Hecke Operators and Distributing Points on S^2. II

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1. Introduction

In Part I of this paper we showed how Hecke operators on $L^2(S^2)$ may be used to generate very evenly distributed sequences of three-dimensional rotations. We defined the Hecke operator T_S, where S is a finite symmetric ($\alpha \in S$ if and only if $\alpha^{-1} \in S$) set of rotations, by

$$T_S f(x) = \sum_{\alpha \in S} f(\alpha x), \qquad f \in L^2(S^2). \tag{1.1}$$

A key ingredient in the analysis was a bound on the absolute value of the next to largest eigenvalue (in absolute value) of T_S which we denote by $\lambda_1(T_S)$. For certain S we have

$$\lambda_1(T_S) \leqq 2\sqrt{p}, \tag{1.2}$$

where $p + 1$ is the number of rotations in S. A simple example of such a set is $S = \{A, B, C, A^{-1}, B^{-1}, C^{-1}\}$, where A, B, C are rotations of $\arccos(-\frac{3}{5})$ about the X, Y, Z-axes, respectively.

In Section 2 of the present paper we give a proof of the inequality (1.2) for a large class of Hecke operators which includes the above example.

In Section 3 we describe a more general scheme for producing very evenly distributed sequences of rotations and analyze the resulting discrepancies. In this case we consider a group Γ diagonally embedded in $G_1 \times \mathrm{SU}(2)$, where $G_1 = \mathrm{PGL}(2, Q_p)$ or $\mathrm{PSL}(2, \mathbb{R})$, and for which the projection of Γ on G_1 is discrete and co-compact. One may then order the elements of Γ by a "lattice type" ordering in G_1. The corresponding projections on SU(2) give the desired sequences. The estimates on the discrepancies are similar to those that were obtained for the sequences in Part I. These estimates are based on certain inequalities which are proved in Section 4. The case treated in Section 2 is a special case of this lattice method. In fact, when X is the tree $\mathrm{PGL}(2, Q_p)/\mathrm{PGL}(2, \mathbb{Z}_p)$ and the discrete group Γ is free with $|\Gamma \setminus X| = 1$, then, by taking the generators of Γ to be the

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elements which take some $x_0 \in X$ to its neighbors, one finds that the lattice method and word length ordering coincide.

In Section 4 we show how these estimates may be derived from the work of Deligne [1]. The notation in this paper is the same as that used in Part I and that work will be referred to as I.

2. Quaternion Groups

Let $H(\mathbf{Z})$ denote the standard quaternion ring with integral entries, i.e., $\{\alpha = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} | a_\nu \in \mathbf{Z}\}$, $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$. For $\alpha \in H(\mathbf{Z})$, $\bar{\alpha} = a_0 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}$ is its conjugate and $N(\alpha) = \alpha\bar{\alpha} \in \mathbf{Z}$ is its norm. It is clear that the units of $H(\mathbf{Z})$ are the eight quaternions ± 1, $\pm\mathbf{i}$, $\pm\mathbf{j}$, $\pm\mathbf{k}$. As is well known (see [6]), the number of representations of a positive integer n as a sum of four squares is

$$r_4(n) = 8 \sum_{\substack{d|n \\ 4 \nmid d}} d. \tag{2.1}$$

Clearly, $r_4(n)$ is also the number of $\alpha \in H(\mathbf{Z})$ with $N(\alpha) = n$.

Consider now the set of $\alpha \in H(\mathbf{Z})$ for which $N(\alpha) = p$, where p is a prime, $p \equiv 1$ (4). In this case only one of the a_i is odd. The units act on this set and it is easy to see that each such α' has a unique associate $\alpha = \varepsilon\alpha'$ for which

$$N(\alpha) = p, \qquad \alpha \equiv 1\ (2) \quad \text{and} \quad a_0 > 0. \tag{2.2}$$

The set of α satisfying (2.2) therefore consists of $p + 1$ elements (by (2.1)) and it clearly splits into $\sigma = \frac{1}{2}(p+1)$ conjugate pairs which we denote by

$$\{\alpha_1, \bar{\alpha}_1, \alpha_2, \bar{\alpha}_2, \cdots, \alpha_\sigma, \bar{\alpha}_\sigma\} = S_p.$$

By a reduced word of length m in S_p, denoted by $R_m(\alpha_1, \bar{\alpha}_1, \cdots, \alpha_\sigma, \bar{\alpha}_\sigma)$, we mean a word in $\alpha_1, \cdots, \bar{\alpha}_\sigma$ in which no subwords $\alpha_j\bar{\alpha}_j$ or $\bar{\alpha}_j\alpha_j$ appear. Clearly, the number of such words of length $l \geqq 1$ is

$$(p+1)p^{l-1}. \tag{2.3}$$

There is a homomorphism of $H(\mathbf{Z})^*$ (the group of invertible elements of $H(\mathbf{Z})$) into SU(2) given by

$$\alpha = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \rightsquigarrow \frac{1}{\sqrt{N(\alpha)}} \begin{pmatrix} a_0 + a_1 i & a_2 + a_3 i \\ -a_2 + a_3 i & a_0 - a_1 i \end{pmatrix}. \tag{2.4}$$

The elements in SU(2) correspond via stereographic projection to rotations in SO(3). The correspondence (2.4) will be used throughout this section and when we refer to α it is either in $H(\mathbb{Z})^*$ or SU(2) or SO(3) as will be clear from the context.

Define the Hecke operator $T_p: L^2(S^2) \to L^2(S^2)$ by

$$T_p f(\zeta) = \sum_{s \in S_p} f(s\zeta) = \frac{1}{2} \sum_{\substack{\alpha \equiv 1(2) \\ N(\alpha)=p}} f(\alpha\zeta) \quad \text{for} \quad \zeta \in S^2. \tag{2.5}$$

A simple calculation shows that

$$S_5 = \{1 + 2\mathbf{i}, 1 - 2\mathbf{i}, 1 + 2\mathbf{j}, 1 - 2\mathbf{j}, 1 + 2\mathbf{k}, 1 - 2\mathbf{k}\}$$

and T_5, when interpreted geometrically via (2.4) and stereographic projection, yields the operator T_5 described in the introduction.

