Dimension theory of self-similar sets and measures

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Abstract

We report on recent results about the dimension and smoothness properties of self-similar sets and measures. Closely related to these are results on the linear projections of such sets, and dually, their intersections with affine subspaces. We also discuss recent progress on the the Bernoulli convolutions problem.

1 Introduction

Consider a random walk $(X_n)_{n=0}^{\infty}$ on $\mathbb{R}^d$, started from a point, and with the transitions given by $X_{n+1} = \xi_n X_n$, where (ξ_n) is an independent sequence of similarity maps, chosen according to a fixed distribution p.

The long term behavior of (X_n) depends on the scaling properties of the ξ_n. If they are expanding, there is no interesting limit. But the other cases are quite interesting. When the ξ_n are isometries, and act in some sense irreducibly on $\mathbb{R}^d$, both a central limit theorem and local limit theorem hold, i.e. $(X_n - \mathbb{E}X_n)/n^{d/2}$ converges in law to a Gaussian, and $X_n - \mathbb{E}X_n$ converges to Lebesgue measure on bounded open sets [52, 19, 43, 53]. Thus, the limiting behavior is universal, and X_n spreads out "as much as possible".

The remaining case, namely, when the ξ_n contract, is our focus here. Then X_n converges in law (without any normalization) to a measure μ, which does not depend on the starting point X_0, but very strongly depends on the step distribution p. An important case is when p is finitely supported. The stationary measure μ is called a **self-similar measure**, and its topological support, which is the set of accumulation points of any orbit of the semigroup generated by $\operatorname{supp} p$, is called a **self-similar set**.

Many mathematical problems surround self-similar sets and measures, and in this paper we survey some of the recent progress on them. Perhaps the most natural problem is to determine the dimension, and, if applicable, smoothness, of μ. Although there is no universal limiting distribution as in the CLT, a weaker universal principle is believed to apply: namely, that μ should be "as spread out as possible" given the constraints imposed by the amount of contraction in the system, and given possible algebraic constraints, such as being trapped in a lower-dimensional subspace. This principle – that in algebraic settings, dynamical processes tend to spread out as much as the algebraic constraints allow – has many counterparts, such as rigidity theorems for horocycle flows and higher rank diagonal actions on homogeneous spaces (for references see e.g. [28]), and stiffness of random walks on homogeneous spaces (e.g. [5, 2]).

Self-similar sets and measures are also natural examples of "fractal" sets; they possesses a rich set of symmetries, and a natural hierarchical structure. This has motivated a number of longstanding conjectures about the geometry of such sets and measures, and specifically, about the dimension of their linear images and their intersections with affine subspaces. Many of these conjectures are now confirmed, as we shall describe below.

Finally, we devote some time to the special case of Bernoulli convolutions, which is a problem with strong number-theoretic connections. This problem also has seen dramatic progress in the past few years.

Due to space constraints, we have omitted many topics, and kept to a minimum the discussion of classical results, except where directly relevant. A more complete picture can be found in the references.

The plan of the paper is as follows. We discuss the dimension problem for self-similar sets and measures in Section 2;. The Bernoulli convolutions problem in Section 3; and projections and slices in Section 4.

2 Self-similar sets and measures

Self-similar sets and measures are the prototypical fractals; the simplest example is the middle-1/3 Cantor set and the Cantor-Lebesgue measure, which arise from the system of contractions $\varphi_0(x) = \frac{1}{3}x$ and $\varphi_1(x) = \frac{1}{3}x + \frac{2}{3}$, taken with equal probabilities. In general, self-similar sets and measures decompose into copies of themselves, just as the Cantor set does. This is most evident using Hutchinson's construction [24], which we specialize to our setting.

An **iterated function system** will mean a finite family $\Phi = \{\varphi_i\}_{i\in\Lambda}$ of contracting similarity maps of $\mathbb{R}^d$. A **self similar set** is the **attractor** of Φ, that is, the unique non-empty compact set $X = X_\Phi$ satisfying

$$X = \bigcup_{i\in\Lambda} \varphi_i(X) \tag{1}$$

The **self-similar measure** determined by $\Phi = \{\varphi_i\}_{i\in\Lambda}$ and a probability vector $p = (p_i)_{i\in\Lambda}$ (which we think of as a measure on Φ) is the unique Borel probability measure $\mu = \mu_{\Phi,p}$ satisfying

$$\mu = \sum_{i\in\Lambda} p_i \cdot \varphi_i \mu \tag{2}$$

where $\varphi\mu = \mu \circ \varphi^{-1}$ is the push-forward measure. When p is strictly positive, the topological support $\operatorname{supp}\mu$ of μ is X.

A similarity has the form $\varphi(x) = rUx + b$ where $r > 0$, U is an orthogonal matrix, and $b \in \mathbb{R}^d$. We call r the **contraction**, U the **orthogonal part**, and b the **translation part** of φ , respectively. We say that Φ, X or μ are

self-homothetic if the φ_i are homotheties, i.e. they have trivial rotations; **uniformly contracting** if φ_i all have the same contraction ratio; and have **uniform rotations** if the φ_i all have the same orthogonal parts. Also, Φ is algebraic if all coefficients defining φ_i are algebraic.

By definition, X and μ decompose into smaller copies of themselves, and by iterating the identities (1) and (2) one gets a decomposition at arbitrarily small scales. For a sequence $\mathbf{i} = i_1 \ldots i_n \in \Lambda^n$ it is convenient to denote $\varphi_{\mathbf{i}} = \varphi_{i_1} \circ \ldots \circ \varphi_{i_n}$. Note that the contraction tends to 0 exponentially as $n \to \infty$. For $\mathbf{i} \in \Lambda^n$, we call $\varphi_{\mathbf{i}} X$ and $\varphi_{\mathbf{i}} \mu$ **generation-n cylinders**. These are the small-scale copies alluded to above, with the corresponding decompositions $X = \bigcup_{\mathbf{i} \in \Lambda^n} \varphi_{\mathbf{i}} X$ and $\mu = \sum_{\mathbf{i} \in \Lambda^n} p_{\mathbf{i}} \cdot \varphi_{\mathbf{i}} \mu$. The last identity shows that the definition above coincides with the earlier description using random walks: For $\mathbf{i} \in \Lambda^n$, the diameter of the support of $\varphi_{\mathbf{i}} \mu$ converges uniformly to zero as $n \to \infty$, so the identity $\mu = \sum_{\mathbf{i} \in \Lambda^n} p_{\mathbf{i}} \cdot \varphi_{\mathbf{i}} \mu$ implies that μ is the limit distribution of $\sum_{\mathbf{i} \in \Lambda^n} p_{\mathbf{i}} \cdot \delta_{\varphi_{\mathbf{i}} x}$ for every $x \in \operatorname{supp} \mu^d$. The last measure is just the distribution of the random walk from the introduction, started from x.

2.1 Preliminaries on dimension

We denote by $\dim(\cdot)$ the Hausdorff dimension for sets, and the lower Hausdorff dimension of Borel probability measures, defined by

$$\dim \mu = \inf\{\dim E \,:\, \mu(E) > 0\}$$

$\mathcal{H}^s$ denotes s-dimensional Hausdorff measure. Absolute continuity (a.c.) is with respect to Lebesgue measure.

There are many other notions of dimension which in general disagree, but for self-similar sets and measures most of them coincide. Specifically, Falconer proved that self-similar sets have equal Hausdorff and box dimensions [10]. For self-similar measures, Feng and Hu [12] proved that a self-similar measure is **exact dimensional**, meaning that for $\alpha = \dim \mu$, as $r \to 0$ we have

$$\mu(B_r(x)) = r^{\alpha(1+o(1))} \qquad \text{for } \mu\text{-a.e. } x \tag{3}$$

(non-uniformly in x).

With regard to smoothness of self-similar measures, in the special case of infinite convolutions it is a classical result of Jessen and Wintner [25] that the measure is of pure type, i.e. is either singular with respect to Lebesgue, or absolutely continuous with respect to it. This is true for all self-similar measures, and is a consequence of Kolmogorov's zero-one law.

2.2 Similarity and Lyapunov dimension

In order to estimate the dimension of a set or measure, one must construct efficient covers of it, or estimate the mass of sets of small diameter. Self-similar sets and measures come equipped with the natural covers given by the cylinder sets of a given generation, or of a given approximate diameter. Counting cylinder sets, one arrives at the following estimates for the dimension:

- The **similarity dimension** is the unique $s = s(\Phi) \geq 0$ satisfying $\sum r_i^s = 1$, where r_i is the contraction constant of φ_i.
- The **Lyapunov dimension** of Φ and a probability vector $p = (p_i)_{i\in\Lambda}$ is $s(\Phi, p) = H(p)/\lambda(p)$, where $H(p) = -\sum p_i \log p_i$ is the Shannon entropy of p, and $\lambda(p) = -\sum p_i \log r_i$ is the asymptotic contraction, i.e. Lyapunov exponent, of the associate random product.

