j + i \cdot \left(x_k - \sum_{i\neq j<k} x_j \right) \in A$$
$$x_{\bar{k}} \in A_k \iff \sum_{j<k} j \cdot x_j \in A$$

Lemma 3. *if $\mu^k \left(\bigcap_{i\leq k} A_i\right) > 0$ then contradiction.*

Proof. First we claim that $\mu^k \left\{\left(a_1, ..., a_{k-1}, \sum_{i<k} a_i\right)\right\} = 0$: Since for every n

(when we view $\left\{\left(a_1, ..., a_{k-1}, \sum_{i<k} a_i\right)\right\} \subset [1, 2n+1]^k$),

$\mu^k \left(\left\{\left(a_1, ..., a_{k-1}, \sum_{i<k} a_i\right)\right\}\right) = \frac{(2n+1)^{k-1}}{(2n+1)^k} = \frac{1}{(2n+1)} \to 0$

So there is some $(a_1, ..., a_k) \in \bigcap_{i\leq k} A_i$ s.t $a_k \neq \sum_{i=1}^{k-1} a_i$.

Define $a = \sum_{i=1}^{k-1} i \cdot a_i$ $d = x_k - \sum_{i=1}^{k-1} a_i$. $d \neq 0$. By definition of $\{A_i\}_{i=1}^k$, $a \in A$ and $a + t \cdot d = \sum_{t\neq j<k} j \cdot a_j + t \left(a_k - \sum_{t\neq j<k} a_j\right) \in A$.

i.e there is a k-arithmetic progression in A. □

Definition 4. $M \subset G$. For $I \subset [1, k]$. $B_{k,I}(M)$ is the boolean algebra of the definable (over M) subsets using variables x_i for $i \in I$. i.e $\left\{(a_i)_{i\in I}\, ;\, \phi\left(\{a_i\}_{i\in I}\right)\right\}$.

$\binom{[1,n]}{t}$ = subsets of $[1, n]$ of size t.

Theorem 5. *Let $t \leq n$ and $A_I \in B^{\sigma}_{n,I}(M)$ for each $I \subset [1, n]$ with $|I| = t$.*
Suppose there exists $\delta > 0$ s.t if whenever $I \in \binom{[1,n]}{t}$ $B_I \in B_{n,I}(M)$ and $\mu^n (A_I \setminus B_I) < \delta$ implies $\bigcap B_I \neq \emptyset$.
Then $\mu^n \left(\bigcap A_I\right) > \emptyset$.

Lemma 6. *Theorem 5 implies $\mu^k \left(\bigcap_{i=1}^k A_i\right) > 0$*

Proof. We need to show the assumption holds for $A_i = A_{\{1,...,i-1,i+1,..,k\}}$.
Take $\delta = \frac{1}{k+1}\mu(A_k)$

Define for $i < k$, $T_i : G^{k-1} \to G^{k-1}$,

$T_i (x_1, ..., x_{i-1}, x_{i+1}, ..., x_k) = \left(x_1, ..., x_{i-1},\, x_k - \sum_{k>j\neq i} x_j,\, x_{i+1}, ..., x_{k-1}\right)$

Claim 7. $\forall i$ $T_i (A_i) = A_k$

Proof. $x_{\bar{i}} \in A_i \implies \sum_{i\neq j<k} j\cdot x_j + i\cdot\left(x_k - \sum_{i\neq j<k} x_j\right) \in A$. $\left(T_i (x_{\bar{i}})\right)_i = x_k - \sum_{j\neq i} x_j$ therefore $\sum_{j<k} j \left(T (x_{\bar{i}})\right)_j \in A$, so $T (x_{\bar{i}}) \in A_k$. Therefore $T_i (A_i) \subset A_k$.
T_i is injective, and for $(x_1, ..., x_{k-1}) \in A_k \implies \sum_{j<k} jx_j \in A$,

$\left(x_1, ..., x_{i-1}, x_{i+1}, ..., x_{k-1}, \sum_{j<k} x_j\right) \in A_i$ and $T\left(x_1, ..., x_{i-1}, x_{i+1}, ..., x_{k-1}, \sum_{j<k} x_j\right) = (x_1, ..., x_{k-1})$. Thus T_i is also surjective. □

Claim 8. $\mu^{k-1}(A_k) = \mu(A) > 0$

Proof. Define $R(x_1, ..., x_{k-1}) = \left(x_1, ..., x_{k-2}, \sum_{i=1}^{k-1} i \cdot x_i\right)$. R is injective therefore $\mu^{k-1}(A_k) = \mu^{k-1}(R(A_k))$.

We want to show that $R(A_k) = \{(a_1, ..., a_{k-2}, a_{k-1}); a_{k-1} \in A\}$. $\subset$ is obvious. $\supset$: For $a_1, ..., a_{k-2} \in G$, $a \in A$, we need to find $a_{k-1} \in G$ s.t $\sum_{i=1}^{k-2} i \cdot a_i + (k-1) a_{k-1} = a$. The first expression can be subtracted from a, and we can also divide by $k-1$ since we took the $2n+1$ in the product to be primes, so it's true in every finite model therefore true in the ultraproduct.

So by Fubini $\mu^{k-1}(R(A_k)) = \mu^{k-2}\left(G^{k-2}\right) \cdot \mu(A) = \mu(A)$. □

Assume we have $B_i \in B_{k,\bar{i}}$, $B_i \subset A_i$, s.t $\mu(A_i \setminus B_i) < \delta$.

(take $T_k(B_k) = B_k$)

$$\mu\left(A_k \setminus \cap_{j\leq k} T_j(B_j)\right) \leq \mu\left(\bigcup_{j\leq k}\left(A_k \setminus T_j(B_j)\right)\right) \leq \sum_{j=1}^{k} \mu\left(A_k \setminus T(B_j)\right)$$
$$\underset{\star}{=} \sum_{j=1}^{k} \mu(A_j \setminus B_j) \leq k \cdot \tfrac{1}{k+1}\mu(A_k) < \mu(A_k)$$

$\star$ holds since T_j are all bijections.

Therefore $\mu\left(\cap_{j<k} T(B_j)\right) > 0$.

Let $(a_1, ..., a_{k-1}) \in \cap_{j<k} T(B_j)$, then $\left(a_1, ..., a_{i-1}, a_{i+1}, ..., a_{k-1}, \sum_{j<k} a_j\right) \in A_i$ thus $\left(a_1, ..., a_{k-1}, \sum_{j<k} a_j\right) \in \bigcap_{j=1}^{k} B_j$ i.e $\bigcap_{j=1}^{k} B_j \neq \emptyset$. □