THEOREM 2.1. *For a prime* $p \equiv 1$ (4),

$$\lambda_1(T_p) \leqq 2\sqrt{p}\,.$$

We prove this theorem by means of the following three lemmas. To begin with we introduce the general Hecke operator:

$$T_n f(\zeta) = \frac{1}{2} \sum_{\substack{\alpha \equiv 1(2) \\ N(\alpha)=n}} f(\alpha\zeta). \tag{2.6}$$

LEMMA 2.2. $T_{p^\nu} = l_\nu(T_p)$, *where the* l_ν *are the Chebychev polynomials of second kind defined in* I, (1.18). *In terms of the complex variable* $\theta = \arccos(\lambda/2\sqrt{p})$,

$$l_\nu(\lambda) = p^{\nu/2}\frac{\sin(\nu+1)\theta}{\sin\theta}.$$

This will follow from

LEMMA 2.3. *Every* $\beta \in H(\mathbb{Z})$ *with* $N(\beta) = p^k$ *has a unique representation*

$$\beta = p^l \varepsilon R_m(\alpha_1, \cdots, \bar{\alpha}_\sigma),$$

where $l \leqq \frac{1}{2}k$, $m + 2l = k$, *and* R_m *is a reduced word of length* m *in* $\alpha_1, \cdots, \bar{\alpha}_\sigma$.

Proof of Lemma 2.2: If $\beta \equiv 1\,(2)$ in Lemma 2.3, then since $\alpha_i \equiv 1\,(2)$ we see that $\varepsilon \equiv 1\,(2)$; whence $\varepsilon = \pm 1$. Thus every $\beta \equiv 1\,(2)$ with $N(\beta) = p^k$ is expressable uniquely in the form

$$\beta = \pm p^l R_m(\alpha_1, \cdots, \bar{\alpha}_\sigma), \tag{2.7}$$

where $2l + m = k$. We can therefore write

$$T_{p^k} f(\zeta) = \frac{1}{2} \sum_{\substack{\alpha \equiv 1(2) \\ N(\alpha) = p^k}} f(\alpha\zeta) \tag{2.8}$$
$$= \sum_{l \leq k/2} \sum_{R_{k-2l}(\alpha_1, \cdots, \bar{\alpha}_\sigma)} f(R\zeta).$$

The inner sum over all reduced words of length $k - 2l$ in $\alpha_1, \cdots, \bar{\alpha}_\sigma$ is just the sum over the shell at distance $k - 2l$ in the tree, using the terminology of I, Chapter 1. That the right-hand side of (2.8) is $l_k(T_p)$ follows from the definition of l_k.

Proof of Lemma 2.3 (see also [5]): We begin with the existence of such a factorization. Dickson [2] has shown that the odd elements (i.e., those with $N(\alpha)$ odd) of $H(\mathbb{Z})$ form a left and right Euclidean ring. Furthermore, $\alpha \in H(\mathbb{Z})$ is prime if and only if $N(\alpha)$ is prime in $\mathbb{Z}$. Now β is odd and $N(\beta) = p^k$. We may therefore write $\beta = \gamma\delta$, where $N(\gamma) = p^{k-1}$, $N(\delta) = p$. By using a unit ε and the definition of the set S_p we have

$$\beta = \gamma\varepsilon\alpha \quad \text{with} \quad \alpha \in S_p.$$

Iterating this we obtain $\beta = \varepsilon s_1 s_2 \cdots s_k$ with $s_j \in S_p$. Carrying out cancellations gives the derived factorization. To prove uniqueness we count the number of such factorizations. By (2.3) this is

$$8\Big(\sum_{0 \leq l < k/2} (p+1) p^{k-2l-1} + \delta(k) \Big),$$

where $\delta(k) = 1$ if k is even and $\delta(k) = 0$ if k is odd. Summing this gives $8((p^{k+1} - 1)/(p - 1))$. However, from (2.1) this is precisely the number of elements of norm p^k. This proves the uniqueness.

Next we check the action of T_p on spherical harmonics of degree m, $H_m(S^2)$. Let $u \in H_m(S^2)$, $m \neq 0$, be fixed.

LEMMA 2.4. *For a fixed $\zeta \in S^2$, the function $\theta(z)$, $z \in \mathfrak{h} = \{z \mid \mathscr{I}m\, z > 0\}$, defined by*

$$\theta(z) = \sum_{\substack{\alpha \equiv 2(4) \\ \alpha \in H(\mathbb{Z})}} N(\alpha)^m u(\alpha\zeta) e^{2\pi i N(\alpha) z/16},$$

is a holomorphic cusp form of weight $2 + 2m$ for the congruence subgroup $\Gamma(4)$ of the modular group.

Proof: We will need a theorem of Schoenberg. For an account of this theorem see Ogg [10] whose notation we adopt here. Let

$$Q(x) = x'x = \frac{1}{2}x'Ax \quad \text{with} \quad A = \begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 2 \end{pmatrix}$$

be the standard quadratic form in 4 variables. The discriminant of this form is 16 and its level (Stuffe) is $N = 4$. Schoenberg's theorem states that the function

$$\theta(z, h) = \sum_{\substack{n \equiv h(N) \\ n \in \mathbf{Z}^4}} P(n) \exp\{2\pi i Q(n) z / N^2\},$$

where $Ah \equiv 0\ (N)$ and $P(x)$ is a homogeneous harmonic polynomial of degree $\nu \geqq 1$ (in four variables), is a cusp form for $\Gamma(N)$ of weight $2 + \nu$. Thus it suffices to show that $u(\alpha\zeta)N(\alpha)^m$ with $\alpha = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ is such a homogeneous polynomial in a, b, c, d. Without loss of generality we may assume that ζ is the South Pole in S^2. A simple calculation then gives

$$u(\alpha\zeta)N(\alpha)^m = N(\alpha)^m u\left(\frac{1}{N(\alpha)}(2(ca - db), 2(da + bc), c^2 + d^2 - a^2 - b^2)\right).$$

Now $u(\zeta)$ is the restriction of a homogeneous harmonic polynomial of degree m in three variables to the unit sphere. Such a polynomial can be written as the sum of polynomials of the form

$$(\xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3)^m \quad \text{with} \quad \sum_{j=1}^{3} \xi_j^2 = 0.$$

We may therefore assume u to be of this form. Then

$$N(\alpha)^m u(\alpha\zeta) = \left(2\xi_1(ca - db) + 2\xi_2(ad + bc) + \xi_3(c^2 + d^2 - a^2 - b^2)\right)^m.$$

This is clearly homogeneous of degree $2m$ in a, b, c, d. It is also harmonic as is easily checked using $\sum_{j=1}^{3}\xi_j^2 = 0$.