Note that $s(\Phi, p)$ is maximal when $p = (r_i^s)_{i\in\Lambda}$ (with $s = s(\Phi)$), and then the similarity and Lyapunov dimensions coincide: $s(\Phi, p) = s(\Phi)$.

These estimates ignore the possibility of coincidences between cylinders. When cylinders of the same generation intersect we say that the system has **overlaps**; it has **exact overlaps** if there exist finite sequences $\mathbf{i}, \mathbf{j} \in \Lambda^*$ such that $\varphi_{\mathbf{i}} = \varphi_{\mathbf{j}}$, or in other words, if the semigroup generated by Φ is not freely generated by it. If this happens, then without loss of generality we can assume that $\mathbf{i}, \mathbf{j}$ have the same length n (otherwise replace them with $\mathbf{ij}$ and $\mathbf{ji}$), and if such pairs exist for some n, then they exist for all large enough n.

In this generality, Hutchinson [24] was the first to show that $\dim X \leq s(\Phi)$ and $\dim \mu \leq s(\Phi, p)$. Furthermore, these are equalities if we assume that there are only mild overlaps. Specifically, Φ is said to satisfy the **open set condition** (OSC) if there exists an open non-empty set U such that $\varphi_i U \subseteq U$ for all $i \in \Lambda$ and $\varphi_i U \cap \varphi_j U = \emptyset$ for all $i \neq j$. A special case of this is when the first generation cylinders are disjoint, which is called the **strong separation condition** (SSC). The OSC allows overlaps, but it implies that the overlaps have bounded multiplicity.

Theorem 2.1 (Hutchinson [24]). *Suppose $\Phi = \{\varphi_i\}_{i\in\Lambda}$ is an IFS in $\mathbb{R}^d$ satisfying the OSC. Then $\dim X_\Phi = s(\Phi)$ and $\dim \mu_{\Phi,p} = s(\Phi, p)$ for every p. Furthermore, writing $s = s(\Phi)$, we have $0 < \mathcal{H}^s(X) < \infty$, and $\mathcal{H}^s|_{X_\Phi}$ is equivalent to the self-similar measure defined by $p = (r_i^s)_{i\in\Lambda}$.*

In fact, for $s = s(\Phi)$, Falconer showed that $\mathcal{H}^s(X) < \infty$ always holds [10]. In general $\mathcal{H}^s$ can vanish on X; Schief [44] (following some special cases [27, 1]) showed that $\mathcal{H}^s(X) > 0$ for $s = s(\Phi)$ is exactly equivalent to the OSC.

It must be emphasized that the OSC allows only "minor" overlaps between cylinders, and since $\dim X \leq d$ for $X \subseteq \mathbb{R}^d$, by Theorem 2.1, the OSC implies $s(\Phi) \leq d$. There exist IFSs with $s(\Phi) > d$, e.g. one can take an IFS $\Phi_n = \{\varphi_i\}_{i=1}^n$ on $\mathbb{R}$ with $\varphi_i(x) = \frac{1}{2}x + i$. The attractor is an interval, but $s(\Phi_n) = \log n / \log 2$. In any case, the dimension d of the ambient space $\mathbb{R}^d$ is also an upper bound on dimension, so whether or not the OSC holds, we have

$$\begin{aligned} \dim X &\leq \min\{d, s(\Phi)\} && (4) \\ \dim \mu &\leq \min\{d, s_p(\Phi)\} && (5) \end{aligned}$$

We say that X or μ exhibits **dimension drop** if the corresponding inequality above is strict. The principle way dimension drop occurs is if there are exact overlaps. Indeed, suppose $\dim X < d$ and $\mathbf{i}, \mathbf{j} \in \Lambda^n$ with $\varphi_{\mathbf{i}} = \varphi_{\mathbf{j}}$. Let $\Phi^n =$

$\{\varphi : \mathbf{u} \in \Lambda^n\}$. Then a short calculation shows that $s(\Phi^n) < s(\Phi)$. Since X is also the attractor of Φ^n, we get $\dim X \leq \min\{d, s(\Phi^n)\} < \min\{d, s(\Phi)\}$, and we have dimension drop.

2.3 The overlaps conjecture, and what we know about it

In this section we specialize to $\mathbb{R}$, where exact overlaps are the only known mechanism that leads to dimension drop. The next conjecture is partly folklore. It seems to have first appeared in general form in [49].

Conjecture 2.2. *In $\mathbb{R}$, dimension drop occurs only in the presence of exact overlaps.*

Thus, non-exact overlaps should not lead to dimension drop. By Theorem 2.1, we know that minor overlaps can indeed be tolerated. Other examples come from parametric families such as the $\{0, 1, 3\}$-problem, which concerns the attractor of the IFS $\Phi_\lambda = \{x \mapsto \lambda x\, , \, x \mapsto \lambda x + 1\, , \, x \mapsto \lambda x + 3\}$. There are only countably many parameters λ with exact overlaps, and Pollicott and Simon showed that for a.e. $\lambda \in [\frac{1}{3}, \frac{1}{2}]$, there is no dimension drop, see also [26].

To go further we must quantify the amount of overlap. Define the distance $d(\cdot, \cdot)$ between similarities $\varphi(x) = ax + b$ and $\varphi'(x) = a'x + b'$ by

$$d(\varphi, \varphi') = |b - b'| + |\log a - \log a'| \tag{6}$$

Alternatively, one can take any left- or right-invariant Riemannian metric on the group of similarities, or the operator norm on the standard embedding of the group into $GL_2(\mathbb{R})$. These metrics are not equivalent, but are mutually bounded up a power distortion, which makes them equivalent for the purpose of what follows.

Given an IFS $\Phi = \{\varphi_i\}_{i \in \Lambda}$ of similarities, let

$$\Delta_n = \min\{d(\varphi_{\mathbf{i}}, \varphi_{\mathbf{j}}) \, : \, \mathbf{i}, \mathbf{j} \in \Lambda^n \, , \, \mathbf{i} \neq \mathbf{j}\}$$

There are exact overlaps if and only if $\Delta_n = 0$ for some n (and hence all large enough n), and contraction implies that $0 \leq \Delta_n \leq r^n$ for some $0 < r < 1$. However, the decay of Δ_n generally need not be faster than exponential. We say that Φ is **exponentially separated** if there is a constant $c > 0$ such that $\Delta_n \geq c^n$ for all n.

Theorem 2.3 (Hochman [20]). *Let Φ be an IFS in $\mathbb{R}$, let $\mu = \mu_{\Phi,p}$ be the self-similar measure, and write $s = s(\Phi, p)$. Then either $\dim \mu = \min\{1, s\}$, or else $\Delta_n \to 0$ super-exponentially. The same statement holds for sets.*[1]

Thus, exponential separation implies no dimension drop. We do not know of any IFS without exact overlaps for which $\frac{1}{n} \log \Delta_n \to \infty$, and it is conceivable that they simply do not exist, which would prove the conjecture.

[1] The statement for sets follows from the measure case applied to $\mu_{\Phi,p}$, with p chosen so that $s(\Phi, p) = s(\Phi)$. The same remark holds for many theorems in the sequel.

Corollary 2.4. *Within the class of algebraic IFSs on $\mathbb{R}$, Conjecture 2.2 is true.*

Indeed, one can choose the metric so that $d(\varphi,\psi)$ is a polynomial in the coefficients of φ,ψ, and then $d(\varphi_{i_1}\ldots\varphi_{i_n},\varphi_{j_1}\ldots\psi_{j_n})$ is a polynomial of degree $O(n)$ in the coefficients of the φ_i. If these are algebraic, such an expression either vanishes, or is bounded below by an exponential c^n for some $c>0$ (see Garsia's Lemma 3.4 below).

