Can divergence of Gibbs measures at temperature zero be observed on finite marginals?

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Let $\varphi : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ be a Hölder continuous function and $\mathcal{M}$ the space of shift-invariant probability measures. For each β, let $\mu_\beta \in \mathcal{M}$ be the Gibbs measure associated to $\beta\varphi$, i.e. the unique measure $\nu \in \mathcal{M}$ maximizing the quantity

$$h(\nu) + \int \beta\varphi \, d\nu$$

Let $\mathcal{G}_\varphi$ denote the set of accumulation points of μ_β as $\beta \to \infty$, and let $\mathcal{M}$ denote the set of φ-maximizing measures, i.e.

$$\mathcal{M}_\varphi = \{\nu \in \mathcal{M} \,:\, \forall \nu' \in \mathcal{M} \; \int \varphi d\nu' \leq \int \varphi d\nu\}$$

Note that $\mathcal{M}$ is compact and convex in the weak-* topology.

It is known that $\mathcal{G}_\varphi \subseteq \mathcal{M}_\varphi$; in fact, if $\nu \in \mathcal{G}_\varphi$ then $h(\nu)$ maximizes $h(\cdot)$ on $\mathcal{M}_\varphi$. Thus if $\mathcal{M}_\varphi$ is a singleton then μ_β converges as $\beta \to \infty$. However, $\mathcal{M}_\varphi$ can contain multiple points maximizing the entropy, indicating that it may be possible for μ_β to diverge weak-*. Jean-René Chazottes and I recently showed that such divergence is possible.

One can try to observe divergence at the level of finite marginals. For $n \in \mathbb{N}$ let

$$\mathcal{M}_{\varphi,n} = \{\nu|_{\{0,1\}^n} \,:\, \nu \in \mathcal{M}_\varphi\}$$

denote the set of measures on $\{0.1\}^n$ obtained by restricting $\mathcal{M}_\varphi$ to coordinates $1, 2, \ldots, n$. This is a closed convex set of probability measures on $\{0,1\}^n$ and since the entropy function $H(\cdot)$ is strictly concave, there is a unique measure

$\mu_n \in \mathcal{M}_{\varphi,n}$ maximizing it. It is clear that any "limit point" of μ_n in $\mathcal{M}_\varphi$ is an entropy-maximizing measure in $\mathcal{M}_\varphi$. In my example with Chazottes, divergence of μ_β is shown by first showing that μ_n diverges, and that $\mu_\beta \sim \mu_n$ when β is related to n in a certain way. This raises the following

Problem: Does the set of accumulation points of μ_n, as $n \to \infty$, coincide with $\mathcal{G}_\varphi$?