We return now to the proof of Theorem 2.1. We write the series

$$\sum_{\substack{\alpha \equiv 2(4) \\ \alpha \in H(\mathbf{Z})}} N(\alpha)^m u(\alpha\zeta) e^{2\pi i N(\alpha) z/16}$$

as

$$\sum_{\nu=1}^{\infty} a_\nu e^{2\pi i \nu z/16},$$

where

$$a_\nu = \nu^m \sum_{\substack{\alpha \equiv 2(4) \\ N(\alpha) = \nu}} u(\alpha\zeta).$$

We may invoke the deep Ramanujan estimates which have been established by Deligne [1] for cusp forms of this type. These give

$$|a_\nu| \ll_\varepsilon \nu^{(2+2m)/2-1/2+\varepsilon}$$

$$= \nu^{m+1/2+\varepsilon} \quad \text{for all} \quad \varepsilon > 0.$$

(The implied constant depends on u and ζ.) Thus we have

$$\left| \sum_{\substack{\alpha \equiv 2(4) \\ \alpha \in H(\mathbf{Z}) \\ N(\alpha) = \nu}} u(\alpha\zeta) \right| \ll_\varepsilon \nu^{1/2+\varepsilon}. \tag{2.9}$$

Writing $\mu = \frac{1}{4}\nu$ we see that $\mu \in \mathbf{Z}$ and, if $\beta = \frac{1}{2}\alpha$ in the above sum, then $\beta \in H(\mathbf{Z})$, $N(\beta) = \mu$ and $\beta \equiv 1$ (2). The relation (2.9) then becomes

$$\left| \sum_{\substack{\beta \equiv 1(2) \\ N(\beta) = \mu}} u(\beta\zeta) \right| \ll_\varepsilon \mu^{1/2+\varepsilon}, \qquad \varepsilon > 0, \tag{2.10}$$

for $\mu \in \mathbf{Z}_+$. In particular, for $\mu = p^k$, $p \equiv 1$ (4) we have, by Lemma 2.2,

$$|T_{p^k} u(\zeta)| = |l_\nu(T_p) u(\zeta)| \ll_\varepsilon p^{k/2+\varepsilon k}. \tag{2.11}$$

We may now complete the proof of Theorem 2.1. Let $u \in H_m$, $m \neq 0$, be an eigenfunction of T_p with eigenvalue λ. Choose $\zeta \in S^2$ such that $u(\zeta) \neq 0$. Applying (2.11) and using Lemma 2.2, we obtain

$$\left| \frac{p^{k/2} \sin \theta(k+1)}{\sin \theta} \right| \ll_\varepsilon p^{k/2+\varepsilon k}$$

for $\varepsilon > 0$, where $\lambda = 2\sqrt{p} \cos \theta$. Hence

$$\left| \frac{\sin \theta(k+1)}{\sin \theta} \right| \ll_\varepsilon p^{\varepsilon k}, \qquad \varepsilon > 0, k > 0.$$

Since ε is arbitrary, this implies that θ is real and hence that

$$|\lambda| \leqq 2\sqrt{p}\,.$$

We end Section 2 with the following remarks.

1. From the discussion and in particular the unique factorization of quaternions into any prescribed order it follows that, for $p, q \equiv 1$ (4), prime $p \neq q$,

$$T_p T_q = T_{pq} = T_q T_p.$$

Thus the Hecke operators T_p on $L^2(S^2)$ commute with one another and hence may be simultaneously diagonalized on each of the spaces H_m.

2. Let σ denote a permutation of the integers (1, 2, 3) and set

$$P_\sigma \alpha = a_0 + a_{\sigma(1)}\mathbf{i} + a_{\sigma(2)}\mathbf{j} + a_{\sigma(3)}\mathbf{k};$$

then it is clear that P_σ permutes the α_i in S_p and hence that

$$P_\sigma T_S = T_S P_\sigma.$$

The action of P_σ in $\mathbb{R}^3$ consists simply of permuting the co-ordinate axes. Thus T_S commutes with the symmetry group of the cube. Since the irreducible unitary representations of this group have orders 1, 2 and 3, we can expect degeneracies in the eigenvalues of the T_p of the same orders.

3. The Lattice Method

In this section we introduce a different scheme for ordering rotations in SO(3) which enables us to obtain many more examples of well-distributed points in S^2. We start with two groups: G_1 which denotes either PSL(2, $\mathbb{R}$) or PGL(2, Q_p), and G_2 which is SO(3). We use the notation Q_∞ to stand for $\mathbb{R}$ (i.e., "$p = \infty$"). Let Γ be a discrete co-compact subgroup of $G = G_1 \times G_2$. Since G_2 is compact the projection of Γ on G_1, call it Γ_1, is discrete while its projection on G_2, call it Γ_2, is typically dense. Since Γ is a subgroup of $G_1 \times G_2$ it acts on the homogeneous space $X \times S^2 \triangleq G_1/K_1 \times G_2/K_2$, where K_1 is a maximal compact subgroup of G_1 and $K_2 = \mathrm{SO}(2)$. The coset space X is the hyperbolic plane in the case $p = \infty$ and it is a homogeneous tree of degree $p + 1$ in the case $p < \infty$ (see Serre [13]). Our method for ordering the elements of Γ_2 is as follows:

(i) For $p < \infty$, fix $x, \xi \in X$ and consider all $\gamma \in \Gamma$ for which $d_1(\gamma x, \xi) \leqq n$. Here d_1 is the distance on the tree. Since Γ_1 is discrete, this set is finite. Our points in G_2 are then the projections of this finite set of γ's in Γ_2. Because this method of ordering depends on the lattice Γ_1 in G_1 and its action on X, we call this the *lattice method*. Only in special cases does this give the ordering of Γ_2 by word length in the generators. Obviously the lattice method is more general.

(ii) For the case of $\mathbb{R}$ we choose $x, \xi \in \mathfrak{h}$, the upper half-plane, and consider all $\gamma \in \Gamma$ for which $d_1(\gamma x, \xi) \leqq T$ (here d_1 is the distance in $\mathfrak{h}$). Again these are finite sets and their projections in Γ_2 furnish us with an effective ordering for Γ_2.

3.1. The p-adic case ($p < \infty$). In order to study the distribution properties of these $\gamma_2 \in \Gamma_2$ we introduce the harmonic analysis of $L^2(\Gamma \setminus X \times S^2)$. That is, consider all f: $X \times S^2 \to \mathbb{C}$ satisfying

(i) $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$ and $z \in X \times S^2$,

(ii) $\sum_{x \in \Gamma_1 \setminus X} \int_{S^2} |f((x, y))|^2 \, d\omega(y) < \infty$.

