Topics in Probability: Random Walks and Percolation

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1 Random Walks - Basics

What is a random walk? hard to define in general. Better start with an example.

Here's THE example from the founder of the field, George Polya:

"... he and his fiancee (would) also set out for a stroll in the woods, and then suddenly I met them there. And then I met them the same morning repeatedly, I don't remember how many times, but certainly much too often and I felt embarrassed: It looked as if I was snooping around which was, I assure you, not the case. I met them by accident - but how likely was it that it happened by accident and not on purpose?"

We model this setup by two walkers, each taking a step north, south, east or west with equal probabilities every second. Will they meet?

On $\mathbb{Z}^2$ this is equivalent to a single walker, asking if and how often he comes back.

Definition. *A **Simple Random Walk** on a graph G is a stochastic process x_i with $Prob(x_{i+1} = v) = \frac{1}{d_{x_i}}$ for $v \sim x_i$, and 0 otherwise.*

Definition. *a graph G is called* ***recurrent*** *if a SRW on G returns to x_0 with probability 1. If G is not recurrent, it is called* ***transient****.*

Exercise. *Show that the definition does not depend on x_0.*

Exercise. *Is the 2 walkers model always equivalent to 1 walker? i.e. is there a recurrent graph where 2 independent SRWs do not meet infinitely often?*

Other interesting questions besides recurrency: Hitting times and hitting probabilities.

Example. *A man plays in the casino fair games with 1\$ bets. How much time till he earns 1\$? how much is he likely to lose before he recovers?*

Example. *A drunkard leaves his house. When is he likely to return?*

Example. *A SRW on $\mathbb{Z}^2$ stops when it hits the set $\{(0,-1),(0,0),(0,1)\}$. What is the distribution? if the walk starts far enough does it matter in which direction?*

Theorem. *(Stirling's formula)*

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}(\frac{n}{e})^n} = 1$$

Proof. For a really short proof see [1]. □

Theorem. *(Polya) SRW on $\mathbb{Z}$ and $\mathbb{Z}^2$ is recurrent.*

Proof. First, consider a SRW on $\mathbb{Z}$. The probability, p_{2n} of the walk visiting 0 at time $2n$ is exactly $\binom{2n}{n}2^{-2n}$. Using Stirling's Formula we get that

$$p_{2n} \sim \frac{\sqrt{2\pi 2n}(\frac{2n}{e})^{2n}}{2\pi n(\frac{n}{e})^{2n}} 2^{-2n} = \frac{1}{\sqrt{\pi n}}$$

Therefore, $\sum_{n=0}^{\infty} p_{2n} = \infty$. This is exactly the expected number of times the SRW visits 0.

Now, suppose that the probability of return to 0 is $P < 1$. The number of times the SRW returns to 0 is distributed geometrically with parameter P, because every time we are at 0 there is a probability P to return again, and $1 - P$ to cease visiting 0. The expectation is this distribution is $1/(1 - P)$. But our calculation showed this expectation to be infinite, so P cannot be less than 1.

Now for $\mathbb{Z}^2$. First, lets take a look at a RW which alternatingly takes a step on the X and Y axis. For this (non simple) RW the probability of visiting 0 at time $4n$ is

$$p_{4n} = (\binom{2n}{n} 2^{-2n})^2 \sim \frac{1}{\pi n}$$

Since $\sum_{n=0}^{\infty} p_{4n} = \infty$, the expected number of returns is infinite and this RW is recurrent, for exactly the same reasons as before.

Now, notice that 2 steps of this alternating RW will take us from (x, y) to either of $(x+1, y+1)$, $(x+1, y-1)$, $(x-1, y+1)$, $(x-1, y-1)$ with equal probabilities. Therefore, our alternating RW is actually a simple RW on the diagonal lattice, which consists of all the integer coordinates (x, y) with $x+y$ even, and with edges between (x, y) and (x', y') iff $|x - x'| = 1$ and $|y - y'|$. The graph of the diagonal lattice is exactly that of $\mathbb{Z}^2$, so a SRW on $\mathbb{Z}^2$ must be recurrent too. □

Theorem. *(Polya)* $\mathbb{Z}^d$ *for* $d \geq 3$ *is transient.*

Proof. □

If G is d-regular then the probability of a path $x_0, .., x_n$ is exactly d^{-n}. In particular, $Pr(x_0, .., x_n) = Pr(x_n, .., x_0)$. If G is not regular then $Pr(x_0, .., x_n) = Pr(x_n, .., x_0) d_{x_0}/d_{x_n}$. If the graph is of bounded degree then the probability of a path and its reversal are the same up to a multiplicative constant. This simple observation turns out to be very powerful.

Definition. *For a graph G and a vertex $v \in V_G$, a set $C \subset V_G$ of vertices is called a **cutest** if the component of v in $G \setminus C$ is finite.*

Definition. *Given two cutsets, C and D we say that C is **nested** in D if S is contained in the (finite) component of v in $G \setminus C$.*

Theorem. *(Nash-Williams [2]) For G a bounded degree graph, if there exists a series of disjoint cutsets C_i, each nested in its successor, such that $\sum_{i=0}^{\infty} |C_i|^{-1} = \infty$ then the graph is recurrent.*

Proof. □

The Nash-Williams criterion is sufficient but not necessary.

Exercise. *Find a bounded degree recurrent graph which does not meet this criterion.*

References

[1] Romik, D. Stirling's approximation for n!: the ultimate short proof? Amer. Math. Monthly 107 (2000), 556-557 http://www.stat.berkeley.edu/ romik/papers.html

[2] Nash-Williams, C. St. J. A., Random walks and electric currents in networks, *Proc. Cambridge Phil. Soc.* **55** (1959), 181-194.