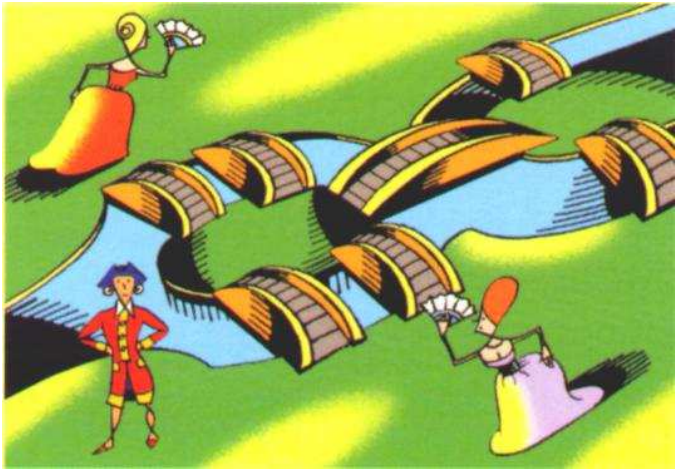
Königsberg Bridges, Periodic Orbits and Ensembles of Truncated Unitary Matrices



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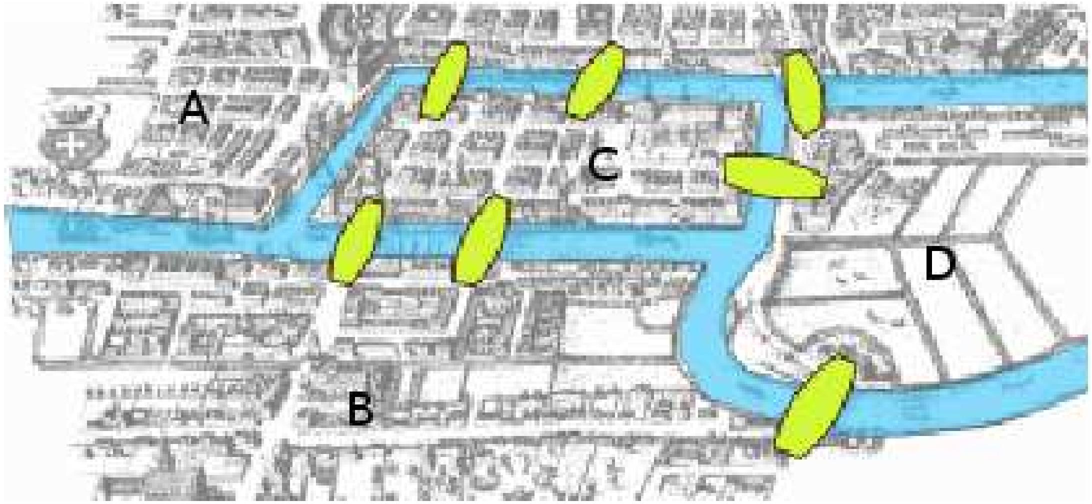
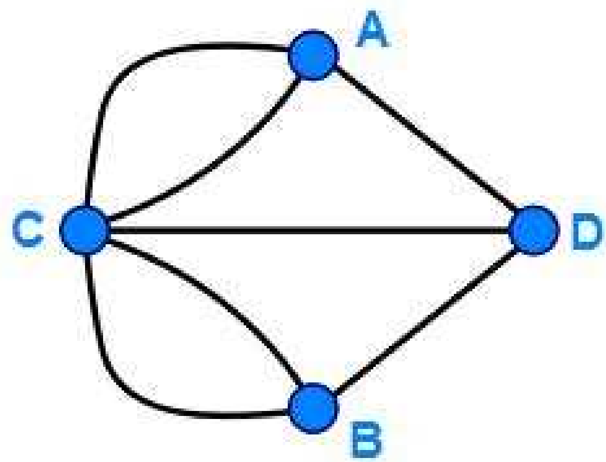
Joint work with V. Osipov

Yosifest, July 2013

Seven bridges of Königsberg





Original problem:
Whether a path exists?

If yes $\Longrightarrow$ **refine question:**
How many Eulerian paths exist?

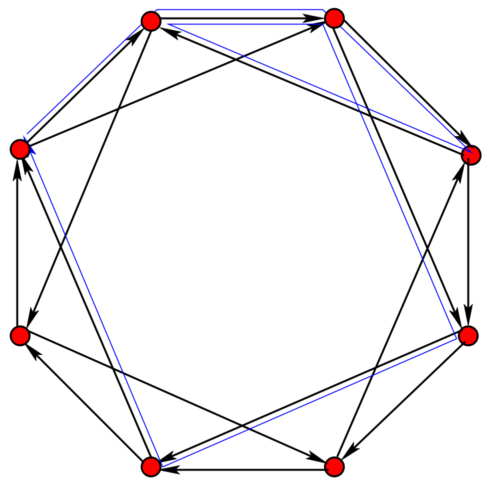
Outline of the talk

- Combinatorial problem. Quantum chaos $\&$ Clustering of periodic orbits

$$\Updownarrow$$

- Spectral universality in ensembles of non-unitary matrices

Cluster of orbits on graphs



Cluster of n-periodic orbits:

$$C_{\vec{n}}, \qquad \vec{n} = (n_1, n_2, \ldots n_N) \quad n = \sum_{i=1}^{N} n_i$$

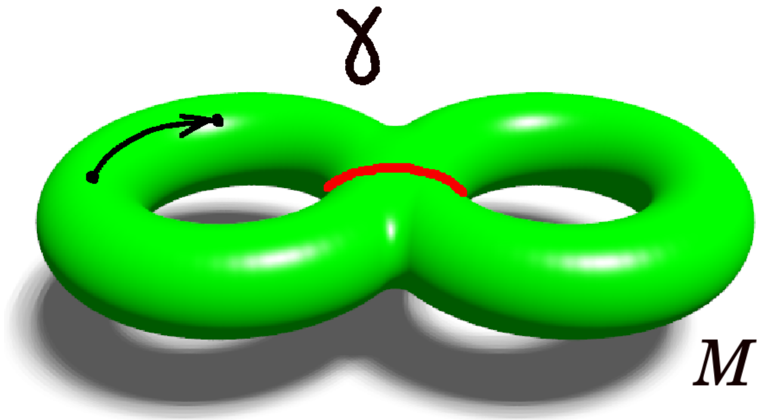
n_i is the number of times an orbit traverses the edge i