Next we choose an orthonormal basis $\phi_j(x, y)$ for this space which is an eigenbasis for the commuting selfadjoint operators, Δ the Laplacian on S^2 and

Δ_p "the Laplacian on the tree":

$$\Delta_p f((x, y)) = \sum_{d(\xi, x)=1} f((\xi, y)); \tag{3.1}$$

here $x, \xi \in X$ and $y \in S^2$; Δ_p is a "Hecke operator" but not a Hecke operator in terms of the generators of Γ. Thus we may write, for $j = 0, 1, \cdots$,

$$\begin{aligned} \Delta\phi_j(x, y) + \mu_j\phi_j(x, y) &= 0, \\ \Delta_p\phi_j(x, y) + \lambda_j\phi_j(x, y) &= 0; \end{aligned} \tag{3.2}$$

$\phi_0(x, y)$ is the constant function so that $\lambda_0 = p + 1$, $\mu_0 = 0$. The numbers μ_j are special in that they are eigenvalues of Δ on S^2 and hence must be of the form $m(m + 1)$, $m \geqq 0$, $m \in \mathbb{Z}$. We may therefore group the ϕ_j according to the values of μ_j and rewrite them as $\phi_{r,m}$, $r = 1, 2, \cdots, l_m$, where $\Delta\phi_{r,m} + m(m + 1)\phi_{r,m} = 0$; l_m will be seen to be $(2m + 1) \cdot |\Gamma_1 \setminus X|$. We expand $\phi_{r,m}$ in a basis of H_m:

$$\phi_{r,m}(x, y) = \sum_{|\nu| \leqq m} a^{\nu}_{r,m}(x) Y^{\nu}_m(y), \tag{3.3}$$

the Y's being the standard spherical and ultra-spherical functions normalized to have $L_2(S^2)$ norm 1. The condition $\phi_{r,m}(\gamma z) = \phi_{r,m}(z)$ translates to

$$\mathbf{a}_{r,m}(\gamma x) = R(\gamma)\mathbf{a}_{r,m}(x), \tag{3.4}$$

where $\mathbf{a}_{r,m}$ is the vector $(a^{-m}_{r,m}, \cdots, a^{m}_{r,m})^{\mathrm{tr}}$ and R is the obvious representation of Γ in H_m. Equations (3.2) then become

$$\Delta_p \mathbf{a}_{r,m} + \lambda_{r,m}\mathbf{a}_{r,m} = 0. \tag{3.5}$$

As in I, we write

$$\lambda_{r,m} = 2\sqrt{p}\cos(\theta_{r,m}); \tag{3.5$'$}$$

Δ_p acts componentwise and this vector-valued operator is selfadjoint with respect to the inner product

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{x \in \Gamma_1 \setminus X} \mathbf{a}(x) \cdot \mathbf{b}(x)^*. \tag{3.6}$$

The two eigenvalue problems (3.2) and (3.5) are easily seen to be the same if (3.2) is restricted to the eigenspace with $\mu_j = m(m + 1)$. From this follows that $l_m = (2m + 1)|\Gamma_1 \setminus X|$. The orthonormality of the $\phi_{r,m}$ is equivalent to

$$\sum_{x \in \Gamma_1 \setminus X} \sum_{|\nu| \leqq m} a^{\nu}_{r,m}(x)\overline{a^{\nu}_{s,m}(x)} = \delta_{r,s}. \tag{3.7}$$

Viewing $a^{\nu}_{r,m}(x)$, indexed by $r = 1, 2, \cdots, l_m$, and the pairs $\{(\nu, x); |\nu| \leq m, x \in \Gamma_1 \backslash X\}$ as an $l_m \times l_m$ matrix (specifically $A_{\alpha\beta} = a^{\nu}_{\alpha,m}(x)$ for $\alpha = 1, \cdots, l_m$, $\beta = (\nu, x)$), (3.7) asserts that A is unitary. We conclude that

$$\sum_r a^{\nu}_{r,m}(x)\overline{a^{\nu_1}_{r,m}(x_1)} = \delta_{x,x_1}\delta_{\nu,\nu_1}. \tag{3.8}$$

For a fixed point $x \in X$ we consider the functions $\phi_{r,m}(x, y)$ as functions on S^2. These are not orthogonal on S^2; however we still have

LEMMA 3.1.
(a) $\sum_{r=1}^{l_m}|\phi_{r,m}(z)|^2 = (2m+1)/4\pi$;
(b) *if* $f, g \in L^2(S^2)$ *and*

$$\hat{f}_x(r, m) = \int_{S^2} f(y)\phi_{r,m}(x, y)\, d\omega(y),$$

$$\hat{g}_x(r, m) = \int_{S^2} g(y)\phi_{r,m}(x, y)\, d\omega(y),$$

then

$$\int_{S^2} f(y)\overline{g(y)}\, d\omega(y) = \sum_{m=0}^{\infty}\sum_{r=1}^{l_m}\hat{f}_x(r, m)\overline{\hat{g}_x(r, m)}.$$

Remark. The relation (a) is a generalization of (2.11) in I.

Proof: Writing ϕ as in (3.3) we have

$$\sum_r |\phi_{r,m}(x, y)|^2 = \sum_r \left|\sum_{|\nu|\leq m} a^{\nu}_{r,m}(x)Y^{\nu}_m(y)\right|^2$$

$$= \sum_r \sum_{\nu_1,\nu_2} a^{\nu_1}_{r,m}(x)\overline{a^{\nu_2}_{r,m}(x)}\, Y^{\nu_1}_m(y)\overline{Y^{\nu_2}_m(y)}.$$

Making use of (3.8) and then of (2.11) in I, we get

$$\sum_r |\phi_{r,m}(x, y)|^2 = \sum_\nu |Y^{\nu}_m(y)|^2 = \frac{2m+1}{4\pi}.$$

To prove (b) we note that

$$\hat{f}(r, m) = \int_{S^2} f(y)\left(\sum_{|\nu|\leq m} a^{\nu}_{r,m}(x)Y^{\nu}_m(y)\right) d\omega(y)$$

$$= \sum_{|\nu|\leq m} a^{\nu}_{r,m}(x)\tilde{f}(\nu, m),$$

$\tilde{f}(\nu, m)$ being the coefficient of f relative to the orthonormal basis Y_m^ν. Similarly,

$$\hat{g}(r,m) = \sum_{|\nu_1| \leq m} a_{r,m}^{\nu_1}(x)\tilde{g}(\nu_1, m).$$

Thus

$$\sum_r \hat{f}(r,m)\overline{\hat{g}(r,m)} = \sum_r \sum_{\nu,\nu_1} a_{r,m}^{\nu}(x)\overline{a_{r,m}^{\nu_1}(x)}\,\tilde{f}(\nu,m)\cdot\overline{\tilde{g}(\nu_1,m)}$$

$$= \sum_\nu \tilde{f}(\nu,m)\overline{\tilde{g}(\nu,m)}.$$

Summing over m we obtain (b).

