Non-uniqueness for specifications in $\ell^{2+\epsilon}$

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The model

For a finite alphabet A, let $P(A)$ be the set of probability distributions on A.

A **specification** (also known as "g-function") is a measurable function from $A^{\mathbf{N}}$ to $P(A)$.

A **Gibbs measure** for a specification g is a probability measure μ on $A^{\mathbf{Z}}$ such that

1. μ is shift-invariant.
2. for $i \in \mathbf{Z}$ and $a \in A$,

$$\mu\left(x_i = a \,|x_{i-1}, x_{i-2}, \ldots\right) = g_{x_{i-1},x_{i-2},\ldots}(a).$$

Example

Every Markov chain is a specification.

Fact: Every ergodic Markov chain has a unique Gibbs measure.

Existence of Gibbs measures

1. Every continuous specification has a Gibbs measure.

2. Every monotone Markov chain has a Gibbs measure.

3. Easy to construct examples of specifications with no Gibbs measure.

Uniqueness of Gibbs measures

Intuitively, the further back the specification looks, the more likely it is to have multiple Gibbs measures. Therefore, we want to define a quantitative notion of "how far back a specification looks".

Variation

For a specification g and $k = 1, 2, \ldots$,

$$\mathsf{var}_k(g) = \sup \left\{ \|g(x) - g(y)\| \;\middle|\; \begin{array}{c} x_1 = y_1 \\ x_2 = y_2 \\ \vdots \\ x_k = y_k \end{array} \right\}$$

Variation

1. $\mathrm{var}_k(g)$ is decreasing.

2. g is a k-step Markov chain $\leftrightarrow$ $\mathrm{var}_k(g) = 0$.

3. g is continuous $\leftrightarrow$ $\mathrm{var}_k(g) \to 0$

Algorithmic significance of the variation.

Classic result

We say that g is **regular** if g is bounded away from zero.

Theorem(Keane 1972, Walters 1975): If var(g) is in ℓ^1 then g admits a unique Gibbs measure.

Incorrect old conjecture: If g is continuous then g admits a unique Gibbs measure.

Bramson and Kalikow's example

Theorem(Bramson and Kalikow, 1993): There exists a continuous and regular specification that admits multiple Gibbs measures.

Construction: Take the alphabet $A = \{-1, +1\}$. For properly chosen $\{p_k\}_{k=1}^{\infty}$ and $\{M_k\}_{k=1}^{\infty}$, take

$$g(x_1, x_2, \ldots) = 0.5 + \sum_{k=1}^{\infty} p_k \text{Maj}\left(x_1, x_2, \ldots, x_{M_k}\right)$$

For g in Bramson-Kalikow's example, $\text{var}_k(g) = \Omega(1/\log g)$.
In particular, $\text{var}(g) \notin \ell^p$ for any p.

Challenge: Find a good condition for uniqueness in terms of the variation sequence.

Theorem(Öberg and Johannsson, 2001): If var(g) is in ℓ^2 then g admits a unique Gibbs measure.

Theorem(B-Hoffman-Sidoravicius, 2003): For every $\epsilon > 0$, there exists a regular specification g that admits multiple Gibbs measures and such that var(g) $\in \ell^{2+\epsilon}$.

Construction of the example

The alphabet is $\{-1,+1\}^2$. We represent (x_0, y_0) as a random function of $\{x_{-1}, y_{-i}\}_{i=1}^{\infty}$.

Main steps

1. Choose y_0 independently of anything, so that $y_0 = 1$ w.p. 0.5 and $y_0 = -1$ w.p. 0.5.

2. Using the values of $\{y_{-i}\}$ choose a set of odd size $S \subset -\mathbf{N}$.

3. Using the values of $\{y_{-i}\}$ choose a value $0 \leq v \leq 0.4$.

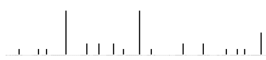
4. Let z be the majority value of $\{X_t : t \in S\}$. We take $x_0 = z$ with probability $0.5 + v$.

Choice of the set S

For every k we define a marker I_k of length 2^k to be

$$I_k = \langle -1, -1, -1, \ldots, -1, +1 \rangle$$

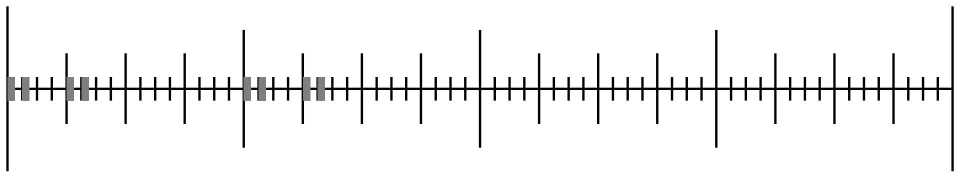
A k-**block** is the interval between two appearances of I_k. Let B_k be the k-block containing 0.



Choice of the set S

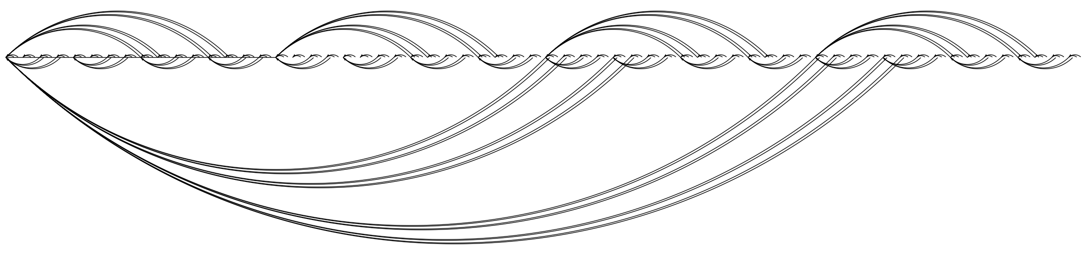
The **beginning** of a k block is the first $\sqrt{2^{2^{k-1}}}$ $k-1$-blocks in it.

The **opening** of a k-block are points that are in the beginning of their j-blocks for $j = 1, \ldots, k$.



Let k_0 be the greatest value of k so that 0 is in the opening of its k-block.

Then S is the opening of the k_0+1-block containing 0.



Choice of v

If the $k_0 + 1$ block containing k is very large, we take $v = 0$.

Otherwise, we take v to be larger than $|S|^{-0.1}$.

v is taken so that

$$v >> 1/\sqrt{|O(B_{k_0})|}, \tag{1}$$

but at the same time

$$v < |B_{k_0}|^{-\frac{1}{100}}. \tag{2}$$

Why does this work?

var(g) is in ℓ^{100}: For any k, if x_0 is influenced by anything further than B_k, then the influence is smaller than $|B_k|^{-\frac{1}{100}}$.

Multiple Gibbs measures: Enough to show that conditioned on $x_{-i} = -1$ for all i, the probability that $x_j = -1$ is bounded away from 0.5 for $k > 0$.

Multiple measures

Fix j.

Let χ_k be the majority value in the opening of the k-block containing j. $\mathbf{P}(\chi_k \neq \chi_{k+1}) < ck^{-2}$.

Therefore, there exist ρ s.t. for every k, $P(x_j = \chi_k) > 0.5 + \rho$.

On the other hand, for large enough k, $\chi_k = -1$. $\square$