

DYNAMICS ON FRACTALS AND FRACTAL DISTRIBUTIONS

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ABSTRACT. We study fractal measures on Euclidean space through the dynamics of “zooming in” on typical points. The resulting family of measures (the “scenery”), can be interpreted as an orbit in an appropriate dynamical system which often equidistributes for some invariant distribution. The first part of the paper develops basic properties of these limiting distributions and the relations between them and other models of dynamics on fractals, specifically to Zähle distributions and Furstenberg’s CP-processes. In the second part of the paper we study the geometric properties of measures arising in these contexts, specifically their behavior under projection and conditioning on subspaces.

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1. INTRODUCTION

This work is a systematic study of a class of measures, called uniformly scaling measures (USMs), and associated distributions on measures, called fractal distributions (FDs), which capture the notion of self-similarity (or “fractality”) of a measure on Euclidean space in terms of the dynamics of rescaling and translation. USMs were introduced abstractly by Gavish [15] though examples were studied earlier by many authors, e.g. Patzschke and Zähle [25], Bandt [2, 1], and Graf [17]. Fractal distributions, which we define here, generalize Zähle’s scale-invariant distributions [30] and are very closely related to Furstenberg’s CP-processes [12, 14]. They may also be viewed as ergodic-theoretic analogues of the scenery flow for sets which was studied, among others, by Bedford and Fisher [3, 4, 5].

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The models we discuss here are sufficiently general so as to unify the treatment of many examples of interest in fractal geometry and dynamics, but at the same time are sufficiently structured that their geometric behavior is far better than general measures.

In particular, this work was motivated by some recent results by Furstenberg [14], Peres and Shmerkin [27], and Hochman and Shmerkin [19], on the geometry of measures which arise from certain combinatorial constructions or as invariant measures for certain dynamics. The behavior of such measures under projections and conditioning on subspaces was shown to be more regular than that of general measures. One of our motivations for the present work was to place these results in a more general framework and to clarify some of the objects involved. We also obtain new results for general measures by relating them to structured measures which arise from them using a limiting procedure.

In the remainder of this introduction we present the main definitions and results. More details discussion, examples and proofs are provided in the subsequent sections.

1.1. Standing notation. Throughout the paper d will be a fixed integer dimension. Equip \mathbb{R}^d with the norm $\|x\| = \sup |x_i|$ and the induced metric. $B_r(x)$ is the closed ball of radius r around x ; in particular, $B_1(0) = [-1, 1]^d$. We abbreviate

$$B_r = B_r(0)$$

Let λ denote Lebesgue measure on \mathbb{R}^d and δ_a the point mass at a .

Let $\mathcal{M} = \mathcal{M}(\mathbb{R}^d)$ denote the space of Radon measures on \mathbb{R}^d with the weak-* topology. For a measurable space X (e.g. a topological space with the Borel structure) let $\mathcal{P}(X)$ denote the space of probability measures X . When X is a compact metric space we give $\mathcal{P}(X)$ the weak-* topology, which is also compact and metrizable.

We reserve the term *distribution* for members of $\mathcal{P}(\mathcal{M})$ and for probability measures on similarly “large” spaces, denoting them by P, Q, R , and use the term *measure* exclusively for members of \mathcal{M} , which are denoted μ, ν, η etc. We generally use brackets to denote operations which produce distributions from measures, e.g. the operations $\langle \mu \rangle_U, \langle \mu \rangle_{x,T}, \langle \mu, x \rangle_N$ defined below.

Write $\mu \sim P$ to indicate that μ is chosen randomly according to the distribution P , and similarly $x \sim \mu$. We also write $\mu \sim \nu$ (or $P \sim Q$) to indicate equivalence of measures i.e. that μ, ν (or P, Q) have the same null sets. Which of these is intended will be clear from the context.

If $\mu \in \mathcal{M}$ and $\mu(A) > 0$ then $\mu|_A \in \mathcal{M}$ is the restricted measure on A , i.e.

$$\mu|_A(B) = \mu(A \cap B)$$

and, assuming $0 < \mu(A) < \infty$, let $\mu_A \in \mathcal{M}$ be normalized version of $\mu|_A$, i.e.

$$\mu_A(B) = \frac{1}{\mu(A)} \mu(A \cap B)$$

Finally, define two (partial) normalization operations $*, \square : \mathcal{M} \rightarrow \mathcal{M}$ by

$$\begin{aligned}\mu^* &= \frac{1}{\mu(B_1)}\mu \\ \mu^\square &= \left(\frac{1}{\mu(B_1)}\mu\right)|_{B_1}\end{aligned}$$

so that $\mu^\square = \mu|_{B_1}$. The operations are defined on the measurable subset $\{\mu \in \mathcal{M} : \mu(B_1) > 0\}$. We apply these operations also to sets and distributions, so

$$\begin{aligned}\mathcal{M}^* &= \{\mu \in \mathcal{M} : \mu(B_1) = 1\} \\ \mathcal{M}^\square &= \{\mu \in \mathcal{M} : \mu \text{ is a probability measure on } B_1\} \\ &\cong \mathcal{P}(B_1)\end{aligned}$$

P^* is the push-forward of the distribution P through $\mu \mapsto \mu^*$, and similarly P^\square , etc.

For $x \in \mathbb{R}^d$ let $T_x : \mathbb{R}^d \rightarrow \mathbb{R}^d$ denote the translation taking x to the origin:

$$T_x(y) = y - x$$

and given a ball $B = B_r(x)$ we write T_B for the orientation-preserving homothety mapping B onto B_1 , i.e.

$$T_B(y) = \frac{1}{r}(y - x)$$

Finally, we define scaling operators on \mathbb{R}^d by

$$S_t(y) = e^t y$$

Note the exponential time scale, which makes $S = (S_t)_{t \in \mathbb{R}}$ into an additive \mathbb{R} -action on \mathbb{R}^d by linear transformations (i.e. $S_{s+t} = S_s S_t$).