When exact overlaps exist, one can get a better bound than the Lyapunov dimension by taking the number of exact overlaps into account. Given Φ and p, let $(\xi_n)_{n=1}^{\infty}$ be i.i.d. elements of Φ with distribution p, and let $\sigma_n=\xi_n\xi_{n-1}\ldots\xi_1$ be the associated random walk on the similarity group. The **random walk entropy** of p is defined by

$$h_{RW}(p)=\lim_{n\to\infty}\frac{1}{n}H(\sigma_n) \tag{7}$$

where $H(\sigma_n)$ is the Shannon entropy of the discrete random variable σ_n. The limit exists by sub-additivity, and if Φ^* is freely generated by Φ, then $h_{RW}(p)=H(p)$. Corresponding to (5) we have the bound

$$\dim\mu\leq\min\{1,\frac{h_{RW}(p)}{\lambda(p)}\}$$

The following is a reasonable extension of Conjecture 2.2:

Conjecture 2.5. *If $\mu=\mu_{\Phi,p}$ is a self-similar measure on $\mathbb{R}$ then $\dim\mu=\min\{1,h_{RW}(p)/\lambda(p)\}$.*

The next theorem is proved by the same argument as Theorem 2.3. The statement first appeared in [53].

Theorem 2.6. *Let $\Phi=\{\varphi_i\}$ be an IFS of similarities in $\mathbb{R}$, and suppose that there is a $c>0$ such that for every $\mathbf{i},\mathbf{j}\in\Lambda^n$ either $\varphi_{\mathbf{i}}=\varphi_{\mathbf{j}}$ or $d(\varphi_{\mathbf{i}},\varphi_{\mathbf{j}})\geq c^n$. Let $\mu=\mu_{\Phi,p}$ be the self-similar measure for Φ. Then $\dim\mu=\min\{1,h_{RW}(p)/\lambda(p)\}$.*

An important strengthening of these results is obtained by replacing Hausdorff dimension with L^q-dimension. To define it, let $\mathcal{D}_n$ denote the dyadic partition of $\mathbb{R}$ into intervals $[k/2^n,(k+1)/2^n)$, $k\in\mathbb{Z}$, and for $q>1$ set

$$D(\mu,q)=\lim_{n\to\infty}-\frac{\log\sum_{I\in\mathcal{D}_n}\mu(I)^q}{(q-1)n}$$

The limit is known to exist for self-similar measures [40], and for such measures, $\dim\mu=\lim_{q\searrow1}D_\mu(q)$. The function $q\mapsto D(\mu,q)$ is non-increasing in q, and has the following property: for every $\alpha<D(\mu,q)$, there is a constant C such that $\mu(B_r(x))\leq C\cdot r^{(1-1/q)\alpha}$, for **every** $x\in\mathbb{R}$. This is in stronger than (3), which holds only for μ-a.e. x, and non-uniformly.

The L^q-analog of the Lyapunov dimension for a self-similar measure $\mu=\mu_{\Phi,p}$, is the solution $s=s^q(\Phi,p)$ of the equation $\sum p_i^q|r_i|^{(q-1)s}=1$, where r_i are the contraction ratios of the maps in Φ. Note that if $p=(r_i^{s(\Phi)})_{i\in\Lambda}$, then $s^q(\Phi,p)=s(\Phi,p)=s(\Phi)$, is independently of q. We always have $D_\mu(q)\leq s^q(\Phi,p)$.

Theorem 2.7 (Shmerkin [47]). *Let $\Phi = \{\varphi_i\}$ be an IFS on $\mathbb{R}$ and $p = (r_i^{s(\Phi)})$. Let $\mu = \mu_{\Phi,p}$ and $s = s(\Phi, p) = s(\Phi)$. Then either $D_\mu(q) = s$ for all $q > 1$, or else $\Delta_n \to 0$ super-exponentially.*

In particular, if Φ is exponentially separated, then for every $t < \min\{1, s(\Phi)\}$, there is a $C = C(t) > 0$ such that $\mu(B_r(x)) \leq Cr^t$ for all $x \in X$ and all $r > 0$.

2.4 Some ideas from the proofs

The main idea is, very roughly, as follows. A self-similar measure μ on $\mathbb{R}$ can be written, locally, as a convolution of a scaled copy of itself with another measure whose "dimension" (in some finitary sense) is proportional to the difference $s(\Phi, p) - \dim \mu$. But convolution is a smoothing operation, and $\mu * \nu$ has larger dimension than μ if the dimension of ν is positive. Hence, if there were dimension drop, at small scales μ would be smoother than itself, which is impossible.

In order to give even a slightly more comprehensive sketch, some preparation is needed. First, by "smoothing", we mean that convolving measures generally results in more "spread out" measures than we started with. The discussion below is very much in the spirit of additive combinatorics, in which one asks when the sum $A + B$ of two finite sets $A, B \subseteq \mathbb{Z}$ is substantially larger than A. "Larger" is often interpreted as $|A + B| > C|A|$ where $C > 0$ is fixed and the sets are large, as in Freiman's theorem (e.g. [51]); but in our setting we mean $|A + B| \geq |A|^{1+\delta}$. Such a growth condition is closely related to the work of several authors on the sum-product phenomenon, notably Bourgain [3, 4]. The version used in Theorem 2.10 and presented below is from [20]. See also the remark at the end of the section.

We measure how "spread out" a measure is using entropy at a finite scale. Recall that $\mathcal{D}_n$ is the level-n dyadic partition of $\mathbb{R}$, whose atoms are the intervals $[k/2^n, (2 + 1)/2^n)$. The **scale-n entropy** of a probability measure ν is the Shannon entropy $H(\nu, \mathcal{D}_n) = -\sum_{I \in \mathcal{D}_n} \nu(I) \log \nu(I)$ of ν with respect to $\mathcal{D}_n$. We refer to [8] for a more thorough introduction to entropy.

Scale-n entropy is a discretized substitute for dimension, and as a first approximation, one can think of it as the logarithm of the number of atoms of $\mathcal{D}_n$ with non-trivial ν mass. In particular if ν is exact dimensional, then

$$\frac{1}{n} H(\nu, \mathcal{D}_n) \to \dim \nu \tag{8}$$

For $m > n$ let $H(\nu, \mathcal{D}_m | \mathcal{D}_n) = H(\nu, \mathcal{D}_m) - H(\nu, \mathcal{D}_n)$ be the conditional entropy, i.e. the entropy increase from scale 2^{-n} to 2^{-m}. Assuming (8), we have

$$\frac{1}{m-n} H(\nu, \mathcal{D}_m | \mathcal{D}_n) = \dim \nu + o(1) \qquad \text{as } n \to \infty \text{ and } m - n \to \infty$$

In general, scale-n does not decrease under convolution,[2] and for generic measures it increases as much as possible, i.e. $H(\theta * \nu, \mathcal{D}_n) \approx H(\nu, \mathcal{D}_n) +$

[2] Actually it could decrease but only by an additive constant, which is negligible with our $1/n$-normalization.

$H(\theta, \mathcal{D}_n)$ assuming that the right hand side does not exceed one. But there certainly are exceptions, even cases in which $H(\theta * \nu) \approx H(\nu)$. But for self-similar measures, some substantial entropy increase must occur.

Theorem 2.8. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds. Let μ be a self-similar measure on $\mathbb{R}$ with $\dim \mu < 1-\varepsilon$, and let θ be a probability measure. Then for n large enough (depending on ε and μ, but not θ),*

$$H(\theta, \mathcal{D}_n) > \varepsilon n \qquad \Longrightarrow \qquad H(\theta * \mu, \mathcal{D}_n) > H(\mu, \mathcal{D}_n) + \delta n$$

This is a consequence of a more general result[3] describing the structure of pairs of measures θ, ν for which $H(\theta * \nu, \mathcal{D}_n) \approx H(\nu, \mathcal{D}_n)$. It says, roughly, that in this case each scale 2^{-i}, $1 \leq i \leq n$, is of one of two types: either ν looks approximately uniform (like Lebesgue measure) on 2^{-i}-balls centered at ν-typical points; or θ looks approximately like an atomic measure on 2^{-i}-balls centered at θ-typical points. The theorem above follows because self similar measures of dimension < 1 are highly homogeneous and don't look uniform on essentially any ball, while if $H(\theta, \mathcal{D}_n) \geq \varepsilon n$ then there is a positive propostion of balls on which θ does not **look** atomic. For the full statement see [20].