With these preliminaries out of the way we can now prove

THEOREM 3.2. *If $\theta_{r,m}$ is real for $(m,r) \neq (0,0)$, then the spherical cap discrepancy of the N-points*[1] $\{\gamma_2 y;\ d(\gamma_1 x, \xi) = n\}$ *satisfies*

$$D \ll \frac{(\log N)^{2/3}}{N^{1/3}}. \tag{3.9}$$

Proof: Let $k(z,\zeta)$ be a point pair invariant function on $X \times S^2$ of the form

$$k(z,\zeta) = k_n(x,\xi)\tilde{k}(y,\eta), \tag{3.10}$$

where $z = (x, y)$, $\zeta = (\xi, \eta)$ and

$$k_n(x,\xi) = \begin{cases} 1 & \text{if} \quad d_1(x,\xi) \leq n, \\ 0 & \quad \text{otherwise;} \end{cases}$$

$\tilde{k}$ is an arbitrary point pair invariant on S^2. The automorphic function

$$K(z,\zeta) = \sum_{\gamma\in\Gamma} k(z,\gamma\zeta) \quad \text{is in} \quad L^2(\Gamma\backslash X \times S^2)$$

and so may be expanded as

$$K(z,\zeta) = \sum_{m=0}^{\infty} \sum_{r=1}^{l_m} h(\lambda_{r,m}, \mu_{r,m})\phi_{r,m}(z)\phi_{r,m}(\zeta). \tag{3.11}$$

Here

$$\int_{\Gamma\backslash X\times S^2} K(z,\zeta)\phi_{r,m}(\zeta)\,d\zeta = h(\lambda_{r,m}, \mu_{r,m})\phi_{r,m}(z),$$

[1] In this case, $N = (p+1)p^{n-1}/|\Gamma_1\backslash X| + O(p^{n/2})$.

and after "unfolding" K, this can be rewritten as

$$\sum_{\xi\in X}\int_{S^2} k_n(x,\xi)\tilde{k}(y,\eta)\phi_{r,m}(\xi,\eta)\,d\omega(\eta) = h(\lambda_{r,m},\mu_{r,m})\phi_{r,m}(x,y). \tag{3.12}$$

By the harmonic analysis on the tree in Section 1 of I applied to Δ_p,

$$\sum_{\xi\in X} k_n(x,\xi)\phi_{r,m}(\xi,\eta) = k_n'(\theta_{r,m})\phi_{r,m}(x,\eta).$$

Hence the left-hand side of (3.12) is

$$k_n'(\theta_{m,r})\int_{S^2}\tilde{k}(y,\eta)\phi_{r,m}(x,\eta)\,d\omega(\eta) = k_n'(\theta_{r,m})\hat{\tilde{k}}(m)\phi_{r,m}(x,y).$$

Thus

$$h(\lambda_{r,m},\mu_{r,m}) = k_n'(\theta_{r,m})\hat{\tilde{k}}(m), \tag{3.13}$$

and inserting (3.13) into (3.11) we get

$$K(z,\zeta) = \sum_{m=0}^{\infty}\sum_{r=1}^{l_m} k_n'(\theta_{r,m})\hat{\tilde{k}}(m)\phi_{r,m}(x,y)\phi_{r,m}(\xi,\eta). \tag{3.14}$$

On the other hand,

$$K(z,\zeta) = \sum_{d(\gamma_1 x,\xi)\leqq n}\tilde{k}(\gamma_2 y,\eta).$$

Combining this with (3.14) we obtain the key identity:

$$\sum_{d(\gamma_1 x,\xi)\leqq n}\tilde{k}(\gamma_2 y,\eta) = \sum_{m=0}^{\infty}\sum_{r=1}^{l_m} k_n'(\theta_{r,m})\hat{\tilde{k}}(m)\phi_{r,m}(x,y)\phi_{r,m}(\xi,\eta). \tag{3.15}$$

The set-up here is now almost exactly the same as that for the proof of Theorem 2.5 in I. Since we have assumed for $(r,m)\neq(0,0)$ (i.e., $\phi_{r,m}$ not the constant function) that $\theta_{r,m}$ is real, we have, as in Section 1 of I,

$$\sum_{d(\gamma_1 x,\xi)\leqq n}\tilde{k}(\gamma_2 y,\eta) = \frac{(p+1)p^{n-1}\hat{\tilde{k}}(0)}{|\Gamma_1\backslash X|} + O\left(p^{n/2}n\sum_{\substack{(r,m)\\ m>0}}\left|\hat{\tilde{k}}(m)\phi_{r,m}(z)\phi_{r,m}(\zeta)\right|\right).$$

The partial isometry result in Lemma 3.1 then allows us to proceed exactly as in the proof of Theorem 2.5 in I to establish the bound (3.9).

The bounds for T_N in Theorem 2.2 of I also hold for this sequence. The proof is the same as before except that now one uses (3.15) and the partial isometry in Lemma 3.1. One also obtains the analogues of Theorems 2.7 and 2.8 of I as follows: Let $\xi \in X$ be fixed. For $f(y)$ an arbitrary function on S^2 we define

$$F(z) = k_n(x, \xi) f(y)$$

and

$$G(z) = \sum_{\gamma \in \Gamma} F(\gamma z) = \sum_{d(\gamma_1 x, \xi) \leq n} f(\gamma_2 y). \tag{3.16}$$

G is Γ automorphic and hence may be expanded as

$$G(z) = \sum_{r, m} \hat{G}(r, m) \phi_{r, m}(x, y), \tag{3.17}$$

where

$$\begin{aligned} \hat{G}(m, r) &= \sum_{x \in \Gamma_1 \backslash X} \int_{S^2} G(z) \phi_{r, m}(z)\, d\omega(y) \\ &= \sum_{x \in X} \int_{S^2} k_n(x, \xi) f(y) \phi_{r, m}(x, y)\, d\omega(y) \\ &= k_n'(\theta_{r, m}) \hat{f}_\xi(r, m). \end{aligned}$$

Thus

$$\sum_{d(\gamma_1 x, \xi) \leq n} f(\gamma_2 y) = \sum_{m, r} k_n'(\theta_{r, m}) \hat{f}_\xi(r, m) \phi_{r, m}(x, y). \tag{3.18}$$

The analogues of Theorems 2.7 and 2.8 of I now follow from the previous arguments and Lemma 3.1.

In Section 4 we shall give examples of quaternion groups Γ for which the assumptions in Theorem 3.2 are satisfied.

3.2. The Hyperbolic case ($p = \infty$). Next we consider the situation where $G_1 = \mathrm{PSL}(2, \mathbb{R})$, so that X is the hyperbolic plane $\mathfrak{h}$. In this case our examples of discrete groups $\Gamma \hookrightarrow G_1 \times G_2$ are as before quaternion groups. Let K be a real quadratic extension of Q. Let D be a quaternion algebra over K which splits at one archimedean place and which ramifies at the other. If Γ is chosen to be the group of elements, in a maximal order of D, of norm 1, then $\Gamma \hookrightarrow \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{SO}(3)$ discretely and is co-compact. Our examples for the rest of this section will be such Γ's or congruence subgroups thereof.