Goal: to estimate the sizes $|C_{\vec{n}}|$ of clusters

Motivation:

Clustering of periodic orbits in chaotic systems

Periodic orbits in chaotic systems



Periodic orbits γ $\Longleftrightarrow$ Closed geodesics

Motivation

Quantum problem: $-\Delta\varphi_n = \lambda_n \varphi_n, \qquad \varphi_n \in L^2(M)$

Semiclassical approach

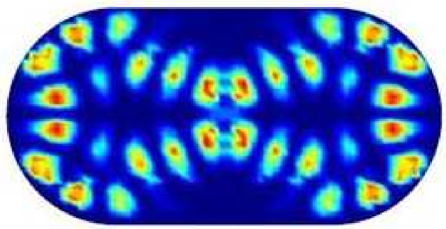
$$\rho(\lambda) = \sum_n \delta(\lambda - \lambda_n) \sim \underbrace{\bar{\rho}(\lambda)}_{Smooth} + \underbrace{\mathrm{Re} \sum_{\gamma \in \mathrm{PO}} \mathcal{A}_\gamma \exp\left(\frac{i}{\hbar} S_\gamma(\lambda)\right)}_{Oscillating}$$

$\mathcal{A}_\gamma$ stability factor, S_γ action of a **periodic orbit** γ

Number of periodic orbits grows **exponentially** with length

Correlations of λ_n $\Longleftrightarrow$ Correlations of S_γ's

Universal spectral statistics

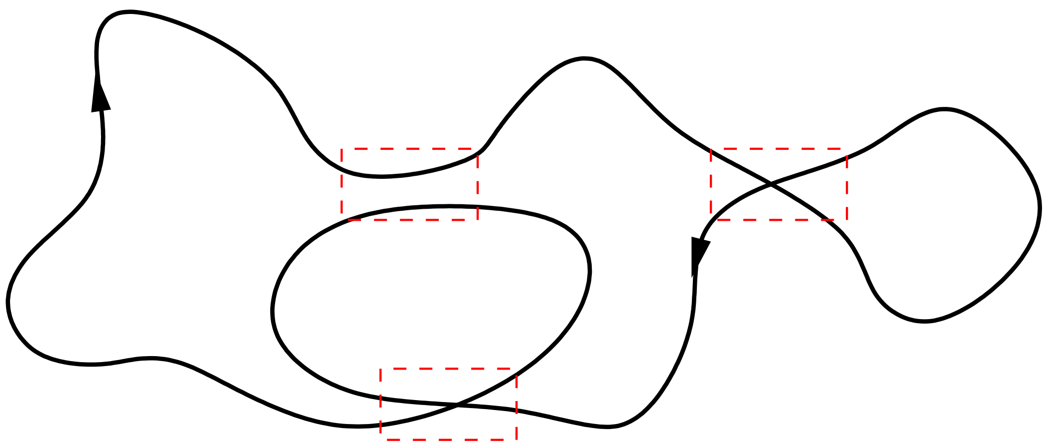


“Chaotic” systems

$$\begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1N} \\ h_{21} & h_{22} & \cdots & h_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N1} & h_{N2} & \cdots & h_{NN} \end{pmatrix}$$

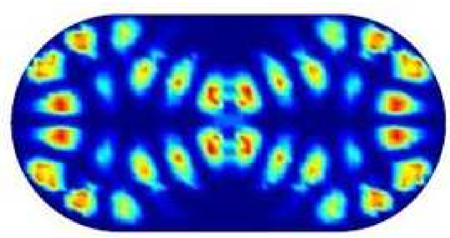
Random matrices

What is the reason?



Clusters of periodic orbits! M. Sieber K. Richter (2001); D. Cohen, H. Primack, U. Smilansky (1998); N. Argaman, et.al (1992); M. Berry (1985)

Universal spectral statistics



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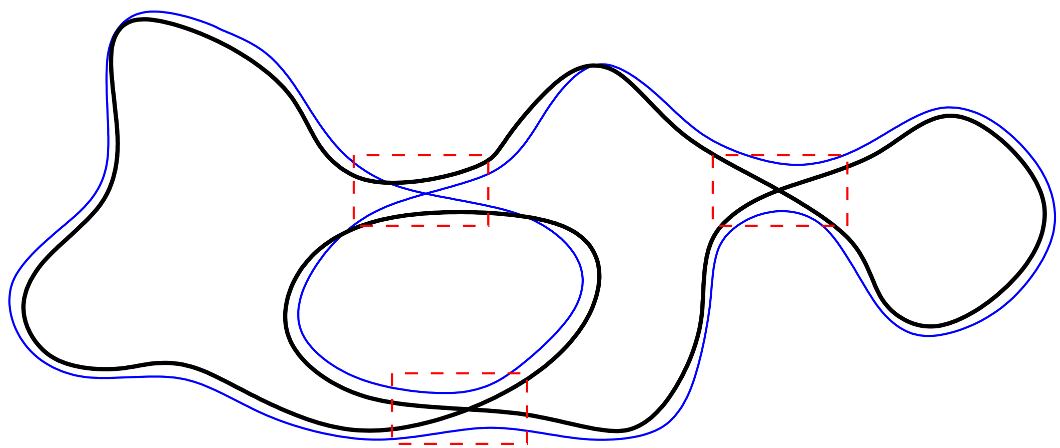