We return to the proof of Theorem 2.3; we assume for contradiction that there is both exponential separation and dimension drop. For simplicity, assume that all the maps in $\Phi = \{\varphi_i\}$ have the same contraction ratio r, and given n write

$$n' = \lfloor n \log(1/r) \rfloor$$

so that $\mathcal{D}_{n'}$ contains atoms of diameter roughly r^n. For $\mathbf{i} \in \Lambda^n$ the map $\varphi_{\mathbf{i}}$ contracts by r^n, and all the generation-n cylinders appearing in the representation $\mu = \sum_{\mathbf{i} \in \Lambda^n} p_{\mathbf{i}} \cdot \varphi_{\mathbf{i}} \mu$ are translates of each other, so this identity can be re-written as a convolution

$$\mu = \mu^{(n)} * S_{r^n} \mu \tag{9}$$

where $\mu^{(n)} = \sum_{i \in \Lambda^n} p_i \cdot \delta_{\varphi_i(0)}$, and $S_t x = tx$ is the scaling operator. Because $S_{r^n}\mu$ is supported on a set of diameter $O(r^n) = O(2^{-n'})$, it contributes to scale-n' entropy only $O(1)$, so

$$H(\mu^{(n)}, \mathcal{D}_{n'}) = H(\mu, \mathcal{D}_{n'}) + O(1) = n' \dim \mu + o(n) \tag{10}$$

Next, chop the measure $\mu^{(n)}$ into a convex combination

$$\mu^{(n)} = \sum_{I \in \mathcal{D}_{n'}} w_I \cdot (\mu_I^{(n)})$$

where $\mu_I^{(n)}$ is $\mu^{(n)}$ conditioned on I. Inserting this in (9) we get

$$\mu = \sum_{I \in \mathcal{D}_{n'}} w_I \cdot \mu_I^{(n)} * S_{r^n} \mu \tag{11}$$

[3]For related work on entropy of convolutions, assuming less growth, see Tao [20] and Madiman [30]. More closely related is Bourgain's work on sumsets, [3, 4].

Since Φ is exponentially separated there is a constant a such that every pair of atoms of $\mu^{(n)}$ is r^{an}-separated, and so lie in different atoms of $\mathcal{D}_{an'}$. A direct calculation shows that

$$H(\mu^{(n)}, \mathcal{D}_{an'}) = -\sum_{\mathbf{i} \in \Lambda^n} p_{\mathbf{i}} \log p_{\mathbf{i}} = nH(p) = n' \dim \mu + \varepsilon n$$

where $\varepsilon = H(p) - \dim \mu \cdot \log(1/r) > 0$ because of dimension drop. Combined with (10) this gives

$$H(\mu^{(n)}, \mathcal{D}_{an'} | \mathcal{D}_{n'}) = \varepsilon n + o(n) \tag{12}$$

By classical identities, this entropy is the average of the entropies of $\mu_I^{(n)}$, so for a μ-large proportion of $I \in \mathcal{D}_{n'}$,

$$H(\mu_I^{(n)}, \mathcal{D}_{an'} | \mathcal{D}_{n'}) = \frac{\varepsilon}{2} n + o(n)$$

Thus,

$$\begin{aligned}
(a-1)n' \cdot \dim \mu &= H(\mu, \mathcal{D}_{an'} | \mathcal{D}_{n'}) \\
&\geq \sum_{I \in \mathcal{D}_{n'}} w_I \cdot H(\mu_I^{(n)} * S_{r^n} \mu, \mathcal{D}_{an'} | \mathcal{D}_{n'}) \\
&= \sum_{I \in \mathcal{D}_{n'}} w_I \cdot H(\mu_I^{(n)} * S_{r^n} \mu, \mathcal{D}_{an'}) - o(n) \\
&\geq (a-1)n' \cdot \dim \mu + \delta n - o(n)
\end{aligned}$$

where in the first inequality we plugged in the identity (11) and used concavity of the entropy function, in the next line we eliminated the conditioning because $\mu_I^{(n)} * S_{r^n \mu}$ is supported on $O(1)$ atoms of $\mathcal{D}_{n'}$, and in the last line, we applied Theorem 2.8 together with (12). This is the desired contradiction.

Theorem 2.7 is proved using a very similar philosophy, but with the L^q-dimension of finite-scale approximations replacing entropy as the measure of smoothness. We refer the reader to the original paper for details.

2.5 Higher dimensions

In higher dimensions, Conjecture 2.2 is false in its stated form. To see this start with two IFSs, Φ_1 and Φ_2, on $\mathbb{R}$. Assume that $s(\Phi_1) > 1$ and the attractor X_1 of Φ_1 is an interval, and the attractor X_2 of Φ_2 satisfies $\dim X_2 = s(\Phi_2)$. We can also assume neither Φ_1 nor Φ_2 have exact overlaps. Let $\Phi = \Phi_1 \times \Phi_2$ be the IFS consisting of all maps of the form $x \mapsto (\varphi(x), \psi(x))$ with $\varphi \in \Phi_1$ and $\psi \in \Phi_2$. Then Φ has attractor $X_1 \times X_2$ of dimension $\dim X_1 + \dim X_2 < s(\Phi_1) + s(\Phi_2) = s(\Phi)$, and Φ has no exact overlaps.

In the example, there are horizontal lines intersecting X in an interval (a copy of X_1) and the family of such lines is preserved by Φ. It seems likely that this is the only new phenomenon possible in higher dimensions. More precisely, let us say that a set X has **full slices** on a linear subspace $V \leq \mathbb{R}^d$, if $\dim X \cap (V+a) = \dim V$ for some $a \in \mathbb{R}^d$. Similarly we say that a measure μ has full slices on V if the system $\{\mu_x^V\}_{x\in\mathbb{R}^d}$ of conditional measure on parallel translates of V satisfies $\dim \mu_x^V = \dim V$ for μ-a.e. x. Finally, we say that V is **linearly invariant under** $\Phi = \{\varphi_i\}$ if $U_i V = V$ for all i, where U_i is the linear part of φ_I. We say that V is non-trivial if $0 < \dim V < d$.

Conjecture 2.9. *Let $X = X_\Phi$ and $\mu = \mu_{\Phi,p}$ be a self-similar measure in $\mathbb{R}^d$. Then dimension drop for X implies that there is a non-trivial linearly invariant subspace $V \leq \mathbb{R}^d$ on which X has full slices, and the analogous statement holds for μ.*

Define a metric d on the similarity group $\mathrm{Aff}(\mathbb{R}^d)$ of $\mathbb{R}^d$ using any of the metrics described after equation (6).

Theorem 2.10 (Hochman). *Let $\mu = \mu_{\Phi,p}$ be a self-similar measure in $\mathbb{R}^d$ with Lyapunov dimension s. Then at least one of the following holds:*

- $\dim \mu = \min\{d, s(\Phi)\}$.
- $\Delta_n \to 0$ *super-exponentially.*
- *There is a non-trivial linearly invariant subspace $V \leq \mathbb{R}^d$ on which μ has full slices.*

In particular, if the linear parts of the maps φ_i act irreducibly on $\mathbb{R}^d$ then dimension drop implies $\Delta_n \to 0$ super-exponentially. If additionally $\{U_i\}$ generate a free group and have algebraic entries, then there is no dimension drop.

The same holds for the attractor.

The analogous statements for L^q dimension are at present not established, but we anticipate that some version of them holds, at least in the case where the linear parts of the contractions are homotheties.

In dimension $d \geq 3$ the orthogonal group is non-abelian, and the random walk associated to the matrices A_i may have a spectral gap. Then a much stronger conclusion is possible:

Theorem 2.11 (Lindenstrauss-Varjú [29]). *Let $U_1, \ldots, U_k \in SO(d)$ and $p = (p_1, \ldots, p_k)$ a probability vector. Suppose that the operator $f \mapsto \sum_{i=1}^k p_i f \circ U_i$ on $L^2(SO(d))$ has a spectral gap. Then there is a number $\widetilde{r} < 1$ such that for every choice $\widetilde{r} < r_1, \ldots, r_k < 1$, and for any $a_1, \ldots, a_k \in \mathbb{R}^d$, the self similar measure with weights p for the IFS $\{r_i U_i + a_i\}_{i=1}^k$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$.*

Contrasting this statement with the previous corollary, in the former we get $\dim \mu = d$ as soon as $s(\Phi) > d$, whereas in the latter we get absolute continuity only when the contraction is close enough to 1, but miss part of the potential parameter range. It is not known if this additional assumption is necessary for the conclusion.

2.6 Parametric families

Many classical problems in geometric measure theory involve parametric families of IFSs, e.g. the Bernoulli convolutions and projection problems discussed below, and the {0,1,3}-problem mentioned earlier. In these problems one wants to show that dimension drop is rare in the parameter space.

