WHAT CAN WE LEARN BY OBSERVING A STATIONARY PROCESS?

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An observation scheme for a class of stationary processes $\mathcal{C}$ is a (real-valued) function g that accepts n-tuples of input symbols from an arbitrary set, and has the property that if $(X_n)_{n=1}^{\infty}$ is any process from $\mathcal{C}$ then $\lim g(X_1, \ldots, X_n)$ exists in probability. If this limit is an isomorphism invariant of $\mathcal{C}$ then this invariant is said to be finitely observable for $\mathcal{C}$. Notice that there are lots of things that we can learn from observing $X_1, X_2, \ldots$, such as their average (if they are numerically valued); this is just the ergodic theorem (let g compute their average). But averages and other estimates are usually not isomorphism invariants. One exception is the entropy of the process, which may be estimated in several ways from observations and is an isomorphism invariant.

Ornstein and Weiss showed that in fact, entropy is the *only* finitely observable isomorphism invariant for the class $\mathcal{C}$ of all processes (D. Ornstein and B. Weiss, *Entropy is the only finitely observable invariant*, Journal of modern dynamics Volume 1 No. 1, 2007).

Notice that as $\mathcal{C}$ gets smaller there are potentially more finitely observable invariants, since the requirement that g converge for every member of the class becomes weaker when the $\mathcal{C}$ is smaller.

Yonatan Gutman and I have extended the Ornstein-Weiss result to cover a very large number of classes $\mathcal{C}$ (Y. Gutmn and M. Hochman, *on processes that cannot be distinguished by finite observation*, Israel journal of Mathematics, Volume 164, Number 1 / March, 2008). In particular,

Theorem. *If $\mathcal{C}$ is any class of ergodic processes which is closed under isomorphism, and such that any two processes in $\mathcal{C}$ have a zero-entropy common joining or factor in $\mathcal{C}$, then entropy is the only finitley observable invariant for $\mathcal{C}$.*

Thus, for example, the class of pure point spectrum systems has no finitely observable invariants.

This gives partial information about the processes arising from an irrational rotation, i.e. pure point spectrum processes whose spectrum is a cyclic subgroup of the circle:

Corollary 0.1. *If $\mathcal{C}$ is the family of ergodic processes arising from irrational rotations, and we parametrize its members by the angle of rotation (up to sign), then there is a set of angles of full measure for which the corresponding processes are not distinguished by the invariant.*

Problem. Are there any non-trivial finitely observable invariants on the class of processes arising from irrational rotations?

This questions becomes even more interesting when one adds regularity assumptions:

Problem. Let $\mathcal{C}$ be the class of processes X_n of the form $X_n(z) = f(e^{2\pi i\alpha n}z)$ for some analytic $f(w`) \neq w^n$ and $\alpha \notin \mathbb{Q}$ (and z is drawn from $[0,1)$ with Lebesgue measure). Is the spectrum – i.e. α – a finitely observable invariant for $\mathcal{C}$?

There is a related problem of distinguishing between two processes. Suppose we observe a process Z_n and are told that it arises from an irrational rotation by one of two angles, α and β. We can then determine with certainty which one: given $X_1, \ldots, X_n$, form the sum $S_n = \frac{1}{n}\sum_{k=1}^{n} e^{2\pi i\alpha n}X_n$ and $T_n = \frac{1}{n}\sum_{k=1}^{n} e^{2\pi i\beta n}X_n$. If $|S_n| < |T_n|$ we guess that the process comes from rotation by β, and otherwise from rotation by α. The Wiener-Wintner theorem says that with probability one, this guess is eventually correct.

There are other ways of distinguishing between two given rotations using the fact that they are disjoint. However, we have not been able to resolve the following:

Problem. Let X_n, Y_n be processes which are disjoint in the sense of Furstenberg. Suppose we observe a process Z_n which is isomorphic to one of the two processes. Is there a procedure to determine which one it is by observing more and more data of Z_n?