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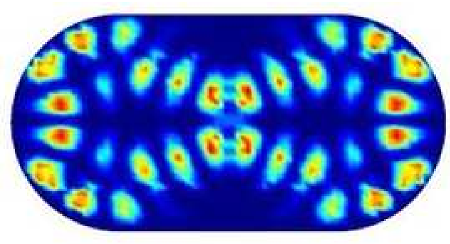

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Random matrices

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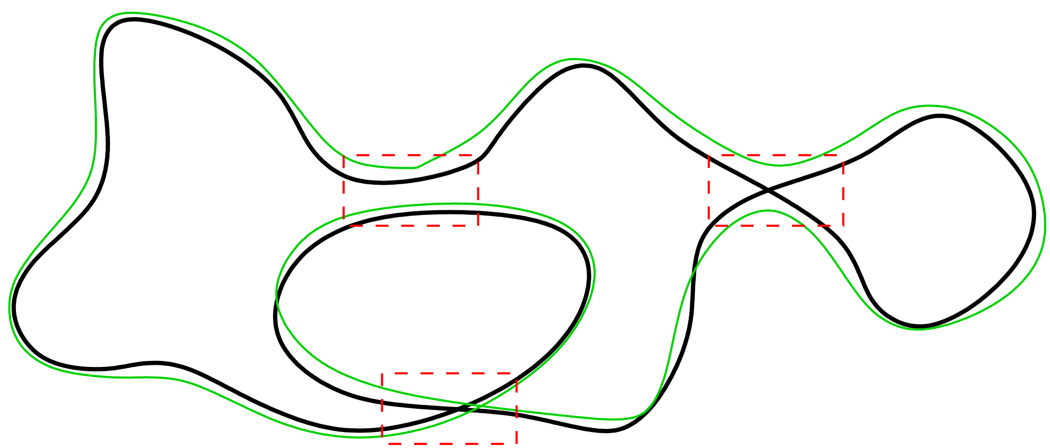

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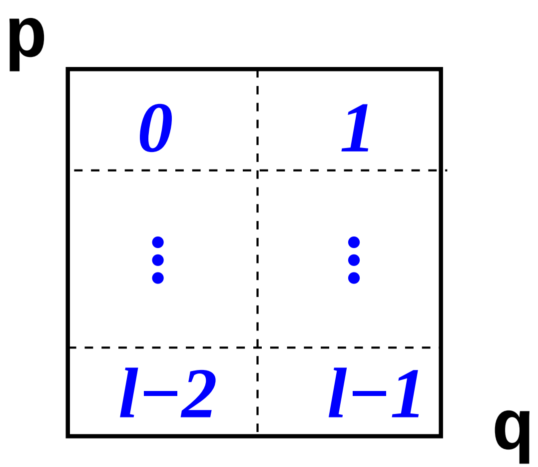
Symbolic Dynamics



Markov partition:

$$V = V_0 \cup V_1 \cup \cdots \cup V_{l-1}$$

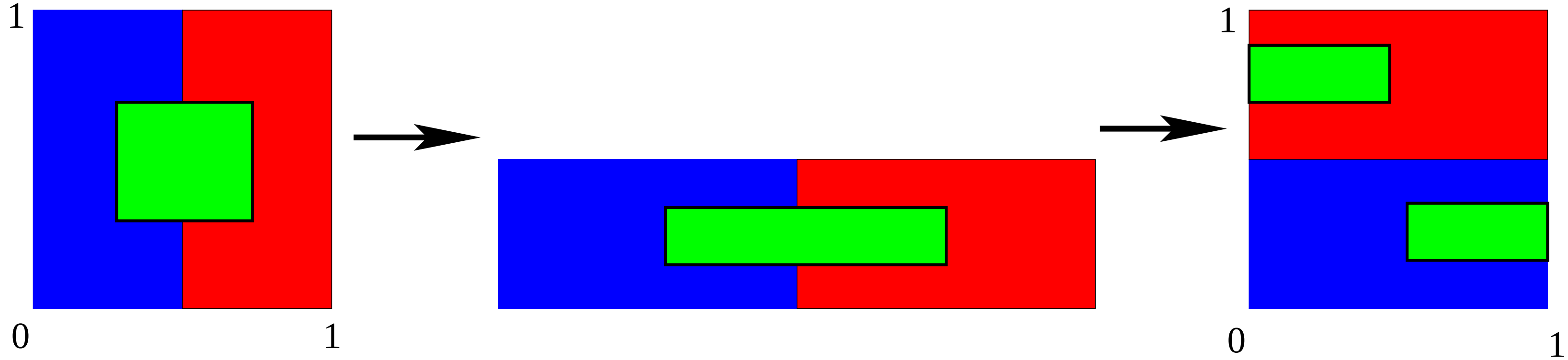
Point in the phase space:

$$x = \underbrace{\ldots x_{-1} x_0}_{past} . \underbrace{x_1 x_2 \ldots}_{future}; \qquad x_i \in \underbrace{\{0, 1, \ldots l-1\}}_{alphabet}$$

$$Tx = \ldots x_{-1} x_0 x_1 . x_2 x_3 \ldots$$

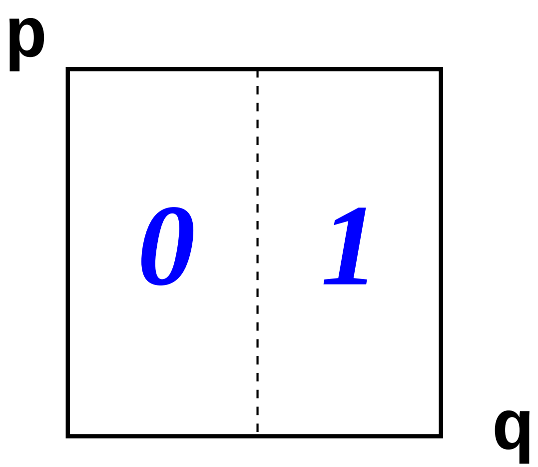
Periodic orbits $\Longleftrightarrow [x_1 x_2 \ldots x_n]$