To set notation, suppose that $\{\Phi_t\}_{t\in I}$ is a parametric family of IFSs on $\mathbb{R}$, so $\varphi_i^t(x) = r_i(t)(x - a_i(t))$ where $r_i : I \to (-1,1) \setminus \{0\}$ and $a_i : I \to \mathbb{R}$ are given functions. For infinite sequences $\mathbf{i}, \mathbf{j} \in \Lambda^{\mathbb{N}}$ set

$$\Delta_{\mathbf{i},\mathbf{j}}(t) = \lim_{n\to\infty} \varphi^t_{i_1\ldots i_n}(0) - \varphi^t_{j_1\ldots j_n}(0)$$

It is clear that if Φ_t has exact overlaps then there exist $\mathbf{i}, \mathbf{j} \in \Lambda^{\mathbb{N}}$ with $\Delta_{\mathbf{i},\mathbf{j}}(t) = 0$, but $\Delta_{\mathbf{i},\mathbf{j}}$ may certainly vanish also when there are non-exact overlaps. However, under an analyticity and non-degeneracy assumption, the zeros of $\Delta_{\mathbf{i},\mathbf{j}}(\cdot)$ will be isolated, and the function will grow polynomially away from its zeros.[4] Furthermore, by a compactness argument, the exponent rate is uniform. Also, assuming analyticity, lower bounds on $\Delta_{\mathbf{i},\mathbf{j}}$ can be translated to lower bounds for $|\varphi^t_{i_1\ldots i_n}(0) - \varphi^t_{j_1\ldots j_n}(0)|$, and hence for $d(\varphi^t_{i_1\ldots i_n}, \varphi^t_{j_1\ldots j_n})$. From these ingredients one obtains efficient covers of the set of parameters for which Φ_t is not exponentially separated. The end result of this analysis is the following.

Theorem 2.12 (Hochman [20]). *Let $I \subseteq \mathbb{R}$ be a compact interval, let $r : I \to (-1,1) \setminus \{0\}$ and $a_i : I \to \mathbb{R}$ be real analytic, and let $\Phi_t = \{\varphi_{i,t}\}_{i\in\Lambda}$ be the associated parametric family of IFSs, as above. Suppose that*

$$\forall \mathbf{i}, \mathbf{j} \in \Lambda^{\mathbb{N}} \quad (\ \Delta_{\mathbf{i},\mathbf{j}} \equiv 0 \text{ on } I \quad \Longleftrightarrow \quad \mathbf{i} = \mathbf{j}\ ).$$

Then the set of $t \in I$ for which there is dimension drop **has** *Hausdorff and packing dimension* 0.

Analogous statements hold in $\mathbb{R}^d$, giving, under some assumptions, that the exceptional parameters have dimension $\le d - 1$. For details see [21].

2.7 Further developments

The same questions can be asked about attractors of non-linear IFSs. The only such case where a version of Theorem 2.3 is known is for linear fractional transformations [23]. Little is known beyond this case.

Another natural problem is to extend the results to self-affine sets and measures, defined in the same way but using affine maps rather than similarities.

[4] In contrast, the classical transversality method for parametric families depends on showing that $\Delta_{\mathbf{i},\mathbf{j}}$ grows linearly away from its zeros, i.e., it requires one to show that all zeros are simple. When this holds one often gets stronger conclusions, e.g. absolute continuity of the measures outside a small (though generally not zero-dimensional) set of parameters. But it is much harder to establish that the zeros are simple, and not always true. For more information we refer to [38, 37]. A major benefit of the method above is that polynomial growth is automatic.

This area is developing rapidly, and it seems likely that analogous results will be established in the near future.

Finally, we mention a result of Fraser, Henderson, Olson and Robinson, showing that if a self-similar set in $\mathbb{R}$ does not have exact overlaps then its Assouad dimension is one [13]. This is a very weak notion of dimension, equal to the maximal dimension of any set which is a Hausdorff limit of magnifications of X. It says nothing about $\dim X$ itself, but it lends moral support to the idea that without exact overlaps, X is "as large as possible".

3 Bernoulli convolutions

In the "supercritical" case $s(\Phi) > 1$, Conjecture 2.2 has a stronger variant:

Conjecture 3.1. *Let $\mu = \mu_{\Phi,p}$ be a self-similar measure on $\mathbb{R}$. If there are no exact overlaps and*[5] *$s(\Phi, p) > 1$, then μ is absolutely continuous with respect to Lebesgue measure.*

The main evidence supporting the conjecture comes from the study of parametric families, the primary example of which are Bernoulli convolutions. For $0 < \lambda < 1$ the **Bernoulli convolution with parameter** λ is the distribution ν_λ of the real random variable $\sum_{n=0}^{\infty} \pm\lambda^n$, where the signs are chosen i.i.d. with $\mathbb{P}(+) = \mathbb{P}(-) = \frac{1}{2}$. The name derives from the fact that ν_λ can be written as the infinite convolution of the measures $\frac{1}{2}(\delta_{-\lambda^n} + \delta_{\lambda^n})$, $n = 0, 1, 2, \ldots$, but it is also a self-similar measure for the IFS $\Phi_\lambda = \{\varphi_{\pm 1}\}$, defined by assigning equal probabilities to each of the maps

$$\varphi_{\pm 1}(x) = \lambda x \pm 1. \tag{13}$$

Let $\Lambda = \{\pm 1\}$. For $\mathbf{i}, \mathbf{j} \in \Lambda^n$, the maps $\varphi_{\mathbf{i}}, \varphi_{\mathbf{j}}$ contract by λ^n, so

$$d(\varphi_{\mathbf{i}}, \varphi_{\mathbf{j}}) = |\varphi_{\mathbf{i}}(0) - \varphi_{\mathbf{j}}(0)| = |\sum_{k=0}^{n} (i_k - j_k)\lambda^k|$$

Hence, since $i_k - j_k \in \{-2, 0, 2\}$, exact overlaps occur if and only if λ is the root of a polynomial in with coefficients $-1, 0, 1$. Write $\nu_\lambda^{(n)}$ for the distribution of the finite sum $\sum_{k=0}^{n} \pm\lambda^k$.

The case $\lambda < \frac{1}{2}$ is simple from the point of view of dimension: Φ_λ satisfies the SSC and $\dim \nu_\lambda = \log \lambda / \log(1/2)$. Also, $\nu_{1/2}$ is uniform on $[-2, 2]$.

Things are more interesting for $\lambda > \frac{1}{2}$. Then the Lyapunov dimension is > 1, the attractor is an interval, and Φ_λ has overlaps. From Conjectures 2.2 and 3.1 one would expect that ν_λ is absolutely continuous (and $\dim \nu_\lambda = 1$) unless there are exact overlaps. Thus, we shall say that λ is **a.c.-exceptional** or **dim-exceptional**, if ν_λ is singular or $\dim \nu_\lambda < 1$, respectively. We denote the sets of these parameters by E_{ac} and $E_{\dim}$.

[5] By Shief's theorem [44], absolute continuity can fail in the "critical" case $s(\Phi, p) = 1$.

It was Erdős who found the first, and so far only, exceptional parameters: if λ^{-1} is a Pisot number[6] then λ is a.c.-exceptional, and Garsia later showed that such λ are also dim-exceptional. Perhaps these are the only ones; some support for this is Salem's theorem that $|\widehat{\nu}_\lambda(t)| \to 0$ as $t \to \infty$ if and only if λ^{-1} is not Pisot.

3.1 Bounds on the size of the exceptional parameters

Much of the work on Bernoulli convolutions has focused on bounding the size of the set of exceptions. The work of Erdős and Kahane implies that $\dim((a,1) \cap E_{\mathrm{ac}}) \to 0$ as $a \nearrow 1$, and Erdős proved that $\dim \nu_\lambda \to 1$ as $\lambda \to 1$ (see also [37]).

A major step forward was Solomyak's proof in 1995 that ν_λ is a.c. for almost every $\lambda \in (\frac{1}{2}, 1)$ [50]. This was one of the early successes of the transversality method. Some improvements, including some bounds on the dimension of exceptions, were later obtained by Peres and Schlag [37].

Theorem 2.12 leads to further improvements:

Theorem 3.2 (Hochman [20]). $\dim \nu_\lambda = 1$ *outside a set of* λ *of Hausdorff and packing dimension* 0.

Currently, these techniques don't give absolute continuity directly, but combined with Fourier-theoretic information, Shmerkin managed to prove

Theorem 3.3 (Shmerkin [45, 47]). *Outside a set of* λ *of Hausdorff dimension* 0, *the measure* ν_λ *is absolutely continuous with density in* L^p *for all* $1 \leq p < \infty$.