For $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\mu \in \mathcal{M}$ write $f\mu$ for the push-forward of μ through f , that is, $f\mu(A) = \mu(f^{-1}A)$. Thus the operators T_x, S_t induce translation and scaling operations on \mathcal{M} , given by

$$\begin{aligned}T_x\mu(A) &= \mu(A + x) \\ S_t\mu(A) &= \mu(e^{-t}A)\end{aligned}$$

We also append $*, \square$ to operations on measures to indicate post-composition with the corresponding normalization operator:

$$\begin{aligned}S_t^*\mu &= (S_t\mu)^* \\ S_t^\square\mu &= (S_t\mu)^\square\end{aligned}$$

and similarly T_x^\square etc. Then $S^* = (S_t^*)_{t \in \mathbb{R}}$ is again an additive \mathbb{R} -action on the measurable set $\{\mu \in \mathcal{M}^* : 0 \in \text{supp } \mu\}$, and $S^\square = (S_t^\square)_{t \in \mathbb{R}^+}$ is an additive \mathbb{R}^+ -action, though note that the restriction to B_1 is not an invertible operations, so the S_t^\square are not invertible. Note also that these actions are discontinuous at measures μ with $\mu(\partial B_1) > 0$.

See Section 1.11 below for a summary of the notation.

1.2. Fractal distributions. As usual in dynamical systems theory, one may study a dynamical system either globally, via the invariant sets or measures of the system, or in terms of the behavior of individual orbits. While individual orbits are often of most interest, it is usually easier to obtain global results or results for typical orbits. Some background in ergodic theory is provided in Section 2, see also [29, 16].

We shall similarly have two perspectives in our study of fractals: a global one, which is concerned with distributions (on measures) which possess certain invariance properties, and an individual one, which deals with individual measures whose “orbits” display some regularity.

Our “global” objects are distributions on measures which exhibit certain invariance under change of scale and translation, mirroring the idea of “self-similarity” of a measure. In this section we describe these objects axiomatically.

Recall that a probability distribution P on \mathcal{M}^* is S^* invariant if $P(S_t^*A) = P(A)$ for all measurable sets A and all $t \in \mathbb{R}$. If P is an S^* -invariant distribution then P^\square (the push-forward of P through $\mu \mapsto \mu^\square$) is an S^\square -invariant distributions called the *restricted version* of P , and P is the extended version of P^\square . This is a 1-1 correspondence between S^* - and S^\square -invariant distributions (see Lemma 3.1).

For a subset $U \subseteq \mathbb{R}^d$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$ define the U -diffusion of μ by

$$\langle \mu \rangle_U = \int_U \delta_{T_x^* \mu} d\mu(x)$$

i.e. $\langle \mu \rangle_U$ is the distribution of the random measure ν obtained by choosing $x \in U$ according to μ and setting $\nu = T_x^* \mu$.

Definition 1.1. A distribution P on \mathcal{M}^* is *quasi-Palm* if for every bounded, open neighborhood U of the origin,

$$P \sim \int \langle \mu \rangle_U dP(\mu)$$

Thus P is quasi-Palm when the following condition holds: $P(\mathcal{A}) = 0$ if and only if the random measure $T_x^* \mu$, obtained by selecting $\mu \sim P$ followed by $x \sim \mu$, is almost surely not in \mathcal{A} . This definition generalizes Palm distributions, which are defined similarly, except that one does not normalize the translated measures and one requires equality (rather than equivalence) of the distributions when U is a symmetric convex neighborhood of the origin.

Definition 1.2. A fractal distribution¹ (FD) is a probability distribution on \mathcal{M}^* which is S^* -invariant and quasi-Palm. An *ergodic fractal distribution* (EFD) is an FD which is ergodic with respect to S^* .

When P is an FD we shall often refer to its restricted version P^\square as an FD as well, even though technically it is not; this is justified by the 1-1 correspondence between restricted and extended versions. See section 3.1 for some remarks on the definition.

¹The terminology derives from Furstenberg’s notion of an ergodic fractal measure [14], which essentially means is a generic measure for a CP-distribution (defined below). Later we shall see that a typical measure for an FD is, up to a translation, an ergodic fractal measure in this sense.

This notion is closely related to the α -scale-invariant distributions of Zähle [30]. These are Palm distributions which are invariant under a family of scaling operations depending on the parameter α . FDs are strictly more general objects and apply in many cases where Zähle distributions are inappropriate. The relationship between them can be described precisely: up to normalization ergodic Zähle distributions are those EFDs which are supported on measures which have second order densities a.e.. See Section 3.3.

As we shall see later there is no shortage of FDs, but constructing a non-trivial one directly requires a little work. At this point we give two trivial examples. One may, first, take $P = \delta_{\lambda^*}$, i.e. the distribution which is concentrated on (normalized) Lebesgue measure λ^* . Clearly λ^* is a fixed point for both translation T_x^* and for the scaling operators S_t^* , so P is an EFD. Second, one can consider the distribution $P = \delta_{\delta_0}$. Since δ_0 is S^* -invariant, and $\langle \delta_0 \rangle_U = \delta_{\delta_0}$ for every neighborhood U of the origin, P is an EFD.

As an example of an S^* -invariant distribution which is not an FD, consider the measure $\eta = (\lambda|_{[-\infty, 0)})^*$ and let $P = \delta_\eta$. Since η is an S^* -fixed point P is S^* -invariant, but for any neighborhood U of the origin, $\langle \eta \rangle_U$ is supported on measures which a.s. give positive mass to $(0, \infty)$ while $\eta((0, \infty)) = 0$, so $\langle \eta \rangle_U \not\sim \delta_\eta$. Therefore δ_η is not an FD.

The ergodic components of an FD with respect to S^* are of course S^* -invariant, but not a-priori quasi-Palm. This is the content of:

Theorem 1.3. *The ergodic components of an FD are EFDs.*

We discuss this and other decompositions in Section 3.4.