In this set-up our operators are Δ_1, Δ_2, where Δ_1 is the Laplacian on $\mathfrak{h}$ and Δ_2 on S^2. As operators on $L^2(\Gamma \setminus \mathfrak{h} \times S^2)$ they have a common set of eigenfunctions which form an orthonormal basis:

$$\begin{aligned} &\Delta_1\phi_j + \lambda_j\phi_j = 0, \\ &\Delta_2\phi_j + \mu_j\phi_j = 0. \end{aligned} \tag{3.19}$$

In Section 4 we show, using the theory of automorphic forms and in particular the Jacquet-Langlands correspondence and a bound of Selberg [12], that for $(\lambda_j, \mu_j) \neq (0,0)$ we have

$$\lambda_j \geqq \tfrac{3}{16}. \tag{3.20}$$

If the general GL(2) "Ramanujan conjectures" were true, then we would have

$$\lambda_1 \geqq \tfrac{1}{4}. \tag{3.20$'$}$$

We turn to the analysis of the distribution of the point $\{\gamma_2 y\}$. One could proceed exactly as we did in the p-adic case. However because the measure on $\mathfrak{h}$ is not atomic we lose considerable leverage and the results are quite weak. A different approach, using the techniques from a paper by Lax and Phillips [8] on the distribution of lattice points, leads to better results. Nevertheless they still are quite a bit weaker than the p-adic results. We now describe this second method.

THEOREM 3.3. *Let Γ be as above, and suppose χ_A is the characteristic function of a set $A \subset S^2$ satisfying condition $(*)$ of Section 2, Part I. Then*

$$\Delta(A,T) \triangleq \left| \frac{1}{N_T} \sum_{d(\gamma_1 x, \xi) < T} \chi_A(\gamma_2 y) - |A| \right| \underset{A}{\ll} T^{1/5} e^{-T/10}. \tag{3.21}$$

Here N_T denotes the number of group elements $\{\gamma_1; d(\gamma_1 x, \xi) \leqq T\}$ which asymptotically is

$$N_T \sim \pi e^T / |\Gamma_1 \setminus \mathfrak{h}|. \tag{3.22}$$

Proof: As before, if we limit ourselves to the eigenspace of Δ_2 made up of spherical harmonics of degree m, we can transform the automorphic conditions (i) and (ii) of subsection 3.1 to the condition (3.4) on square integrable vector-valued functions in $\Gamma_1 \setminus \mathfrak{h}$ with values in a $(2m+1)$-dimensional vector space V. We then consider the action of the wave operator on automorphic functions of this kind:

$$\begin{aligned} &u_{tt} = Lu, \qquad L = \Delta_1 + \tfrac{1}{4}, \\ &u(x,0) = 0, \qquad u_t(x,0) = 2\pi f(x). \end{aligned} \tag{3.23}$$

For f we choose the automorphic vector-valued function

$$f(x) = \sum_{\beta_1 \in \Gamma_1} R(\beta) k_\varepsilon(\beta_1^{-1} x, \xi)\mathbf{e}, \tag{3.24}$$

where k_ε is again a two-point invariant function on $\mathfrak{h}$ approximating the δ-function, $\xi \in \Gamma_1 \setminus \mathfrak{h}$ and $\mathbf{e}$ is a fixed vector in V.

Denote the eigenpairs of L, corresponding to eigenvalues of Δ_1 of magnitude less than $\frac{1}{4}$, by

$$\left\{ \left(\phi_j, \nu_j^2 \right), \ j = 1, \cdots, r_m \right\}.$$

Note that $\nu_j^2 = \frac{1}{4} - \lambda_j$ for $j \leq r_m$. Then the solution of (3.23) can be written as

$$u(x, t) = \sum \left(a_j \exp\{\nu_j t\} + b_j \exp\{-\nu_j t\} \right) \phi_j(x) + v(x, t), \tag{3.25}$$

where

$$a_j = -b_j = \frac{\pi}{\nu_j} \int_{\Gamma_1 \setminus \mathfrak{h}} f(x) \phi_j(x)\, dx = \frac{\pi}{\nu_j} \phi_j(\xi) + O(\varepsilon). \tag{3.26}$$

We note that the constant associated with $O(\varepsilon)$ depends only on $\nabla \phi_j / \nu_j$, which can be estimated by Sobolev inequalities in terms of

$$\frac{1}{\nu_j} \|\nabla \phi_j\| = \frac{1}{\nu_j} \left(\Delta \phi_j, \phi_j \right)^{1/2} = \|\phi_j\| = 1$$

and

$$\frac{1}{\nu_j} \|\Delta^2 \phi_j\| = \nu_j^3 \leqq \tfrac{1}{8}.$$

It is therefore independent of both j and m; $v(x, t)$ is a solution of the wave equation orthogonal to the ϕ_j.

Next we avail ourselves of the relation (5.12) of [8]:

$$\begin{aligned} I(T, \varepsilon) &= \frac{\sqrt{2}}{\pi} \int_0^T (\cosh T - \cosh t)^{-1/2} \sinh t \, u(x, t)\, dt \\ &= \int_{d(x', x) \leqq T} f(x')\, dx' \\ &= \sum_{d(\gamma_1 x, \xi) \leqq T} R(\gamma)\mathbf{e} + O(\varepsilon e^T) + O(e^{2T/3}). \end{aligned} \tag{3.27}$$

The error term results from the fact that k_ε is an approximation to the δ-function with support of radius ε and that in an annulus of radius T and width ε there can be $O(\varepsilon e^T)$ lattice points to within an error of $O(e^{2T/3})$. Since R is a unitary representation, this too is independent of m.

Substituting (3.25) and (3.26) for $u(x, t)$ in (3.27), we find (see page 321 of [8]) that, for $m > 0$,

$$I(T, \varepsilon) = \Sigma(T) + r_m O(\varepsilon \exp\{(\nu_1 + \tfrac{1}{2})T\}) + w(x, T), \tag{3.28}$$

where

$$\Sigma(T) = \sum_{j=1}^{r_m} \sqrt{\pi} \frac{(\nu_j - 1)!}{(\nu_j + 1)!} \phi_j(\xi)\phi_j(x)\exp\{(\nu_j + \tfrac{1}{2})T\} \tag{3.29}$$

and

$$w(x, T) = \frac{\sqrt{2}}{\pi} \int_0^T (\cosh T - \cosh t)^{-1/2} \sinh t \, v(x, t)\, dt. \tag{3.29$'$}$$

The main contribution to the error term comes from (3.26) and this time the error does depend on m, having r_m as a factor.