Here is the idea of the proof. Fix an integer k, and split the random sum as

$$\sum_{n=1}^{\infty} \pm\lambda^n = \sum_{n=0 \bmod k} \pm\lambda^{kn} + \sum_{n\neq 0 \bmod k} \pm\lambda^n \tag{14}$$

The first term on the right has distribution ν_{λ^k}. Write τ_{λ^k} for the distribution of the second term. Since the two series are mutually independent, we get $\nu_\lambda = \nu_{\lambda^k} * \tau_{\lambda^k}$. Next, using an energy-theoretic argument, it is shown that the convolution will be absolutely continuous provided that the Fourier transform $\widehat{\nu}_{\lambda^k}$ has power decay (i.e. $|\widehat{\nu}_{\lambda^k}(t)| \leq t^{-c}$ for some $c = c(\lambda) > 0$) and $\dim \tau_{\lambda^k} = 1$. Now, a classical result of Erdős and Kahane (see [38]) says that outside a zero-dimensional set E' of parameters, the Fourier transform $\widehat{\nu}_{\lambda^k}$ indeed has power decay; on the other hand, τ_{λ^k} is itself a parametric family of self-similar measures, and Theorem 2.12 implies that there is dimension drop for a set E'' of parameters of dimension zero. For $\lambda \notin E''$ we will have $\dim \tau_{\lambda^k} = 1$ whenever the Lyapunov dimension is ≥ 1, which by a short calculation happens when $\lambda \in ((1/2)^{1-1/k}, 1)$. Thus, ν_λ is absolutely continuous for $\lambda \in [(1/2)^{1-1/k}, 1] \setminus (E' \cup E'')$. Taking the union over k gives the claim.

In order to obtain densities in L^p, an analogous argument is carried out using Theorem 2.7 instead of Theorem 2.3.

[6] A Pisot number is an algebraic integer greater than one, all of whose conjugates lie in the interior of the unit disk.

There remains a difference between Theorems 3.2 and 3.3. In the former, a parameter is "good" if Φ_λ is exponentially separated, which gives new explicit examples, e.g. all rational parameters. The set E'' in the above is similarly explicit. But E' is completely ineffective. Consequently, to get new examples of absolutely continuous ν_λ requires other methods, see Section 3.3.

3.2 Mahler measure

The **Mahler measure** of an algebraic number λ is $M_\lambda = |a| \cdot \prod_{|\xi|>1} |\xi|$ where the product is over all Galois conjugates ξ of λ, and a is the leading coefficient of its minimal polynomial. This is a standard measure of the size or complexity of an algebraic number. It first appeared in connection with Bernoulli convolution in the following lemma of Garsia:

Lemma 3.4 (Garsia [17]). *Let $\lambda > 1$ be algebraic with conjugates $\lambda_1, \ldots, \lambda_s \neq \lambda$, of which σ lie on the unit circle. Let $p(x) = \sum_{i=0}^n a_i x^i$ be an integer polynomial and $A = \max\{|a_i|\}$. Then either $p(\lambda) = 0$, or*

$$|p(\lambda)| \geq \frac{\prod_{|\lambda_i|\neq 1} ||\lambda_i| - 1|}{A^s(n+1)^\sigma \left(\prod_{|\lambda_1|>1} |\lambda_i|\right)^n} \geq \frac{C_\lambda}{A^s(n+1)^\sigma M_\lambda^n}$$

In particular, any distinct atoms of the n-th approximation $\nu_\lambda^{(n)}$ of ν_λ are separated by at least $C_\lambda((n+1)^\sigma M_\lambda)^{-1}$.

Recently, Mahler measure has been related to the random walk entropy h_λ associated to ν_λ in Section 2.3:

Theorem 3.5 (Breuillard-Varjú [6]). *There exists a constant $c > 0$ such that for any algebraic number $\lambda \in (\frac{1}{2}, 1)$,*

$$c \min\{1, \log M_\lambda\} \leq h_\lambda \leq \min\{1, \log M_\lambda\}$$

In particular, if $\lambda > \min\{2, M_\lambda\}^{-c}$ (for c a as in the theorem), then $\dim \nu_{\lambda=1}$. We do not discuss the proof of the theorem here, the reader may consult [6].

3.3 Absolute continuity for algebraic parameters

By the identity $\nu_\lambda = \nu_{\lambda^k} * \tau_{\lambda^k}$, which follows from (14), $\nu_{1/2^{1/k}} = \nu_{1/2} * \tau_{1/2}$, hence, since $\nu_{1/2}$ is Lebesgue measure on an interval, $\nu_{1/2^{1/k}}$ is absolutely continuous. Garsia identified a less trivial class of examples: those λ such that λ^{-1} is an algebraic integer, and $M_\lambda = 2$. Such numbers are not roots of $0, \pm 1$-polynomials, so $\sum_{i=0}^n \pm\lambda^n$ takes 2^n equally likely values, and by Lemma 3.4 the values are $c \cdot 2^{-n}$-separated (for some c). This implies that $\nu_\lambda = \lim \nu_\lambda^{(n)}$ is absolutely continuous. Until recently these were the only examples. We now have the following, which gives many more. For example, it applies to every rational number close enough to one in a manner depending on their denominator [53, Section 1.3.1].

Theorem 3.6 (Varjú [53])**.** *For every $\varepsilon > 0$ there is a $c > 0$ such such if $\lambda \in (\frac{1}{2}, 1)$ is algebraic and satisfies*

$$\lambda > 1 - c\min\{\log M_\lambda, (\log M_\lambda)^{-(1-\varepsilon)}\}$$

Then ν_λ is absolutely continuous with density in $L\log L$.

The proof relies on the following, which goes back to Garsia:

Theorem 3.7 (Garsia [18])**.** *ν_λ is absolutely continuous with density in $L\log L$ if and only if $H(\nu_\lambda^{(n)}) = n - O(1)$.*

The argument in Theorem 2.3 gives $H(\nu_\lambda^{(n)}) = n - o(n)$; in order to get the $O(1)$ error required by Garsia's theorem, Varjú proves two quantitative variants of the general entropy-growth result underlying Theorem 2.8. Roughly speaking,[7] the first shows that if α is small enough, and measures θ_1, θ_2 satisfy $H(\theta_i, \mathcal{D}_{n+1}|\mathcal{D}_n) > 1 - \alpha$, then $H(\theta_1 * \theta_2, \mathcal{D}_{n+1}|\mathcal{D}_n) > 1 - \alpha^2$. The second is analogous to Theorem 2.8 but with $\delta = c\varepsilon$. Now, fixing N, split the series $\sum_{n=0}^{N} \pm\lambda^n$ into $k = [\log(N^2)]$ finite sums $\sum_{n\in I_i} \pm\lambda^n$ of distribution $\nu_\lambda^{(N,i)}$ respectively, so that $\nu_\lambda^{(N)} = \nu_\lambda^{(N,1)} * \ldots * \nu_\lambda^{(N,k)}$. If we can choose $I_1, \ldots, I_k$ so that $H(\nu_\lambda^{(N,i)}, \mathcal{D}_N|\mathcal{D}_{N-1}) > 1 - \alpha_0$, with α_0 small, we can apply the first entropy growth result iteratively, and get $H(\nu_\lambda^{(N)}, \mathcal{D}_N|\mathcal{D}_{N-1}) > 1 - \alpha_0^{2^{\log k}} = O(1/N^2)$; summing over $0 \le N \le M$ gives $H(\nu_\lambda^{(M)}) = M - O(1)$, as desired. In order to find $I_1, \ldots, I_k$ as above, one uses a similar argument, relying on the second entropy-growth theorem to amplify the random walk entropy provided by Theorem 3.5. For more details, we refer to the original paper.