1.3. Uniformly scaling measures. Next, we examine individual measures which display dynamical regularity upon “zooming in” to typical points. This idea has a long history; one may view the density theorems of Lebesgue and Besicovitch as early manifestations of it, and similarly the work of D. Preiss on tangent measures. More recently the dynamical perspective has been taken up by many authors [25, 2, 17, 3, 1, 4, 21, 23, 24, 5, 26, 11, 15, 19].

Given a measure $\mu \in \mathcal{M}$ and $x \in \text{supp } \mu$ one may translate x to the origin, forming $T_x\mu$, and consider the orbit of this measure under S^* , which is called the scenery of μ at x . We are interested in measures for which for typical x the scenery equidistributes in the space of measures, i.e. the uniform measure on the orbit up to time T converges, as $T \rightarrow \infty$, to some distribution. Unfortunately, the space of measures on \mathcal{M} does not carry a good topology.² We therefore work in \mathcal{M}^\square , which, when identified with $\mathcal{P}([-1, 1]^d)$, is compact and metrizable in the weak-* topology.

Definition 1.4. Let $\mu \in \mathcal{M}$ and $x \in \text{supp } \mu$. The parametrized family $(\mu_{x,t}^\square)_{t>0}$ given by

$$\mu_{x,t}^\square = S_t^\square(T_x\mu)$$

²Some authors have used vague convergence, but this causes various complications, such as degenerate limit points and “inseparability” of the topology (though it is not really a topology). We prefer to avoid these.

is called the *scenery* of μ at x . The *scenery distributions* is obtained by placing length measure on initial segments of the scenery:

$$\langle \mu \rangle_{x,T} = \frac{1}{T} \int_0^T \delta_{\mu_{x,t}^\square} dt$$

Similar definitions have appeared previously in the literature. Our method of normalization appears in Bandt [2, 1], Graf [17], and Gavish [15]. Other authors [30, 24, 26, 25] have studied the normalization in which $\mu_{x,t}^\alpha = e^{\alpha t} S_t(T_x \mu)$, which is appropriate when μ has second-order density α at x , but does not apply to many common cases, e.g. when second-order densities do not exist. See section 3.3.

There is in general no reason the scenery distributions should converge as $T \rightarrow \infty$. In the case where they do we introduce the following terminology (Gavish [15]):³

Definition 1.5. Let $\mu \in \mathcal{M}$ and $x \in \text{supp } \mu$.

- (1) If $\langle \mu \rangle_{x,T} \rightarrow P$ as $T \rightarrow \infty$ we say that μ *generates* P at x .
- (2) If μ generates a distribution P_x at μ -a.e. x , we say that μ is a *scaling measure* (SM).
- (3) If μ generates the same distribution P at a.e. point then μ is a *uniformly scaling measure* (USM) and μ generates P .

There is a close relation between FDs and SMs, similar to the relation between an invariant measure on a dynamical system and a generic point for the measure. In one direction, the following claim is an easy consequence of the definitions and the ergodic theorem:

Theorem 1.6. *If P is an FD then P -a.e. measure is a USM generating the ergodic component of P to which μ belongs.*

For the proof see Section 3.5. The converse is less trivial:

Theorem 1.7. *If $\mu \in \mathcal{M}$ then for μ -a.e. x , every accumulation point of $\langle \mu \rangle_{x,T}^\square$ is an FD. In particular, if μ generates P then P is an FD.*

See section 5.

While it is fairly obvious⁴ that an accumulation point of the scenery distributions is S^\square -invariant, it is remarkable that the distribution should have the additional spatial invariance of an FD. This fact is analogous to a similar one discovered by P. Mörters and D. Preiss [24], namely that Zähle distributions arise as the limiting distributions of sceneries of measures with average local density α using the scaling operation S^α .

Note also that in general the distributions generated by scenery distributions of a measure need not be S^* -ergodic.

One important property of the limiting distributions of sceneries (and more generally their accumulation points) is that they behave nicely under linear maps, and, since they are infinitesimal notions, under local diffeomorphisms. The following propositions follow easily from the definitions, and we record them here for later use.

³Gavish uses the term “measures with uniformly scaling sceneries”, but we prefer the shorter name.

⁴Actually this is somewhat tedious to prove directly, due to the discontinuity of the action of S^\square .

Proposition 1.8. *If $U \in GL(\mathbb{R}^d)$ is a linear map then the induced map $U^* : \mathcal{M}^* \rightarrow \mathcal{M}^*$ maps EFDs to EFDs.*

Proposition 1.9. *Let $\mu \in \mathcal{M}$ be supported on an open set U and let $f : U \rightarrow V \subseteq \mathbb{R}^d$ be a diffeomorphism. Then for every $x \in U$, if $\langle \mu \rangle_{x, T_i} \rightarrow P$ for some sequence of times T_i , then $\langle f\mu \rangle_{f x, T_i} \rightarrow (D_x f)^* P$. In particular if μ is a scaling measure then so is $f\mu$.*

1.4. CP-distributions. Another closely related family of “self-similar” distributions are Furstenberg’s CP-distributions [14]. We recall the definition.

Fix an integer $b \geq 2$ and let $\mathcal{D}_b = \mathcal{D}_b(\mathbb{R})$ denote the partition of \mathbb{R} into intervals of the form $[\frac{k}{b}, \frac{k+2}{b})$ for $k = b \bmod 2$. This partition was chosen so that it divides $[-1, 1]$ into b equal intervals of length $2/b$ each.⁵ For $d > 1$ define $\mathcal{D}_b = \mathcal{D}_b(\mathbb{R}^d)$ to be the partition of \mathbb{R}^d into cells of the form $I_1 \times \dots \times I_d$, $I_i \in \mathcal{D}_b(\mathbb{R})$, so B_1 is partitioned into b^d cells. We denote by $\mathcal{D}_b(x)$ the unique element of \mathcal{D}_b containing x .

