GROUPS OF AUTOMORPHISMS OF SFTS

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Let Σ be a finite alphabet and $\Sigma^{\mathbb{Z}}$ the space of bi-infinite sequences over it, i.e. the full shift. Let S be the shift map,

$$(Sx)_n = x_{n+1}$$

An automorphism of the full shift is a hoemomorphism $\varphi : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$ that sommutes with the shift: $S\varphi = \varphi S$. Such a map is given by a block code. This means there is some integer $R > 0$ and function $\varphi_0 : \Sigma^{[-R,R]} \to \Sigma$, so that

$$(\varphi x)_n = \varphi_0(x_{n-R}, x_{n-R+1}, \ldots, x_n, \ldots x_{n+R})$$

or, in other words, $(\varphi x)_n$ is determined by applying φ_0 to the symmetric block of $2R + 1$ symbols around n. The minimal R for which φ can be presented in this way is called the window width of φ.

Thus the automorphism group is countable, but it is very large. Hedlund established that it contains free groups and all finite groups; later on Boyle, Lind and Rudolph generalized this to automorphism groups of mixing shifts of finite type (*The automorphism group of a shift of finite type*, Trans. Amer. Math. Soc., Volume 306, 1988 No. 1, pages 71–114). More recently a rich algebraic theory has developed around this group.

However, there are still some basic questions questions. The following are taken from the paper of Boyle, Lind and Rudolph:

Problem. Is the automorphism group of the full shift on two symbols isomorphic (as a group) to the automorphism group of the full shift on three symbols?

The automorphism group of a full shift is not amenable, since it contains free groups. Nevertheless it contains a natural family of increasing finite subsets that exhaust the group, namely the sets F_n of automorphisms defined by block codes of window width n. There is also unique measure μ on $\Sigma^{\mathbb{Z}}$ which is invariant under all automorphisms, namely the unique measure of maximal entropy (the product measure giving $1/|\Sigma|$ mass to each symbol). The following problem is frmo the same paper:

Problem. Let f be a continuous function on $\Sigma^{\mathbb{Z}}$. Do the averages

$$\frac{1}{|F_n|}\sum_{\varphi\in F_n} f\circ\varphi_n$$

converge to the mean of f with respect to the measure μ?

Here "converge" could mean in $L^2(\mu)$ or, even better, at every non-periodic point.

There are some fascinating questions about automorphism groups of higher-dimensional shifts. I studied some basic properties of these groups in *On the automorphism group of multidimensional shifts of finite type* (currently in review).

For any d we can define the full shift $\Sigma^{\mathbb{Z}^d}$ and the associated shift action, which is generated by d homeomorphisms $T_1,\dots,T_d$, corresponding to shifting by each of the d generators of $\mathbb{Z}^d$. An automorphism is again a shift-commuting homeomorphism.

Problem. For $m\neq n$, are the automorphism groups of $\{0,1\}^{\mathbb{Z}^m}$ and $\{0,1\}^{\mathbb{Z}^n}$ isomorphic?

By a theorem of Kim and Roush, the $\operatorname{Aut}\Sigma^{\mathbb{Z}}$ is embeddable, as a group, in any $\operatorname{Aut}\Delta^{\mathbb{Z}}$. One can show that it is also embeddable in $\operatorname{Aut}\Delta^{\mathbb{Z}^d}$ for any d.

Problem. Let $d>1$. Which automorphism groups $\operatorname{Aut}\Sigma^{\mathbb{Z}^d}$ can be embedded in each other?

If one moves away from full shifts the multidimensional situation becomes more complicated. For positive entropy SFTs the automorphism group remains large, but can be degenerate in some ways; for example, it does not have to be residually finite, as it does in dimension 1, not does it have to contain any elements of infinite order after moding out the shift action itself. There is also little connection between mixing and the size of the atomorphism group.

One property of $\operatorname{Aut}X$ when X is a one-dimensional SFT is that any finitely-generated subgroup has decidable word problem. This means that if $g_1,\dots,g_n\in\operatorname{Aut}X$ and $G=\langle g_1,\dots,g_n\rangle$, then there is an algorithm that decides, for a given expression $w(x_1,\dots,x_n)$ involving the variables x_i, multiplication and the inverse operation, whether $w(g_1,\dots,g_n)=\mathrm{id}$ or not. This is because, using the block code representation of the g_i we can represent $w(g_1,\dots,g_n)$ as a block code, and then we need to test whether it acts as the identity on long finite inputs. Since we can find all blocks of a given length that appear in X, we can check whether $w(g_1,\dots,g_n)-\mathrm{id}$.

For higher dimensional SFTs this breaks down because, by Berger's theorem, we cannot produce all finite patterns in the system. A compactness argument shows that if $w(g_1, \dots, g_n)$ is the identity then we can determin this in finite time, but if it is not then it's not clear how we can detect this.

Problem. If G is a finitely generated group of automorphisms of a multidimensional SFT, does it have decidable word problem?

One can be more ambitious. The argument above shows that for a finitely generated group G of automorphisms, the relation

$$R_G = \{(a, b) \in G \times G \times G : ab^{-1} = 1_G\}$$

is a recursively enumerable set.

Problem. Suppose that G is a countable group such that R_G is recursively enumerable. Can G be embedded in the automorphism group of a d-dimensional shift of finite type?

In [??], we showed that there are many parameters of higher dimensional symbolic systems which are restricted only by recursive conditions like the above. Thus it seems at least possible that the answer to the above is positive, but it would be even more interesting if it were not.