It remains to estimate $w(x, T)$ and here the development[2] in [8] yields, after a few obvious modifications,

$$|w(x, T)| = O\left(T^{1/2} \frac{|\log \varepsilon|}{\varepsilon^{1/2}} e^{T/2}\right); \tag{3.30}$$

moreover, this bound does not depend on m. Combining this with (3.27) and (3.28) we get

$$\begin{aligned} \left| \sum_{d(\gamma_1 x, \xi) \leqq T} R(\gamma)\mathbf{e} \right| &\leqq |\Sigma(T)| + O(\varepsilon e^T) + O(e^{2T/3}) \\ &\quad + r_m O(\varepsilon \exp\{(\nu_1 + \tfrac{1}{2})T\}) + O\left(T^{1/2} \frac{|\log \varepsilon|}{\varepsilon^{1/2}} e^{T/2}\right). \end{aligned} \tag{3.31}$$

For $m > 0$,

$$|\Sigma(T)| \leqq r_m O(\exp\{(\nu_1 + \tfrac{1}{2})T\}). \tag{3.31$'$}$$

[2] The relation (5.80) in [12] is not quite correct and should be replaced by

$$\|M^{1/2+\varepsilon}V\| \leq \text{const.} \frac{e^{T/2}|\log \alpha|^{1/2}}{\alpha^{1/2+\varepsilon}} \left[\frac{e^{T/2}\alpha^{1/2}}{(\cosh T - \cosh S)^{1/2}} + \frac{Te^{-T/2}(\cosh T - \cosh S)^{1/2}}{\alpha^{1/2}} \right].$$

Hence setting $\varepsilon = Te^{-T/3}$ we see that, for $m > 0$,

$$\Big|\sum_{d(\gamma_1 x,\xi)\leqq T} R(\gamma)\mathbf{e}\Big| = O(Te^{2T/3}) + r_m O\big(\exp\{(\nu_1 + \tfrac{1}{2})T\}\big). \tag{3.32}$$

Using the Selberg bound (3.20) for ν_1, i.e., $\nu_1 \leqq \frac{1}{4}$, we see that

$$\Big|\sum_{d(\gamma_1 x,\xi)\leqq T} R(\gamma)\mathbf{e}\Big| = r_m O(e^{3T/4}). \tag{3.33}$$

In terms of the orthonormal basis $\{Y_m^i\}$ in H_m, the vector $\mathbf{e} = \{e_i\}$ corresponds to

$$f_m(y) = \sum e_i Y_m^i(y) \in H_m$$

and $R(\gamma)\mathbf{e}$ to $f_m(\gamma y)$. Hence making use of the relation (2.11) of I we see that

$$\Big|\sum_{d(\gamma_1 x,\xi)\leqq T} f_m(\gamma y)\Big| = \Big\|\sum_{d(\gamma_1 x,\xi)\leqq T} R(\gamma)\mathbf{e}\Big\|\sqrt{\frac{2m+1}{4\pi}} \leqq \sqrt{m}\, r_m O(e^{3T/4}). \tag{3.34}$$

We also need an upper bound for r_m, that is the number of automorphic vector-valued eigenvalues of Δ_1 of magnitude less than $\frac{1}{4}$. By the minimax principle this will be less than the number of eigenvalues for vector-valued functions on $\Gamma_1/\mathfrak{h}$ with each component having a free boundary. Thus

$$r_m \leqq c(2m+1). \tag{3.35}$$

We now proceed as in the proof of Theorem 2.7 of I. We begin by mollifying χ_{A_ν}:

$$\chi_{A_\nu}^\varepsilon(y) = \int k_\varepsilon(y, y)\chi_{A_\nu}(y)\,dy$$

and then project $\chi_{A_\nu}^\varepsilon$ into H_m:

$$\begin{aligned} F_m(\chi_{A_\nu}^\varepsilon) &= \frac{2m+1}{4\pi}\int P_m(y\cdot\eta)\chi_{A_\nu}^\varepsilon(\eta)\,d\omega(\eta) \\ &= \frac{2m+1}{4\pi}\int \frac{\Delta^\alpha P_m(y\cdot\eta)}{[m(m+1)]^\alpha}\chi_{A_\nu}^\varepsilon(\eta)\,d\omega(\eta) \\ &= \frac{1}{[m(m+1)]^\alpha}F_m(\Delta^\alpha\chi_{A_\nu}^\varepsilon); \end{aligned} \tag{3.36}$$

here we have invoked the selfadjointness of Δ^α. Making use of (2.7) of I, it follows from (3.34) (3.35) and (3.36) that

$$I_T = \left| \frac{1}{N_T} \sum_{d(\gamma_1 x, \xi) \leqq T} \chi_A(\gamma y) - |A| \right|$$

$$\leqq c|A_2 \setminus A_1| + \sum_{\nu=1}^{2} e^{-T/4} \sum_{m=1}^{\infty} \frac{m^{3/2}}{m^{2\alpha}} \| F_m(\Delta^\alpha \chi^\varepsilon_{A_\nu}) \| \tag{3.37}$$

$$\leqq c'\varepsilon + \frac{c^4}{\sqrt{\alpha - 1}} \left(\sum_{\nu=1}^{2} \| \Delta^\alpha \chi^\varepsilon_{A_\nu} \| \right) e^{-T/4}.$$

Interpolating $\|\Delta^\alpha \chi^\varepsilon_{A_\nu}\|$ between $\alpha = 1$ and $\alpha = 2$, we find that

$$\|\Delta^\alpha \chi^\varepsilon_{A_\nu}\| \leqq \varepsilon^{1/2 - 2\alpha}.$$

Inserting this into (3.37) and setting $\alpha = 1 + 1/T$ and $\varepsilon = T^{1/5} e^{-T/10}$ we see that

$$I_T = O(T^{1/5} e^{-T/10}). \tag{3.38}$$

If the Ramanujan conjecture holds so that $\nu_1 = 0$ and $r_m = 0$, then

$$\left\| \sum_{d(\gamma_1 x, \xi) \leqq T} R(\gamma) \mathbf{e} \right\| \leqq O(T e^{2T/3}), \tag{3.33)$'$}$$

$$\left| \sum_{d(\gamma_1 x, \xi) \leqq T} f_m(\gamma y) \right| \leqq \sqrt{m}\, O(T e^{2T/3}), \tag{3.34)$'$}$$

$$I_T \leqq c'\varepsilon + \frac{c''}{\sqrt{2\alpha - 1}} \|\Delta^\alpha \chi^\varepsilon_{A_\pm}\| \, e^{-T/3}. \tag{3.37)$'$}$$

Interpolating between $\alpha = \frac{1}{2}$ and 1, we get $\|\Delta^\alpha \chi^\varepsilon_{A_\pm}\| \leqq c/\varepsilon^{1/4 + \alpha/2}$. Setting $\alpha = \frac{1}{2} + 1/T$ and $\varepsilon = T e^{-2T/9}$ finally gives

$$I_T = O(T e^{-2T/9}). \tag{3.38)$'$}$$

Although this is an improvement on (3.38) it is still considerably worse than the analogous p-adic result.