Definition 1.10. The base- b magnification operator $M_b : \mathcal{M}^\square \times B_1 \rightarrow \mathcal{M}^\square \times B_1$ is given by

$$M_b(\mu, x) = (T_{\mathcal{D}_b(x)}\mu, T_{\mathcal{D}_b(x)}x)$$

We denote by M_b^* , M_b^\square the associated operators in which, after M_b is applied, $*$, \square are applied, respectively, to the measure component of the output.

Both M_b and its domain are measurable, though M_b is discontinuous at pairs (μ, x) when $\mu(\partial\mathcal{D}_b(x)) > 0$. Starting from (μ, x) , the orbit $(M_b^\square)^n(\mu, x) = M_{b^n}^\square(\mu, x)$, $n = 0, 1, 2, \dots$ may be viewed as the scenery obtained by zooming in to x along \mathcal{D}_b -cells. We call this the b -scenery at x . Note that this is a discrete time sequence, and the point x is generally not in the “center of the frame” as it is for sceneries.

Definition 1.11. A distribution Q on $\mathcal{M}^\square \times B_1$ is *adapted* if, conditioned on the first component being ν , the second component is distributed according to ν . Equivalently, for every $f \in C(\mathcal{M}^\square)$,

$$\int f(T_x\nu) dQ(\nu, x) = \int \left(\int f(T_x\nu) d\nu(x) \right) dQ(\nu)$$

Definition 1.12. A restricted base- b CP-distribution⁶ is an adapted probability distribution on $\mathcal{M}^\square \times B_1$ which is invariant under M_b^\square

The projection of a CP-distribution to the first component is a distribution on \mathcal{M}^\square , and we sometimes refer to this distribution also as a CP-distribution.⁷

⁵It would be more natural to work with the partition of \mathbb{R} into b -adic intervals, i.e. intervals of the form $[\frac{k}{b}, \frac{k+1}{b})$, which divides $[0, 1]$ into b subintervals. The definitions of CP-distributions and processes in [14, 19] follow this path. This would make sense if we used $[0, 1]$ as our “basic” interval instead of $B_1 = [-1, 1]$, but much of our notation is adapted to B_1 , e.g. the maps T_B and normalization operators, and because of this it is more efficient to adopt the present definition.

⁶The terminology follows [14]. “CP” stands for conditional probability, hinting at the role of adaptedness in the definition.

⁷Given b it is easy to see that the measure component of a base- b CP-distribution determines the original distribution. If b is not given this may not hold: certain distributions supported on Lebesgue measure and atomic measures are CP-distributions for several bases. It is not clear if there are any other examples. This may be regarded as a version of Furstenberg’s $\times 2, \times 3$ problem.

To every restricted CP-distribution P there is an associated extended version on $\mathcal{M}^* \times B_1$ which is invariant under the M_b^* and projects to Q under $(\mu, x) \mapsto (\mu^\square, x)$. This distribution is not unique but the construction is canonical; see Section 3.2.

The relation of CP-processes and USMs was previously examined by Gavish [15], who showed that CP-processes typically give rise to USMs. We strengthen this to show that they give rise to FDs. Define $\text{cent}_0 : \mathcal{M}^* \times B_1 \rightarrow \mathcal{M}^*$ by

$$\text{cent}_0(\mu, x) = T_x^* \mu$$

Definition 1.13. The *discrete centering* of an extended base- b CP-distribution Q is the distribution $\text{cent}_0 Q$. The *continuous centering* $\text{cent} Q$ is the distribution

$$\text{cent} Q = \frac{1}{\log b} \int_0^{\log b} S_t^* \text{cent}_0 Q dt$$

Theorem 1.14. *The continuous centering $P = \text{cent} Q$ of a CP-distribution Q is an FD, and the measure-preserving system (P, S^*) is a factor of the suspension of (Q, M_b) by the function with constant height $\log b$.*

For the proof see Section 3.5. There is also a converse:

Theorem 1.15. *If P is an FD and $b \geq 2$ then P is the continuous centering of some base- b CP-distribution.*

See Section 5.

The CP-distribution in the theorem above is highly non-unique. We record one useful manifestation of this:

Proposition 1.16. *The CP-distribution in Theorem 1.15 may be chosen so that its second component is distributed according to Lebesgue measure on B_1 .*

The centering operation establishes (by definition) a measure-preserving map from a CP-distribution to the corresponding FD, under which each measure ν of the measure component is mapped to a translated, scaled and normalized version of ν . These operations preserve most geometric properties of ν . This is significant because CP-distributions, while being powerful analytic tools, are tied inflexibly to a particular coordinate system and base. For example, the results of Furstenberg on dimension conservation [14] are for the projection of a CP-distribution onto one of the coordinate planes. One would like to be able to change the coordinates and base. This is made possible by the two theorems above, which allow one to pass from a CP-distribution to an FD, the latter object being defined in a coordinate-free way.⁸ Thus one can pass from a CP-distribution to the associated FD, change the coordinate system, and pass to a new CP-distribution in a base of our choosing, and still the new distribution has the same underlying measures as the old one. In particular, a geometric property of measures which is unchanged by translations and scaling of measures will hold a.s. for the old CP-distribution if and only if it holds almost surely for the new one. We summarize this as follows:

⁸In fact there is a mild dependence on coordinates in FDs since B_1 depends on the coordinates and is used to define normalization, but this is insignificant. See Section 3.1.

Corollary 1.17. *For any extended FD or CP-distribution Q and for any choice of base b and coordinate system, there is a CP-distribution Q' in this base and coordinate system such that, for any Borel set $A \subseteq \mathcal{M}$ which is invariant under translations, scaling and normalization, we have $Q(A) = Q'(A)$.*

Finally, we note that in certain situations it is useful to generalize the notion of CP-distributions to allow for more complicated partitions than \mathcal{D}_b . Some examples are provided in Section 4. See also [19].

1.5. Dimension. We now turn to the geometric properties of USMs and EFDs. We recall below some background on dimension. Falconer's books [7], [8] are good introductions to the topic.