3.4 Dimension results for other parameters

Let $\mathcal{P}$ be the set of polynomials with coefficients $0, \pm 1$ and set $\mathcal{P}_n = \{f \in \mathcal{P} : \deg f \le n\}$. Suppose that $\dim \nu_\lambda < 1$. Then by Theorem 2.3, $\Delta_n \to 0$ super-exponentially, i.e. there exist $p_n \in \mathcal{P}_n$ such that $p_n(\lambda) \to 0$ super-exponentially. This does not force λ to be algebraic, but using transversality arguments or Jensen's formula, one can find roots λ_n of p_n such that $|\lambda_n - \lambda| \to 0$ super-exponentially, so λ_n, λ_{n+1} are super-exponentially close. If the roots of elements of $\mathcal{P}_n$ were sufficiently (i.e. exponentially) separated, this would force the sequence λ_n to stabilize, and λ would be algebraic, in fact a root of some $p_{n_0} \in \mathcal{P}$. Thus, an affirmative answer to the following problem would imply $\dim \nu_\lambda = 1$ for all λ without exact overlaps:

Question 3.8. *Does there exist a constant $c > 0$ such that if $\alpha \ne \beta$ are roots of (possibly different) polynomials in $\mathcal{P}_n$, then $|\alpha - \beta| > c^n$?*

The best current bound, due to Mahler, is of the form $|\alpha - \beta| > n^{-cn}$ for some constant $c > 0$ [31]. In order for this to be useful, one needs to get a similar

[7]We have omitted many assumptions, logarithmic factors, and even then the entropy inequalities are false using Shannon entropy; one must use spatially averaged entropy, see [53].

rate for the decay of Δ_n in Theorem (2.3). This is essentially the content of the following:

Theorem 3.9 (Breuillard-Varjú [7]). *Suppose that* $\dim \nu_\lambda < 1$ *for some* $\lambda \in (\frac{1}{2}, 1)$. *Then there exist arbitrarily large* n *for which there is an algebraic number* ξ *that is a root of a polynomial in* $\mathcal{P}_n$, *such that*

$$|\lambda - \xi| < \exp(-n^{\log\log\log n})$$

and

$$\dim \nu_\xi < 1.$$

Compared to the argument at the start of the section, notice that the numbers ξ are only guaranteed to exist for infinitely many n, not all large enough n. Therefore, even though the rate is better than Mahler's bound, one cannot conclude that the approximants stabilize. But, most importantly, the algebraic number ξ are guaranteed to be themselves exceptional for dimension. The latter has a dramatic implication:

Theorem 3.10 (Breuillard-Varjú [7]). $E_{\dim} = \overline{E_{\dim} \cap \overline{\mathbb{Q}}}$, *where* $\overline{\mathbb{Q}}$ *denotes the algebraic closure of* $\mathbb{Q}$.

This reduces the question of the dimension of Bernoulli convolutions to the algebraic case. In particular, it is known that the Pisot numbers form a closed set, so if these were shown to be the only algebraic parameters with dimension drop, these would be the only exceptions altogether. Also, recall that Lehmer's famous problem asks whether $\inf\{M_\lambda : M_\lambda \neq 1\} > 1$. If this were true, then Theorem 3.5 would imply that there is an $\varepsilon > 0$ such that $\dim \nu_\lambda = 1$ for every algebraic $\lambda \in (1 - \varepsilon, 1)$; combined with Theorem 3.10, this gives:

Theorem 3.11 (Breuillard-Varjú [7]). *If the answer to Lehmer's problem is affirmative, then there is an* $\varepsilon > 0$ *such that* $\dim \nu_\lambda = 1$ *for every* $\lambda \in (1 - \varepsilon, 1)$.

There is no known converse, but for a related result see [38, Proposition 5.1].

Finally, using the information gained about algebraic approximants of exceptional parameters, Breuillard and Varjú have managed to find the first explicit transcendental parameters for which ν_λ has full dimension; e.g. $e, 1/\ln 2$, and other natural constants. For details see [7].

4 Projection and slice theorems

A basic principle in (fractal) geometry is that projections of a set typically should be "as large as possible", and slices should be correspondingly small. By a projection of a set we mean its image under an orthogonal projection π_V to a linear subspace V, and by a slice we mean its intersection with an affine subspace. The trivial bound for projections is $\dim \pi_V X \leq \min\{\dim X, \dim V\}$ (because π_V is Lipschitz and has range V). The following theorem shows that this is generally the right bound, and that slices behave dually. Let $G(d, k)$ denote the manifold of k-dimensional affine subspaces of $\mathbb{R}^d$.

Theorem 4.1 (Marstrand [32], Mattila [33]). *Let $E \subseteq \mathbb{R}^d$ be Borel and $1 \leq k < d$. Then for a.e. $V \in G(d,k)$,*

$$\dim \pi_V E \quad = \quad \min\{k, \dim E\}$$

If in addition $\dim E > k$ then[8] *$\pi_V(E)$ has positive k-dimensional volume for a.e. $V \in G(d,k)$, and for a.e. $y \in V$ with respect to the volume,*

$$\dim E \cap \pi_V^{-1}(y) \leq \max\{0, \dim E - k\}$$

Kaufman, Falconer and Mattila (see e.g. [34]) also bounded the dimension of the set of exceptional $V \in G(d,k)$, e.g. for $d = 2$ and $k = 1$, if $\dim E < k$ the exceptions have dimension $\leq \dim E$, and if $\dim E > k$ it is at most $d - \dim E$.

What these general results for generic directions fail to give is any information at all about particular directions. For "natural", well-structured sets, one would expect to be able to be more precise. What one expects in such cases is that the projections will be as large as possible unless there is some combinatorial or algebraic obstruction; and that slices be correspondingly small.

In the coming discussion we restrict attention to self-similar sets in $\mathbb{R}^2$, where results are more complete. We mention measures only occasionally, and multi-dimensional analogues, which require more assumptions. We also do not discuss results on randomly generated fractals, for this see [41, 36, 48].

4.1 Dimension conservation

Heuristically, projections to V and slices in direction $V^\perp$ are complementary, in the sense that having a large image forces most slices to be small, and vice versa. This is exactly true for finite-scale entropy, and combinatorial versions can also be formulated. For dimension, this duality does not always hold. The following "dimension conservation", a relative of the Ledrappier-Young formula, result marked the start of the current phase of research.

Theorem 4.2 (Furstenberg [16]). *Let $X \subseteq \mathbb{R}^d$ be a self-homothetic self-similar set. Then for every $V \in G(d,k)$, we have*

$$\dim \pi_V X + \sup_{y \in V} \dim(X \cap \pi_V^{-1}(y)) \geq \dim X$$

For self-homothetic measures μ there is in fact equality for $\pi_V\mu$-a.e. y, and until recently it was not known whether this, or the theorem might apply also outside of homotheties. It turns out that

Theorem 4.3 (Rapaport [42]). *There exists a self-similar measure μ on $\mathbb{R}^2$ with $\dim \mu > 1$, uniform contractions and uniform dense rotations, such that for a dense G_δ set of directions V the conditional measures on translates of V are a.s. atomic, hence*

$$\dim \pi_V \mu + \operatorname*{esssup}_{y \sim \pi_V \mu} \dim \mu_{\pi^{-1}(y)} < \dim \mu$$

[8] Although unrelated to our discussion, it is worth mentioning that if $\dim E = k$, then $\pi_V E$ will have positive k-dimensional volume depending on whether it is rectifiable or purely unrectifiable. See [34].

Problem 4.4. For is self-similar set in $\mathbb{R}^2$ with dense rotations, do all projections have positive length?

4.2 Projections of self-homothetic sets

Self-homothetic self-similar sets X have the special property that each projection $\pi_V X$ is also self-homothetic, being the attractor of $\Phi_V = \{\varphi_{i,V}(x) = r_i x + \pi_V a_i\}$. Assuming that Φ has strong separation, and fixing a self-similar measure on X, Theorem 2.12 applies to the parametric family $\{\Phi_V\}$. In the case $\dim X > 1$ an argument similar to that in Theorem 3.3 also applies. All in all, we get

Theorem 4.5 (Hochman [20], Shmerkin [46])**.** *[46]If $X \subseteq \mathbb{R}^2$ is self-homothetic then $\dim \pi_V X = \min\{1, \dim X\}$ for all but a zero dimensional set of $V \in G(2,1)$. If also $\dim X > 1$, the projection will have positive length outside a set of V of dimension zero.*

The reason no separation condition is needed is that a self-similar set X always contains smaller self-similar sets of dimension arbitrarily close to $\dim X$, and satisfying strong separation and uniform contraction [39], and it is enough to show that these subsets have large projections.

More is true when the maps φ_i are algebraic, by which we mean that the parameters r_i, a_i are algebraic. The conjecture that this is the case was raised by Furstenberg for the so-called 1-dimensional Sierpinski gasket, first appearing in the work of Kenyon [27]. The next theorem follows from Corollary 2.4 and an argument (originally due to Solomyak) showing that if the projected IFS Φ_V does not have exact overlaps, then it is exponentially separated.

Theorem 4.6 ([20])**.** *If X is the attractor of an algebraic IFS consisting of homotheties, then $\dim \pi_V X = \min\{1, \dim X\}$ for all except at most countably many V, which are among the V which collapse cylinders, i.e. $\pi_V \varphi_{\mathbf{i}} = \pi_V \varphi_{\mathbf{j}}$ for some n and $\mathbf{i}, \mathbf{j} \in \Lambda^n$.*

There certainly can exist exceptional directions, but they have not been entirely characterized (a special case was analyzed by Kenyon [27]). Currently, no analogous result exists for the Lebesgue measure of the projection in the regime $\dim X > 1$. Finally, note that the result for measures seems to require strong separation, since for measures, there is no analog of the trick of passing to a sub-self-similar set.