Paradigm problem: Baker's map



$$\mathcal{T} \cdot (\mathsf{q}, \mathsf{p}) = \begin{cases} (2\mathsf{q}, \frac{1}{2}\mathsf{p}) & \textsf{if } \mathsf{q} \in [0, \frac{1}{2}) \\ (2 - 2\mathsf{q}, 1 - \frac{1}{2}\mathsf{p}) & \textsf{if } \mathsf{q} \in [\frac{1}{2}, 1) \end{cases}$$

Baker's map. Markov partition



Alphabet: $x_i \in \{0, 1\}$

$$x = \ldots x_{-2}x_{-1}x_0.x_1x_2x_3\ldots$$

$$(\mathsf{q}, \mathsf{p}) \leftrightarrow x$$

$$q(x) = 0.x_0x_1x_2\ldots, \qquad p(x) = 0.x_{-1}x_{-2}x_{-3}\ldots$$

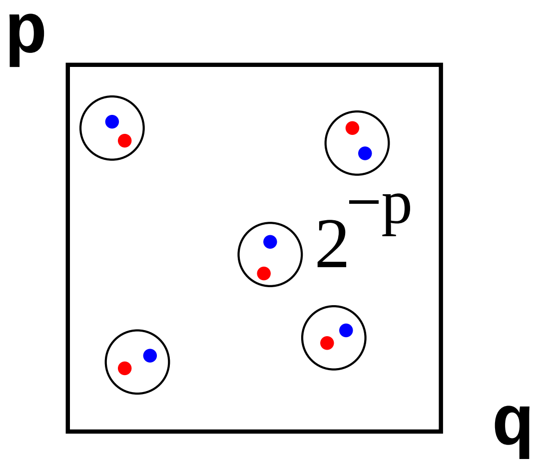
Periodic orbits:

$$\gamma = [x_1x_2\ldots x_n], \qquad x_i \in \{0, 1\}$$

Number of periodic orbits - $2^n/n$

“Sieber Richter pairs"

Definition: Two sequences x, y are p-**close** if each subsequence $a_1 \ldots a_p$ of length p appears the same number of times in x and y



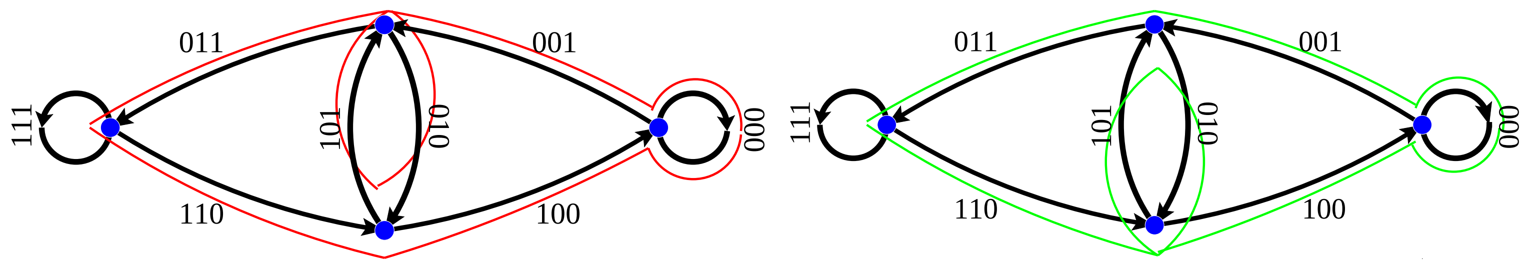
“Metric” distance $\leq 2^{-p}$

Example: $x = [1101000]$, $y = [1100010]$ are 3-close.
$x = [1101000]$ and $z = [1100100]$ are 2-close.

Main property: if x is p-close to y and x is p-close to z, then y is p-close to $z \Longrightarrow$ Clustering of periodic orbits

De Brujin graphs

$$[a_1 \dots a_{p-1}] \to [a_1 \dots a_{p-1}0], \quad [a_1 \dots a_{p-1}1]$$



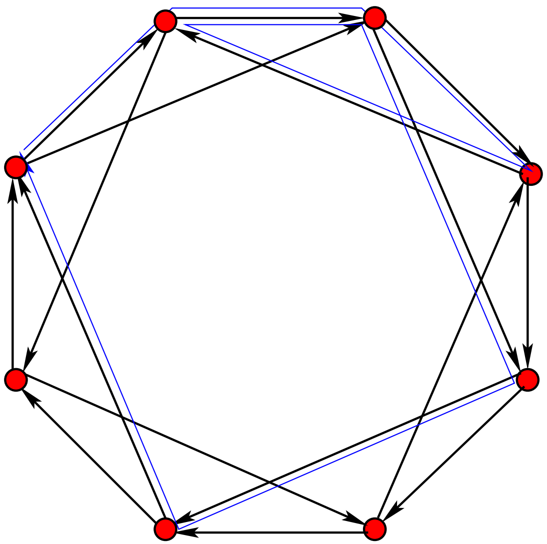
$$x = [0010110] \Longleftrightarrow \gamma_x, \qquad y = [0011010] \Longleftrightarrow \gamma_y$$

x and y are p-close iff γ_x and γ_y pass **the same number of times** through the edges of $G_p \Longrightarrow$

Cluster of p-close P.O. = Cluster of P.O. on G_p

Spectral problem

Cluster sizes



Q **- Connectivity matrix**

$$\Lambda(\boldsymbol{\phi}) = \mathbf{diag}(e^{i\phi_1}, e^{i\phi_2} \ldots e^{i\phi_N})$$

$$\mathrm{Tr}(Q\Lambda(\boldsymbol{\phi}))^n = \sum_{\vec{n}} |C_{\vec{n}}| \exp\left(i(\vec{n}, \boldsymbol{\phi})\right), \qquad (\vec{n}, \boldsymbol{\phi}) = \sum n_a \phi_a$$