Denote the Hausdorff dimension of a set A by $\dim A$. For a measure $\mu \in \mathcal{M}$ and $x \in \text{supp } \mu$ let

$$\begin{aligned}\overline{D}_\mu(x) &= \limsup_{r \rightarrow 0} \frac{\log \mu(B_r(x))}{\log r} \\ \underline{D}_\mu(x) &= \liminf_{r \rightarrow 0} \frac{\log \mu(B_r(x))}{\log r}\end{aligned}$$

These are called, respectively, the *upper and lower local dimension* of μ and x . Let

$$\begin{aligned}\overline{\dim} \mu &= \text{esssup } \overline{D}_\mu(x) \\ \underline{\dim} \mu &= \text{essinf } \underline{D}_\mu(x)\end{aligned}$$

These are the upper and lower⁹ dimensions of μ . If $\underline{D}_\mu(x) = \overline{D}_\mu(x)$ we denote their common value by $D_\mu(x)$. If $D_\mu(x)$ exists and is constant μ -a.e. its almost-sure value is the *exact dimension* $\dim \mu$ of μ , and we say that μ is *exact dimensional*.

Lemma 1.18. *If P is an EFD then P -a.e. μ is exact dimensional and the dimension is P -a.s. constant. Writing $\dim P$ for the a.s. value of the dimension, for every $0 < r < 1$ we have*

$$\dim P = \int \frac{\log \mu(B_r(0))}{\log r} dP(\mu)$$

The fact that P -a.s. the dimension is constant follows from ergodicity of P by noticing that the map $\mu \mapsto \dim \mu$ is S^* -invariant.¹⁰ Existence of exact dimension is proved in Section 6.2.

If P is an FD with ergodic decomposition P_ν , $\nu \sim P$, we define

$$\dim P = \int \dim P_\nu dP(\nu)$$

Note that when P is not ergodic it is not true that P -a.e. ν has dimension $\dim P$; rather, P -a.e. μ has dimension $\dim P_\nu$ where P_ν is the ergodic component of ν .

For $\mu \in \mathcal{M}$ let \mathcal{V}_x denote the accumulation points in $\mathcal{P}(\mathcal{M}^\square)$ of $\langle \mu \rangle_{x,T}$ as $T \rightarrow \infty$. We have the following characterization of the local dimension of μ :

⁹The lower dimension may also be characterized as $\underline{\dim} \mu = \inf\{\dim A : \mu(A) > 0\}$.

¹⁰One may check that the map is measurable. We shall generally omit these routine verifications.

Proposition 1.19. *If $\mu \in \mathcal{M}$ then for μ -a.e. x*

$$\underline{D}_\mu(x) \geq \inf_{P \in \mathcal{V}_x} \dim P$$

and

$$\overline{D}_\mu(x) \leq \sup_{P \in \mathcal{V}_x} \dim P$$

In particular, if μ is a USM generating an EFD P then μ is exact dimensional and $\dim \mu = \dim P$.

The last conclusion fails if P is not ergodic. See Section 6.2.

1.6. Projections of measures. One of the fundamental facts in fractal geometry is that if one projects a set or measure through a typical linear map $\pi : \mathbb{R}^d \rightarrow \mathbb{R}^k$ (or pushes it through a typical C^1 map) then the image measure has dimension which is “as large as it can be”. This result has many variants but is often referred to generally as Marstrand’s theorem. More precisely, let $\Pi_{d,k}$ denote¹¹ the space of linear maps $\mathbb{R}^d \rightarrow \mathbb{R}^k$. Note that $\Pi_{d,k}$ is a smooth manifold and carries a natural measure class. Marstrand’s theorem for measures is:

Theorem 1.20 (Hunt-Kaloshin, [20]). *If μ is an exact-dimensional finite measure on \mathbb{R}^d , then for a.e. $\pi \in \Pi_{d,k}$, the image measure $\pi\mu$ is exact dimensional and*

$$\dim \pi\mu = \min\{k, \dim \mu\}$$

This general statement has two shortcomings. First, it is an almost-everywhere statement, and gives no information about $\pi\mu$ for particular $\pi \in \Pi_{d,k}$. Second, in general the map $E_\mu : \Pi_{d,k} \rightarrow \mathbb{R}$ given by

$$E_\mu(\pi) = \dim \pi\mu$$

is Borel, but does not have any regularity.

In the context of CP-distributions, however, Hochman and Shmerkin [19] recently proved that some regularity exists. To state this, we note first that for $\mu \in \mathcal{M}$ the projection $\pi\mu$ may not be Radon, so we define¹²

$$\underline{\dim} \pi\mu = \lim_{R \rightarrow \infty} (\underline{\dim} \mu|_{B_R})$$

and similarly for $\overline{\dim}$ and \dim . Next, if P is a distribution on \mathcal{M} , write

$$E_P(\mu) = \int \underline{\dim} \pi\mu dP(\mu)$$

If P is S^* -ergodic then the map $\mu \mapsto \underline{\dim} \pi\mu$ is S^* -invariant and hence constant, so the integral in the definitions of $E_P(\mu)$ trivializes and $E_P(\mu)$ is just the almost-sure value of $\underline{\dim} \pi\mu$.

Theorem 1.21 (Hochman-Shmerkin [19], Theorem 1.10). *Let P be an ergodic CP-distribution.*

¹¹We shall mostly be interested in the elements of $\Pi_{d,k}$ only up to change of coordinates in the range. Some authors identify $\Pi_{d,k}$ modulo this relation with the space of orthogonal projections from \mathbb{R}^d to k -dimensional subspaces.

¹²Although it would be more precise to write $\dim(\pi, \mu)$ and not $\dim \pi\mu$, no confusion should arise.

- (1) $E_P(\cdot)$ is lower semi-continuous and equal almost everywhere to $\min\{k, \dim P\}$.
- (2) For P -a.e. μ and every regular $f \in C^1(\mathbb{R}^d, \mathbb{R}^k)$,

$$\underline{\dim} \varphi\mu \geq \operatorname{ess\,inf}_{x \sim \mu} E_P(D\varphi(x))$$

In particular, $\underline{\dim} \pi\mu \geq E_P(\pi)$ for $\pi \in \Pi_{d,k}$.