All statements above hold if instead of self-homothetic sets we allow IFSs whose orthogonal parts generate a finite group of rotations.

4.3 Projections of sets and measures with rich symmetries

Let $\Phi = \{\varphi_i\}$ be an IFS in $\mathbb{R}^2$ with $\varphi_i(x) = r_i O_i x + a_i$, with $0 < r_1 < 1$, O_i an orthogonal matrix, and $a_i \in \mathbb{R}^2$. We say that Φ has **irrational rotations** if at least one O_i is an irrational rotation (has infinite order). Notice that $\pi_V \varphi_i X \subseteq \pi_V X$, and $\pi_V \varphi_i$, up to change of coordinates, is projection to $O_i^{-1} V$. Iterating this and using the fact that $\{O_{i_1}^{-1} \dots O_{i_n}^{-1} V\}_{\mathbf{i} \in \Lambda^*}$ is dense, we see that

$\pi_V X$ contains subsets approximating every other projection $\pi_W X$. This can be used to prove that there are no exceptional directions:

Theorem 4.7 ([39], [35], [22]). *Let $X \subseteq \mathbb{R}^2$ be a self-similar set with dense rotations. Then $\dim \pi_V X = \min\{1, \dim X\}$ for every V. The same holds for self-similar measures assuming the open set condition.*

This was first proved for sets by Peres and Shmerkin [39] for sets. Nazarov-Peres-Shmerkin [35] proved an analog for measures assuming uniform rotations. Hochman-Shmerkin proved the general version [22]. See also Farkas [11].

We briefly sketch the proof of Hochman-Shmerkin. A central ingredient is the method of local entropy averages. Suppose that ν is any probability measure on $\mathbb{R}^d$. Let $\mathcal{D}_n(x)$ denote the unique atom of $\mathcal{D}_n$ containing x and let

$$\nu_{x,n} = \frac{1}{\nu(\mathcal{D}_n(x))}\nu|_{\mathcal{D}_n(x)}$$

Thus the sequence $(\nu_{x,n})_{n=1}^{\infty}$ is what you see when you "zoom in" to x along dyadic cells.

Theorem 4.8 (Hochman-Shmerkin [22]). *Let ν be a Borel probability measure on $\mathbb{R}^d$, and $V \in G(d,k)$. If for some $\alpha \geq 0$ and $m \in \mathbb{N}$*

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} H(\nu_{x,n}, \mathcal{D}_{n+m}) \geq \alpha \tag{15}$$

then $\dim \pi_V \nu \geq \alpha - O_{d,k}(1/m)$.

If μ is self-similar with OSC, then $\mu_{x,n}$ is a piece (or a combination of boundedly many pieces) of a cylinder measure of diameter approximately 2^{-n}, and if there are dense rotations, for μ-a.e. x the rotations observed along the sequence $(\mu_{x,n})_{n=1}^{\infty}$equidistribute in the circle. Now, by Marstrand's theorem, $\dim \pi_V \mu = \min\{1, \dim \mu\}$ for a.e. line V, so by (8), for large enough m, with high probability over V we have $\frac{1}{m}H(\pi_V \mu, \mathcal{D}_m) > \min\{1, \dim \mu\} - \varepsilon$. This condition on V is essentially open, so $(\mu_{x,n})_{n=1}^{\infty}$ consists predominantly of measures that, after re-scaling by 2^n, satisfy this inequality. Theorem 4.8 now completes the argument.

This method also gives the following, which was conjectured by Furstenberg:

Theorem 4.9 (Hochman-Shmerkin [22]). *Let $Y_a, Y_b \subseteq [0,1]$ be closed and invariant under $x \mapsto ax \bmod 1$, $x \mapsto bx \bmod 1$, and assume $\log a/\log b \notin \mathbb{Q}$. Then $\dim \pi_V(Y_a \times Y_b) = \min\{1, \dim Y_a + \dim Y_b\}$ for all V except the horizontal and vertical directions.*

The methods also apply in non-linear cases. This has some overlap with the work of Moreira on nonlinearly generated fractals, which predates all the results above, but does not apply in the linear case. See [9].

Problem 4.10. If Y_a, Y_b are as in the theorem, and $\dim Y_a \times Y_b > 1$, does $\pi_V(Y_a \times Y_b)$ have positive Lebesgue measure for all V not parallel to the axes?

Surprisingly the analogous problem for products of self-similar measures has a negative answer, see Nazarov-Peres-Shmerkin [35].

4.4 Slices

Dual to the projections problem is that of slices. When projections achieve their maximal value, one might expect slices not to exceed their typical value. Furstenberg [15] conjectured this for non-vertical/horizontal slices of products as in Theorem 4.9. Very recently two independent proofs of this emerged:

Theorem 4.11 (Shmerkin [47], Wu [54]). *Let $X = Y_a \times Y_b$ be as in Theorem 4.9. Then $\dim(X \cap \ell) \leq \max\{0, \dim X - 1\}$ for all lines $\ell \subseteq \mathbb{R}^2$ not parallel to the axes. Similarly if X is self-similar set with uniform contraction and uniform dense rotations, the bounds holds for all ℓ.*

The case $\dim X < 1/2$ was proved by Furstenberg. He showed that if $\dim X \cap \ell = \alpha$, then there exists a stationary ergodic process $Z = (x_n, \theta_n)_{n=1}^{\infty}$ with $x_n \in X$ and $\theta_n \in [0,1]$, such that a.s. the line $\ell(x_n, \theta_n)$ of slope θ_n through x_n satisfies $\dim(X \cap \ell(x_n, \theta_n)) = \alpha$, and the process (θ_n) has pure point spectrum. Thus θ_1 is uniform on $[0,1]$, so in a.e. direction there is a pair $x', x'' \in X$ with $x' - x''$ pointing in that direction. This implies $\dim X \geq 1/2$.

Wu's proof is ergodic-theoretic and begins with the same construction. Now, if there were a point $\xi \in X$ such that the distribution of θ_1 is uniform given $x_1 = \xi$, this would give a "bouquet" of α-dimensional slices passing through ξ and pointing in a 1-dimensional set of directions, and imply $\dim X \geq 1 + \alpha$, the desired bound on α. To find such ξ, first apply a classical theorem of Sinai to get a Bernoulli factor W of the process $Z = (x_n, \theta_n)$ exhausting the entropy, i.e. $h(Z|W) = 0$. Let P_w denote the disintegration of the distribution of Z over $w \in W$ and let $Q_w = \mathbb{E}(x_1|w)$ be the image of P_w in X. Next, self-similarity gives an expanding conformal dynamics on X, and the process Z can be constructed such that $\dim Q_w$ is proportional to $h(Z|W)$, hence $\dim Q_w = 0$. Finally, $\Theta = (\theta_n)$ is a rotation and W is Bernoulli, by Furstenberg's disjointness theory [14], Θ is independent of W, hence θ_1 is distributed uniformly conditioned on w. Thus, there is a family of slices of dimension α, with uniformly distributed directions, passing through the points of the zero-dimensional measure Q_w. From this one can construct (approximations of) the desired bouquets.

Shmerkin's proof is entirely different. Consider the self-similar case with the natural self-similar measure μ on X. It is a basic fact that if $\inf_{q>1} D(\pi_V \mu, q) = \alpha$, then $\dim(X \cap \pi_V^{-1}(y)) \leq \dim X - \alpha$ for all y. Thus the goal is to show that the L^q dimension is maximal in all directions. Now, $\pi_V \mu$ are not self-similar but it has a convolutions structure, because all cylinders of a given generation in μ differ only by translations. By an argument similar to Theorems 2.7 and 2.12, we conclude that $\inf_{q>1} D(\pi_V \mu, q) = \min\{1, \dim X\}$ for a large set of V. To extend this for all directions, one uses (among many other things) unique ergodicity of a certain cocycle arising from the rotational symmetry of X.

Finally, Theorem 2.7 implies an L^q version of Theorem 4.6, which gives a dual result for the slices, confirming another old conjecture of Furstenberg:

Theorem 4.12 (Shmerkin [47]). *Let $X \subseteq \mathbb{R}^2$ be a self-homothetic algebraic self-similar measure. Then outside a countable set of directions, there are no exceptional slices.*

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