Moments of cluster distribution

$$Z_k = \sum_{\vec{n}} |C_{\vec{n}}|^k \iff \text{Traces of } Q\Lambda(\boldsymbol{\phi})$$

Second moment

$$Z_2 = \langle |\mathrm{Tr}(Q\Lambda(\phi))^n|^2\rangle_\phi = |q_{\max}|^{2n} \langle | \sum_{i=1}^{N} z_i^n |^2 \rangle_\phi$$

$\langle\cdot\rangle_\phi = \int(\cdot)\prod d\phi_i$; $z_i,\ i = 1,\ldots N$ are eigenvalues of sub-unitary matrix $\frac{1}{q_{\max}} Q\Lambda(\phi)$, $q_{\max}$ - is largest eigenvalue of Q

Spectral form-factor for “non-unitary“ quantum graphs

A. N is fixed, $n \to \infty \Longrightarrow$ Need to know the distribution of **largest eigenvalue** $z_{\max}(\phi)$

B. $N, n \to \infty \Longrightarrow$ Need to know the **density and correlations of** z_i's

Baker's map. Cluster sizes

For De Brujin graphs $q_{\max} = 2$,

$$Q = \begin{pmatrix} 1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 1 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 1 & 0 & 0 & \dots & 1 \end{pmatrix}, \Lambda(\boldsymbol{\phi}) = \begin{pmatrix} e^{i\phi_1} & 0 & 0 & \dots & 0 \\ 0 & e^{i\phi_2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & e^{i\phi_{2p-1}} & 0 \\ 0 & 0 & \dots & 0 & e^{i\phi_{2p}} \end{pmatrix}$$

Need information on the **spectral form factor** for ensemble of matrices $\frac{1}{2}Q\Lambda(\boldsymbol{\phi})$ with **flat measure** $\prod_{i=1}^{N} d\phi_i$

$2^p = N$ is fixed $n \to \infty$

$$Z_2 = \left(\prod_{i=1}^{N} \int_0^{2\pi} \frac{d\phi_i}{2\pi} \right) \cdot \exp \mathcal{F}_n(\boldsymbol{\phi})$$

n is large $\Longrightarrow$ $\mathcal{F}_n(\boldsymbol{\phi}) = \log |\mathrm{Tr}(Q\Lambda(\boldsymbol{\phi}))^n|^2 \sim 2n \log |z_{\mathrm{max}}(\boldsymbol{\phi})|$
we can apply saddle point approximation

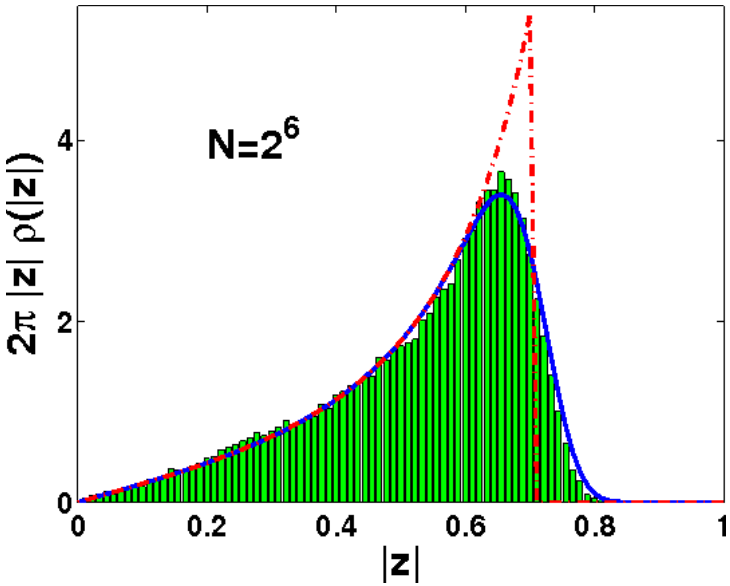
– The global maximum of $\mathcal{F}_n(\boldsymbol{\phi})$ is at $\phi_i = 0$, $i = 1, \ldots N$ and $\mathcal{F}_n(\mathbf{0}) = n \log 2$

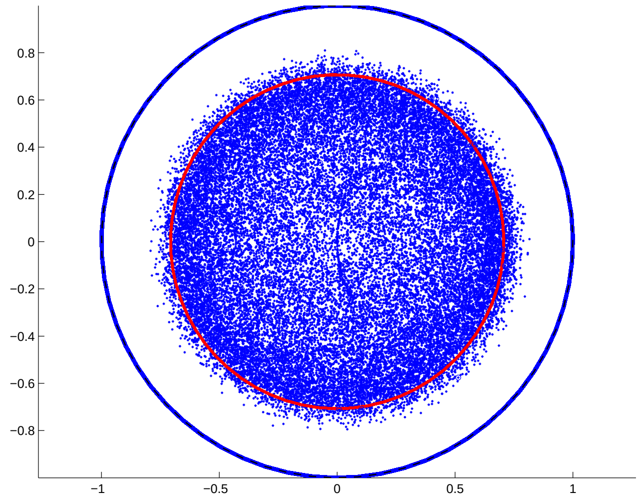
$$Z_2(n) = 2^{2n} \left(\frac{N}{2\pi n} \right)^{N/4} \left(1 + O\left(\frac{1}{n} \right) \right)$$

B.G, V. Osipov (2011)