4. Lattice Method Groups

In this section we present examples of groups $\Gamma \hookrightarrow \mathrm{PGL}(2, Q_p) \times \mathrm{SO}(3)$ and $\Gamma \hookrightarrow \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{SO}(3)$ which satisfy the conditions of Theorems 3.2 and 3.3. We shall make use of the general representation-theoretic formulation as well as results of the theory of automorphic forms (as described for example in Gelbart [4]).

We begin with the p-adic case: Let D be a definite quaternion algebra defined over Q, p a prime at which D splits and let G' be the algebraic group of the invertible elements of D. Set $\tilde{\Gamma}$ equal to the group $G'(\mathbb{Z}[1/p])$. The diagonal map

$$G\left(\mathbb{Z}\left[\frac{1}{p}\right]\right) \to G'(Q_p) \times G'(\mathbb{R}) = \mathrm{PGL}(2, Q_p) \times (\mathrm{SU}(2) \times \mathbb{R}^*)$$

furnishes us with a map $\tilde{\Gamma} \to \mathrm{PGL}(2, Q_p) \times \mathrm{SO}(3)$. We denote the image of $\tilde{\Gamma}$ under this map by Γ. Such a Γ is discrete and co-compact (see Vignéras [14]).

We claim that Γ meets the requirements of Theorem 3.2. To see this, let $f(x, y)$ be a function on $\Gamma \backslash X \times S^2$ satisfying

$$\Delta_x f(x, y) = \lambda f(x, y), \qquad \Delta_y f(x, y) = \mu f(x, y). \tag{4.1}$$

Recall that X is the $p + 1$ regular tree identified with $\mathrm{PGL}(2, Q_p)/\mathrm{PGL}(2, \mathbb{Z}_p)$ and that Δ_x and Δ_y are, respectively, Laplacians on the tree and S^2.

THEOREM 4.1. *If $\mu \neq 0$ or if $\mu = 0$ and $\lambda \neq \pm(p + 1)$, then $|\lambda| \leqq 2\sqrt{p}$.*

Proof: Let $G'_{\mathbb{A}}$ be the adelic points of G'. The function f may be used to define an automorphic form on $G'_{\mathbb{A}}$ as follows: By the strong approximation theorem for the reduced norm 1 quaternions G_1 (cf. Kneser [7]) together with

$$\mathbb{A}_Q^* = Q^* \mathbb{R}_+^* \prod_p \mathbb{Z}_p^*, \tag{4.2}$$

we have

$$G'_{\mathbb{A}} = G'_Q G'_\infty \times G'_p \times \prod_{q \neq p} K_q, \tag{4.3}$$

where $K_q = G'(\mathbb{Z}_q)$, G'_p stands for $G'(Q_p)$, etc. Further since

$$\tilde{\Gamma} = G'_Q \cap \prod_{q \neq p} K_q, \tag{4.4}$$

we have

$$\Gamma \backslash X \times \mathrm{SU}(2) \cong \Gamma \backslash G'_p \times G'_\infty / K_p \mathbb{Z} \cong G'_Q \backslash G'_{\mathbb{A}} / \prod_q K_q \mathbb{Z}_A, \tag{4.5}$$

where Z denotes the center.

Hence the function f may be extended to be defined on $\mathbb{Z}_A G'_Q \backslash G'_{\mathbb{A}}$ and is an eigenfunction of the Hecke operators at p and ∞. By using the other Hecke operators (for the other primes) we can define a function on $\mathbb{Z}_A G'_Q \setminus G'_{\mathbb{A}}$ which is an eigenfunction of all the Hecke operators and the eigenvalues λ and μ will be unchanged.

By standard methods (see [3]) one may construct from this function an automorphic representation of $G'_{\mathbb{A}}$, call it $\pi' = \pi'_f$, whose π_p component corresponds to λ_p and is of class 1. This representation is not one-dimensional if $\lambda_p \neq \pm(p+1)$ or if $\mu \neq 0$. By the Jacquet-Langlands correspondence (Gelbart [3], Theorem 10.5) we may associate with π' a cuspidal representation π of $\mathrm{GL}(2, \mathbb{A})$. Furthermore, $\pi'_p \cong \pi_p$ and π_∞ is in the discrete series of $\mathrm{GL}(2, \mathbb{R})$ and is of weight $2 + 2m$, where m is the degree of the spherical harmonic $f(\cdot, y)$.

Thus π corresponds to a holomorphic cusp form $\tilde{f}$ for $\Gamma_0(N)$ of weight $2 + 2m$ (cf. Gelbart [3]); here $\Gamma_0(N)$ is some Hecke congruence subgroup of level N, where N is the conductor of π. Since π_p is of class 1, $p \nmid N$. Now the Hecke operator T_p will leave $\tilde{f}$ invariant, in fact

$$T_p\tilde{f} = \lambda p^{m-1}\tilde{f} \tag{4.6}$$

since $\pi'_p \cong \pi_p$. We may conclude by Deligne's theorem (see [1]) that

$$|\lambda| \leqq 2\sqrt{p}\,.$$

Next we consider the case $p = \infty$. In this case, we let D denote a quaternion algebra over a real quadratic field k of class number one, which splits at one infinite place and ramifies at the other. Let G' be the group of norm one quaternions of D and let $\tilde{\Gamma}$ be the group of elements of D with integer entries and of norm one.

As before we have

$$\tilde{\Gamma} \to G'_{\infty_1} \times G'_{\infty_2} = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SU}(2) \tag{4.7}$$

and hence

$$\tilde{\Gamma} \to \mathrm{PSL}(2, \mathbb{R}) \times \mathrm{SO}(3). \tag{4.8}$$

We set Γ equal to the image of $\tilde{\Gamma}$. Finally suppose $f(x, y)\colon \mathfrak{h} \times S^2 \to \mathbb{C}$ satisfies

$$f(\gamma z) = f(z) \quad \text{for all} \quad \gamma \in \Gamma, \quad z = (x, y), \tag{4.9}$$

$$\Delta_x f = \lambda f, \qquad \Delta_y f = \mu f. \tag{4.10}$$

THEOREM 4.2. *If* $(\lambda, \mu) \neq (0,0)$, *then*

$$\lambda \geqq \tfrac{3}{16}. \tag{4.11}$$

Proof: As before we can construct out of f an automorphic function on $G'_{\mathbb{A}}$ (here $\mathbb{A}$ is the adele ring of k) and then an automorphic representation of $G'_{\mathbb{A}}$. Again by the Jacquet-Langlands correspondence we obtain an automorphic representation π of $\mathrm{GL}(2, \mathbb{A})$. This representation gives rise to a modular form of the Hilbert modular group which is holomorphic in one variable and non-holomorphic in the other. The assumption $(\lambda, \mu) \neq (0,0)$ implies that it is a cusp form. It follows from the work of Gelbart-Jacquet [4] or the method of Selberg (cf. Sarnak [11]) that $\lambda \geqq \frac{3}{16}$, as needed.

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