In view of Theorem 1.15, this transfers immediately to EFDs. We can also improve it in several ways. First, Furstenberg [14] showed that if Q is a CP-distribution and π is the projection to a coordinate plane, i.e. $\pi(x) = (x_{i_1}, \dots, x_{i_k})$ for some indices $1 \leq i_1 < \dots < i_k \leq d$, then $\pi\mu$ is exact dimensional for Q -a.e. μ . Using Corollary 1.17, we have

Theorem 1.22. *If P is a CP-distribution or an FD then for every $\pi \in \Pi_{d,k}$, for P -a.e. μ the image $\pi\mu$ is exact dimensional.*

The last two theorems are still a.e results, the uncertainty being about the measure rather than the projection. Instead, one would like to obtain information about individual measures. For example, if Q is a CP-distribution, is it true for Q -a.e. μ that $\pi\mu$ is exact dimensional and $E_\mu = E_Q$? Since a typical measure for an FD is a USM, one approach to these questions is to explore the validity of the results above for USMs and, more generally, SMs.

In this spirit, one can get lower bounds for projections of scaling measures. For an $\mu \in \mathcal{M}$ let \mathcal{V}_x again denote the accumulation points of $\langle \mu \rangle_{x,T}$ as $T \rightarrow \infty$, and let

$$(1.1) \quad E_x(\cdot) = \inf_{P \in \mathcal{V}_x} E_P(\cdot)$$

Theorem 1.23. *Let $\mu \in \mathcal{M}$. Then for regular $f \in C^1(\mathbb{R}^d, \mathbb{R}^k)$,*

$$\underline{\dim} f\mu \geq \operatorname{ess\,inf}_{x \sim \mu} E_x(Df(x))$$

In particular, if μ is a USM generating an EFD P , then for all $\pi \in \Pi_{d,k}$,

$$\underline{\dim} \pi\mu \geq E_P(\pi)$$

See Section 6.3. However, one cannot hope for equality or exact dimension:

Proposition 1.24. *Let μ be a USM generating an EFD P and $\pi \in \Pi_{d,k}$. Then it is possible that $\pi\mu$ is not exact dimensional and that $\underline{\dim} \pi\mu > E_P(\pi)$.*

We give such examples in Section 6.4. As a consequence, since USMs are exact dimensional we have:

Corollary 1.25. *The projection of a USM need not be a USM.*

We have not been able to settle the following:

Problem 1.26. *If P is an EFD on \mathbb{R}^d , is it true that for P -a.e μ , for every $\pi \in \Pi_{d,k}$ the image $\pi\mu$ is exact dimensional with $\dim \pi\mu = E_P(\pi)$?*

1.7. Conditional measures on subspaces. Another classical question, which is in some sense dual to the problem of understanding projections, is to understand the conditional measures of μ on the fibers of a projection. More precisely, given a measure μ on \mathbb{R}^d and a map $f : \mathbb{R}^d \rightarrow \mathbb{R}^k$, define for $x \in \mathbb{R}^d$ the f -fiber through x by

$$[x]_f = f^{-1}(f(x))$$

and the corresponding partition into fibers,

$$\begin{aligned} \mathcal{F}_\pi &= \{f^{-1}(y) : y \in \mathbb{R}^k\} \\ &= \{[x]_f : x \in \mathbb{R}^d\} \end{aligned}$$

We write $\mu_{[x]_f}$ for the conditional measure on $[x]_f$, which is defined μ -a.e. (note that when f is not finite these fiber measures are well defined only up to a multiplicative constant).

The classical result about fibers is the following:

Theorem 1.27 (Matilla [22]). *If μ is an exact-dimensional measure on \mathbb{R}^d then, for a.e. $\pi \in \Pi_{d,k}$ and μ -a.e. $x \in \mathbb{R}^d$, the measure $\mu_{[x]_\pi}$ is exact dimensional and*

$$(1.2) \quad \dim \mu_{[x]_\pi} = \max\{0, \dim \mu - k\}$$

and in particular for a.e. $\pi \in \Pi_{d,k}$,

$$(1.3) \quad \dim \pi\mu + \dim \mu_{[x]_\pi} = \dim \mu \quad \text{for } \mu\text{-a.e. } x$$

The last equation is a ‘‘dimension conservation’’ phenomenon: for an exact-dimensional measure μ and a.e. projection $\pi \in \Pi_{d,k}$, the sum of dimensions of the image $\pi\mu$ and a typical fiber $\mu_{[x]_\pi}$ is precisely $\dim \mu$.

We now turn to the behavior of FDs under conditioning, which is discussed in Section 1.7. Let us first make some remarks about the global picture. Although the conditional measures of μ with respect to π are defined only for μ -a.e. fiber, if P is a d -dimensional EFD and $\pi \in \Pi_{d,k}$ then, due to the quasi-Palm property, the conditional measure of a P -typical μ on the fiber $\pi^{-1}(0)$ is well defined. The fiber can be identified with \mathbb{R}^{d-k} , and one may verify without difficulty that the map $\mu \mapsto (\mu_{\pi^{-1}(0)})^*$ intertwines S^* and that the image Q of P by this map is a k -dimensional EFD. In particular,

Proposition 1.28. *If P is a d -dimensional EFD and $\pi \in \Pi_{d,k}$ then for P -a.e. μ the conditional measures of $\mu_{[x]_\pi}$ are exact-dimensional and the dimension is a.s. independent of μ and x .*

In [14], Furstenberg established a version of Theorem 1.27 in which the measure is a typical measures for CP-processes, but for concrete (rather than generic) projections, namely the *coordinate projections*: $\pi(x) = (x_{i_1}, \dots, x_{i_k})$ for some indices $1 \leq i_1 < \dots < i_k \leq d$.