Spectrum of sub-unitary matrices

Truncated unitary matrices





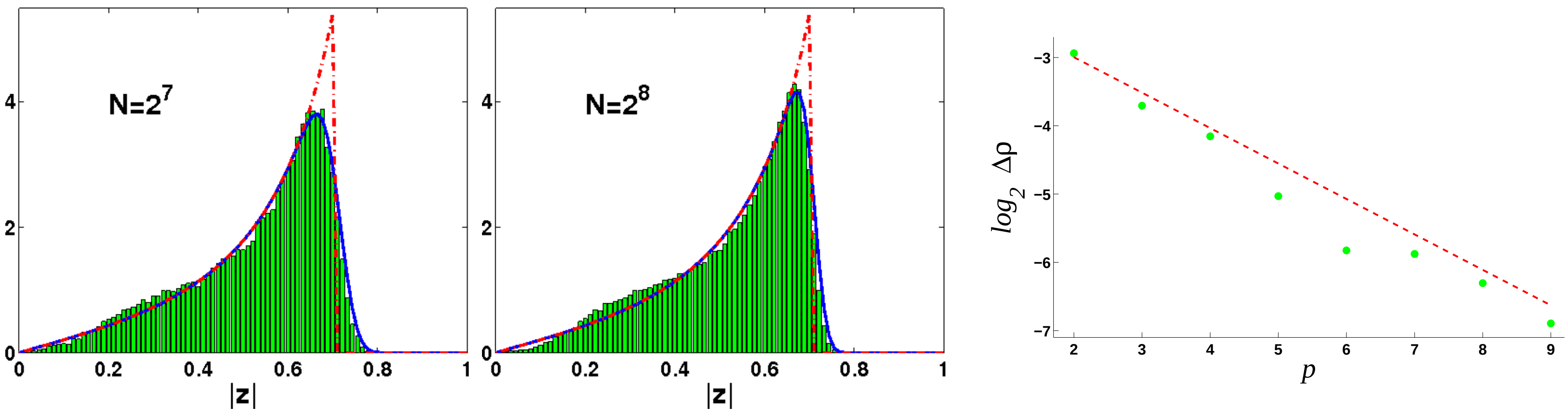
$$\frac{1}{2}Q\Lambda(\phi) \to U_0 \frac{1}{2} Q\Lambda(\phi) U_0^* = PU(\phi)$$

$U(\phi)$ - unitary; $P = \text{diag}\{1, \ldots 1, 0, \ldots 0\}$- projection
Compare with invariant ensembles:

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}_{N \times N}, \quad U \in \mathbf{CUE}$$

K. Zyczkowski, H.-J. Sommers (2000)

Truncated unitary matrices



Asymptotics at the edge:

$$\rho(1/\sqrt{2} - s/\sqrt{N}) \sim \left(1 - \frac{1}{2}\text{erfc}\left(\sqrt{2}s\right)\right)$$

Same asymptotics holds for other non-unitary ensembles with invariant measure e.g., Ginibre, but with different scaling

Spectral Universality

B.G, V. Osipov (2013)

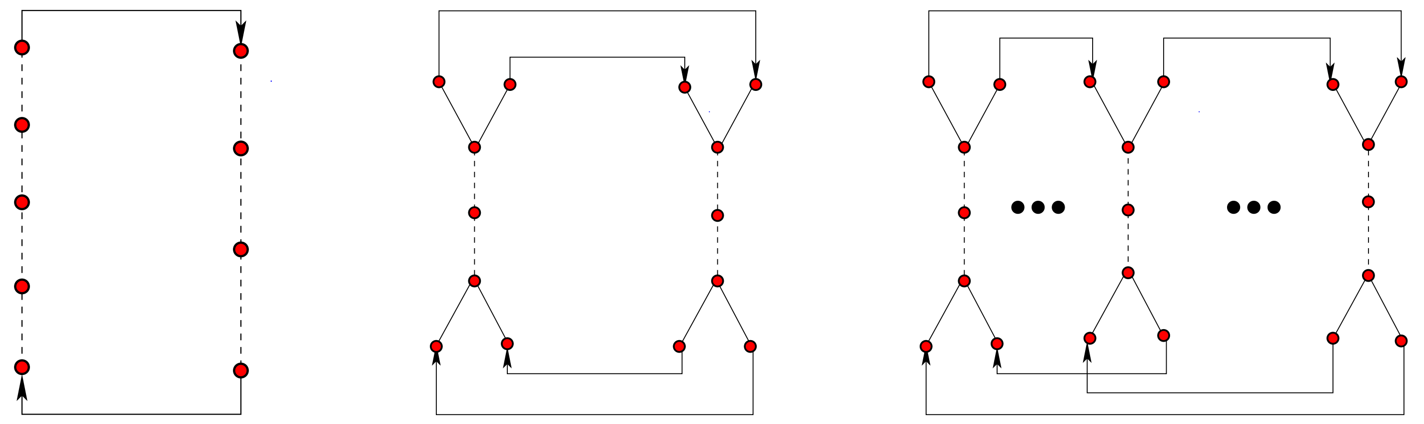
1) *Spectral edge of* $Q\Lambda(\phi)$ *is at* $r = \sqrt{\mu_1}$, *where* μ_1 *is the largest eigenvalue of* Q_2, $(Q_2)_{i,j} = |Q_{i,j}|^2$

2) *Spectral density* $\&$ *correlations in ensembles* $S_\phi = \frac{1}{\sqrt{\mu_1}} Q\Lambda(\phi)$ *are universal at* $N^{-\frac{1}{2}}$ *neighborhood of edge*

Necessary conditions:
(A) Large gap in the spectrum of Q_2,
$\mu_1 - \mu_2 = O(N^{-\kappa}), \quad \kappa < 1/2$
(B) “Strong” non-unitarity of Q

Examples: Random regular graph, $Q_{i,j}$ – connectivity matrix, Friedman (2003). For $Q_{i,j} = |U_{i,j}|$, where U is random unitary, G. Berkolaiko (2001)