Theorem 1.29 (Furstenberg [14]). *If P is an ergodic CP-distribution and π is a coordinate projection, then for P -a.e. μ ,*

$$\dim \pi\mu + \dim \mu_{[x]_\pi} = \dim \mu \quad \text{for } \mu\text{-a.e. } x$$

To this we apply the fact that an FD can be represented as a CP process in any coordinate system, at the cost of changing P on a measure zero set (Corollary 1.17). We obtain exact dimension of fiber measures and dimension conservation for typical measures and arbitrary projections:

Theorem 1.30. *Let P be an EFD and $\pi \in \Pi_{d,k}$. Then for P -a.e. μ and μ -a.e. x , the fiber measure $\mu_{[x]_\pi}$ is exact dimensional, and*

$$(1.4) \quad \dim \pi\mu + \dim \mu_{[x]_\pi} = \dim \mu \quad \text{for } \mu\text{-a.e. } x$$

Turning to USMs, we have a dual version of 1.23, giving an upper bound on the dimension of fibers:

Theorem 1.31. *If $\mu \in \mathcal{M}$ and E_x is defined as in equation (1.1), then for every regular $f \in C^1(\mathbb{R}^d, \mathbb{R}^k)$,*

$$\overline{\dim} \mu_{[x]_f} \leq \text{esssup}_{x \sim \mu} (\dim \mu - E_x(Df(x))) \quad \text{for } \mu\text{-a.e. } x$$

In particular, if μ is a USM generating an EFD P and $\pi \in \Pi_{d,k}$ then ,

$$\overline{\dim} \mu_{[x]_f} \leq \dim P - E_P(\pi) \quad \text{for } \mu\text{-a.e. } x$$

In light of the fact that there can be a strict inequality in Theorem 1.23 (and also as is evident from the construction in that proof), dimension conservation can fail for USMs. Furthermore, it can fail in a rather dramatic fashion:

Proposition 1.32. *There exists a USM μ on \mathbb{R}^2 generating an EFD P , and such that $\dim(\mu) > 1$ for P -a.e. μ the projection $\pi(x, y) = x$ is an injection.*

We are left the dual version of Problem 1.26:

Problem 1.33. *If P is a USM on \mathbb{R}^d is it true that for P -a.e μ , for every $\pi \in \Pi_{d,k}$ a.e. fiber $\mu_{\pi^{-1}(y)}$ is exact dimensional with $\dim \mu_{\pi^{-1}(y)} = \dim \mu - E_P(\pi)$?*

Finally, the same construction as in Proposition 1.32 provides a counterexample to another question of Furstenberg. Following [14], for a compact set $X \subseteq \mathbb{R}^d$ define a microset to be a set of the form $B_1 \cap T_E X$, where E is a ball, and define a microset to be a limit of minisets, with respect to the Hausdorff metric on the space of closed subsets of B_1 . The set X is homogeneous if every microset is a subset of some miniset. For such sets Furstenberg constructed a CP-process on X of the same dimension as X , and used this to show that, for linear projections, a dimension conservation similar to [14] holds for X . He also asked [13] whether this holds for smooth maps. The answer is negative:

Proposition 1.34. *There exists a homogeneous set $X \subseteq \mathbb{R}^2$ and a C^∞ map $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ which is injective on X (so each fiber is a singleton) and such that $\dim fX < \dim X$.*

1.8. FDs with additional invariance. For FDs which enjoy certain additional invariance properties one can draw stronger conclusions than the above. We briefly mention these applications now, and present them in more detail in Section 7.

Our first example uses the notion of a homogeneous measure, which is a modification of a similar notion of Gavish [15].¹³

Definition 1.35. A point x is *homogeneous* for a measure $\mu \in \mathcal{M}$ if for every accumulation point ν of $(\mu_{x,t}^\square)_{t>0}$ there is a ball B with $\mu \ll T_B \nu$. A measure μ is homogeneous if μ -a.e. point is homogeneous for μ .

Examples of homogeneous measures include self-similar measures for iterated function with strong separation whose contractions are homotheties or, more generally, if the linear part orthogonal of contractions generate a finite group (see Section 4).

Proposition 1.36. *If μ is a homogeneous measure then it is a USM and generates an EFD P supported on homogeneous measures.*

With these facts in hand it is not hard to use our results from the previous sections to deduce:

Theorem 1.37. *If μ is a homogeneous measure then $E_\mu(\cdot)$ is lower semi-continuous; for every $\pi \in \Pi_{d,k}$ the image $\pi\mu$ and a.e. conditional fiber measure is exact dimensional, and furthermore a.e. fiber measure is a USM; and μ satisfies dimension conservation for every $\pi \in \Pi_{d,k}$.*

We next consider EFDs with additional geometric invariance. For any k there is an action of $GL(\mathbb{R}^d)$ on $\Pi_{d,k}$ given by $U : \pi \rightarrow \pi \circ U^{-1}$, and a $GL(\mathbb{R}^k)$ -action given by $V : \pi \rightarrow V \circ \pi$. These actions commute, giving a $GL(\mathbb{R}^k) \times GL(\mathbb{R}^d)$ -action.

Let $A \subseteq GL(\mathbb{R}^d)$ be a group of linear transformations. A induces an action on measures and hence an action A^* on distributions. An EFD P is non-singular with respect to A if $a^*P \sim P$ for every $a \in A$.

As a consequence of the semi-continuity of $E_P(\cdot)$, we obtain the following:

Proposition 1.38. *Let P be an EFD which is non-singular with respect to a group $A \subseteq GL_n(\mathbb{R}^d)$. Then $E_P(\cdot)$ is constant on \overline{A} -orbits of $\Pi_{d,k}$. In particular if an orbit $\Lambda \subseteq \Pi_{d,k}$ of $GL(\mathbb{R})^k \times \overline{A}$ has non-empty interior then $E_P|_\Lambda = \min\{k, \dim \alpha\}$.*

This allows us to recover the main results from [19] (see Theorems 7.1 and 7.2 below).