“Semiclassical” proof

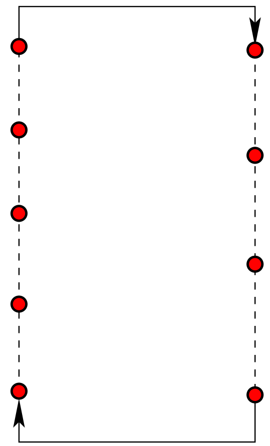


Consider limit $t\sqrt{N} = n \to \infty$, t is fixed

$$\frac{1}{n}\langle|\mathrm{Tr}\,(S_{\boldsymbol{\phi}})^{n}\,|^2\rangle =$$

$$= \left\langle \left| \sum_{\Gamma} A_{\Gamma} e^{i(\boldsymbol{n},\boldsymbol{\phi})} \right|^2 \right\rangle_{\boldsymbol{\phi}} = \underbrace{\sum_{\Gamma} |A_{\Gamma}|^2}_{D^{(0)}\text{ -diagonal}} + \underbrace{\sum_{\Gamma,\Gamma'} A_{\Gamma} A^{*}_{\Gamma'}}_{D^{(2)} + D^{(4)} \ldots}$$

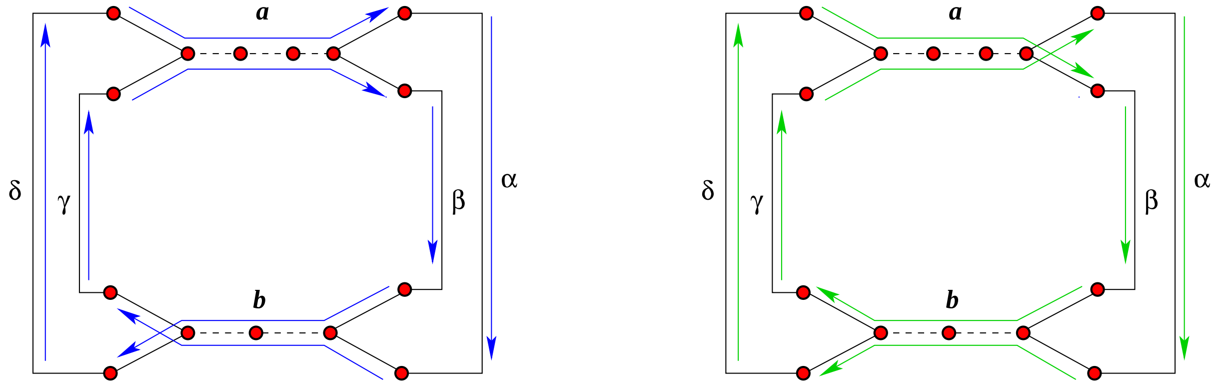
Diagonal approximation



$$D^{(0)} = \frac{1}{\mu_1^n} \sum_{i_1,\ldots i_n} |Q_{i_1,i_2}|^2 |Q_{i_2,i_3}|^2 \ldots |Q_{i_n,i_1}|^2 =$$

$$= \frac{1}{\mu_1^n} \mathrm{Tr} Q_2^n = 1 + O\left(N^{-\frac{1}{2}+\kappa}\right)$$

Second order



$$D^{(2)} = \underbrace{\Big(\text{Loops}\Big)}_{\frac{1}{4!}\left(\frac{n}{N}\right)^4} \underbrace{\Big(\text{Encounters}\Big)}_{(N\mu_N)^2} \underbrace{\Big(\text{Structure}\Big)}_{1}$$

$$\mu_N = \frac{1}{N}\mathrm{Tr}\left((QLQ^\dagger \bar{L})^2 - (L\bar{L})^2\right)$$

$L = \text{diag}\{\ell_1, \ldots, \ell_N\}, \bar{L} = \text{diag}\{\bar{\ell}_1, \ldots, \bar{\ell}_N\}$. $\ell_i, \bar{\ell}_i$ elements of highest left (right) eigenvector of Q_2, $\langle \ell | \ell \rangle = \langle \bar{\ell} | \bar{\ell} \rangle = 1$

All orders

$$D^{(k)} = \left(\frac{1}{(2k)!} \left(\frac{n}{2N}\right)^{2k} \right) \left((N\mu_N)^k \right) \left(\frac{(2k)!}{k!} \right) + O\left(N^{-\frac{1}{2}+\kappa}\right)$$

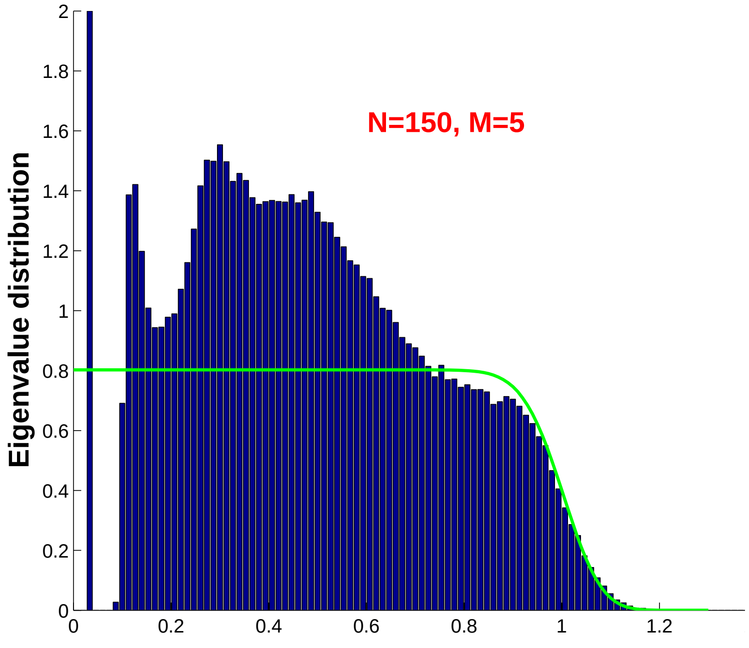
Assuming $\lim_{N\to\infty} \mu_N = \mu > 0$ **(Strong non-unitarity)**:

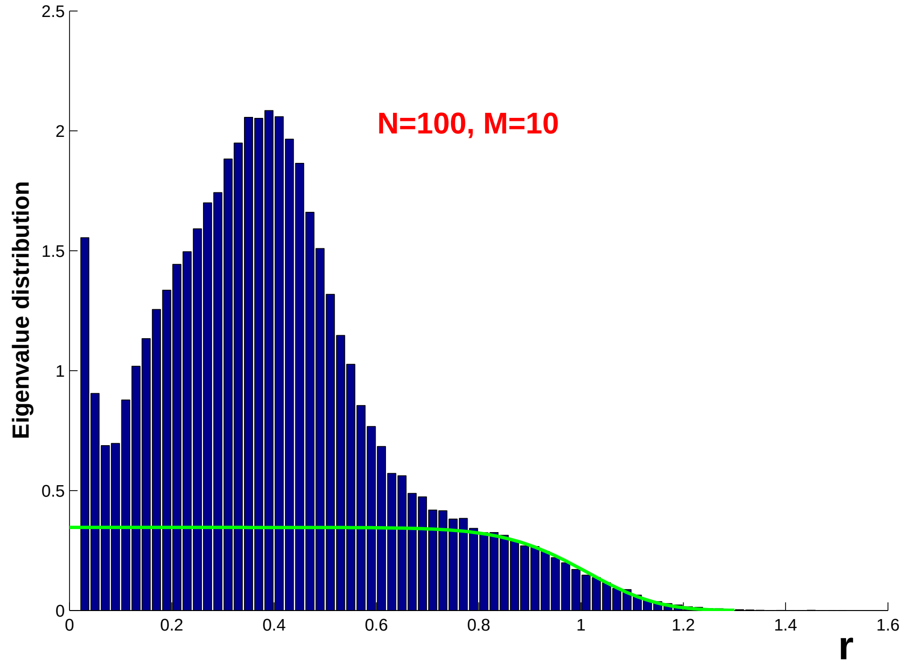
$$\frac{1}{n}\langle|\mathrm{Tr}S_\phi^n|^2\rangle = \sum_{\text{even k}} D^{(k)} = \frac{2N}{\mu n^2}\sinh\left(\frac{\mu n^2}{2N}\right) + O\left(N^{-\frac{1}{2}+\kappa}\right)$$

$$= \underbrace{\frac{1}{\mu t^2}\exp(\mu t^2/2)}_{\langle|z_i|^{2n}\rangle=\int\rho(r)r^{2n}dr} - \underbrace{\frac{1}{\mu t^2}\exp(-\mu t^2/2)}_{\langle z_i^n z_j^{*n}\rangle} + O\left(N^{-\frac{1}{2}+\kappa}\right)$$

The result can be also obtained by SUSY

Density of states





Spectral density:

$$\rho(1 - s/\sqrt{N}) = \frac{2}{\mu}\left(1 - \frac{1}{2}\text{erfc}\left(s\sqrt{\frac{2}{\mu}}\right)\right) + O\left(N^{-\frac{1}{2}+\kappa}\right)$$

In particular, at the edge:

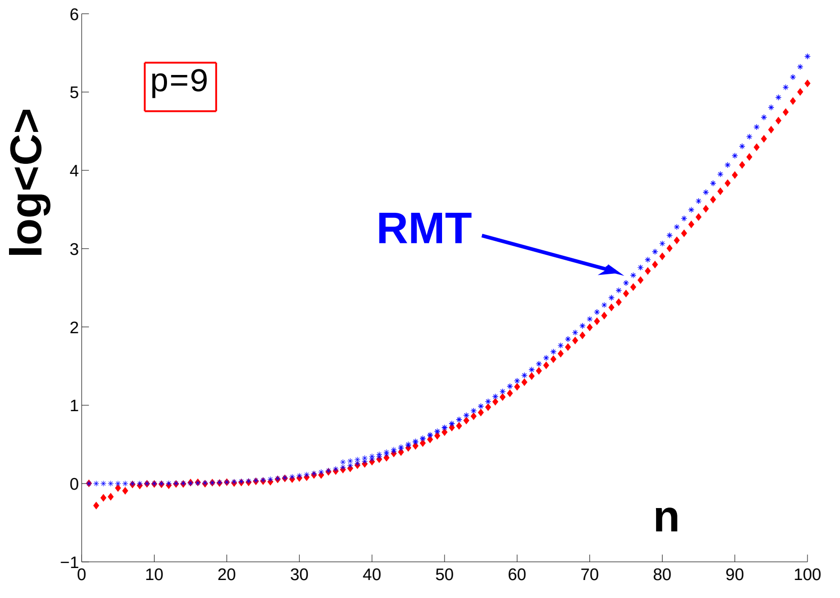
$$\rho(1) = 1/\mu + O\left(N^{-\frac{1}{2}+\kappa}\right)$$

Application to periodic orbits ($N = 2^p$)

For d-regular graph $\mu_1 = d$, $L = \bar{L} = I \Longrightarrow \mu = d - 1$
Average size of cluster for Backer's map ($d = 2$):

$$< C >= \frac{\sum_i |C_i|^2}{\sum_i |C_i|} = \frac{Z_2}{n 2^n} = \frac{\sinh \nu}{\nu} + O(1/\sqrt{N})$$

$\nu = n^2/2N$ is average number of encounters



Application to periodic orbits ($N = 2^p$)

For short periodic orbits $n \lesssim \sqrt{N}$, $< C >\sim 1 \Longrightarrow$ Most of the clusters contain just one periodic orbit.

At $n \sim \sqrt{N}$ starts exponential growth of cluster sizes:
$\log < C >\sim n^2$

At very large times $n \gtrsim N$ (beyond Heisenberg time)
$\log < C >\sim n \Longrightarrow$ periodic orbits “condense” into huge clusters, whose sizes are comparable with total number of periodic orbits. Every point of the phase space is encounter.

Summary

- **Edge** spectral **universality** in ensembles of non-unitary matrices. Asymptotics of spectral density at the edge

- Non-unitary $\to$ Unitary. Leading order diagrams vanish. Scale $n \sim \sqrt{N}$ changes to $n \sim N$

- Application: Clustering of periodic orbits $\Longleftrightarrow$ Spectral analysis of ensembles of sub-unitary matrices

- **Congratulations!**