1.9. Open problems. Let us mention some problems which we have been so far unable to resolve.

- (1) Is it true that if P is an FD then P -a.e. μ has exact dimensional projections for every $\pi \in \Pi_{d,k}$? Exact dimensional conditionals? dimension conservation?
- (2) What are the limits of scenery distributions of self-affine measures and how do they relate to the structure of the measure?

¹³Gavish's definition is flawed in a number of ways. Most measures of interest are not homogeneous in his sense; it is simple, for example, to show that any measure on a cantor set in the line has atomic micromeasures and hence if the original measure is non-atomic it cannot be homogeneous in Gavish's sense. Our definition seems to capture better Gavish's intention.

- (3) What class of EFDs has absolutely continuous projections? We note that the semi-continuity result for dimension of projections does not have an analogue for absolute continuity, as is evident from [26]. However there are various special measures for which absolute continuity has been verified. It seems possible that some mixing assumption, perhaps quantitative, on the FD might yield similar results (perhaps what is required is some form of temporal or spatial mixing, or both).
- (4) More generally, what dynamical properties of an EFD as a measure preserving system have implications for the geometry of the fractal? For one result of this type see [18].

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1.11. Summary of notation

d	Dimension of the ambient Euclidean space.
$B_r(x)$, B_r	The closed ball of radius r around x (if not specified, $x = 0$).
\mathcal{M} , $\mathcal{M}(\mathbb{R}^d)$	Space of Radon measure on \mathbb{R}^d .
$\mathcal{P}(X)$	Space of probability measures on X .
μ, ν, η, θ	Measures (elements of \mathcal{M}).
P, Q, R	Distributions (elements of $\mathcal{P}(\mathcal{M})$).
$\mathcal{U}, \mathcal{V}, \mathcal{W}$	Subsets of large spaces, e.g. \mathcal{M} or $\mathcal{P}(\mathcal{M})$.
μ^*	μ normalized to mass 1 on $[-1, 1]^d$.
μ^\square	μ restricted and normalized to mass 1 on $[-1, 1]^d$.
μ_A	Conditional measure of μ on A .
T_x, T_x^*, T_x^\square	$T_x(y) = y - x$, and normalized variants.
S_t, S_t^*, S_t^\square	$S_t(x) = e^t x$, and normalized variants.
$\mathcal{D}_b, \mathcal{D}_b(x)$	Partition into b -adic cells and the cell containing x .
M_b, M_b^*, M_b^\square	Base- b magnification operator, normalized variants.
$\mu_{x,t}$	Scenery of μ at x at scale T .
$\langle \mu \rangle_{x,T}, \langle \mu, x \rangle_T$	Continuous and b -scenery distributions at x and scale T .
$\Pi_{d,k}$	Space of linear maps $\mathbb{R}^d \rightarrow \mathbb{R}^k$
$\underline{D}_\mu(x), \overline{D}_\mu(x)$	Lower and upper local dimension of μ at x .
$\underline{\dim}, \overline{\dim}, \dim$	Lower, upper and exact dimension of a measure.

2. ERGODIC-THEORETIC PRELIMINARIES

Our perspective in this work is ergodic-theoretic, although we will not require very much beyond the basic definitions and the ergodic theorem. Most relevant material is summarized here. For more information see [29, 16].

2.1. Measure preserving systems. A measure preserving system is quadruple $(\Omega, \mathcal{B}, P, S)$, where (Ω, \mathcal{B}, P) is a standard probability space and S is a semigroup or group acting on Ω by measure-preserving transformations: if $s\omega$ denotes the action of $s \in S$ on $\omega \in \Omega$, then

$s : \Omega \rightarrow \Omega$ is measurable and $P(s^{-1}A) = P(A)$ for every $A \in \mathcal{B}$ and $s \in S$. We usually drop \mathcal{B} in our notation, and sometimes abbreviate the system to (P, S) .

We will only encounter the cases $S = \mathbb{Z}$ or \mathbb{Z}^+ , which we call discrete time systems, and $S = \mathbb{R}$ or \mathbb{R}^+ , which we call continuous time systems, or flows. In the discrete time case one often writes the system as $(\Omega, \mathcal{B}, P, s)$ where s is a generator of S .

2.2. Ergodicity and ergodic decomposition. A measure preserving system is *ergodic* if the only invariant sets are trivial, i.e. if

$$(\forall s \in S \quad P(s^{-1}A\Delta A) = 0) \quad \implies \quad (P(A) = 0 \text{ or } 1)$$

The ergodic decomposition theorem asserts that for any measure preserving system $(\Omega, \mathcal{B}, P, S)$ there is a map $\Omega \rightarrow \mathcal{P}(\Omega)$, denoted $\omega \mapsto P_\omega$, which is (i) measurable with respect to the sub- σ -algebra $\mathcal{I} \subseteq \mathcal{B}$ of S -invariant sets, (ii) $P = \int P_\omega dP(\omega)$, (iii) P -a.e. P_ω is invariant and ergodic for S and supported on the atom of \mathcal{I} containing ω . Both P_ω and the atom containing ω are called the ergodic components of ω . The map $\omega \rightarrow P_\omega$ is unique up to changes on a set of P -measure zero.

2.3. Time-1 maps and suspensions. Given a continuous time system $(\Omega, \mathcal{B}, P, S)$ and $s_0 \in S$ one can consider the discrete-time systems $(\Omega, \mathcal{B}, P, s_0)$ in which the action is by the semigroup (or group if s_0 is invertible) generated by s_0 . This system need not be ergodic even if the S -system is.

Conversely there is a standard construction to go from a discrete-time system to a continuous-time one. Given a measure-preserving transformation s of (Ω, \mathcal{B}, P) , consider the product system $\Omega \times [0, c]$ with measure $P \times \lambda_{[0, c]}$, where $\lambda_{[0, c]}$ is normalized Lebesgue measure on $[0, c)$, and for $t \in \mathbb{R}^+$ define $s_t(x, r) = (s^{\lfloor t/c \rfloor}x, c\{\frac{1}{c}(r+t)\})$, where $[u]$, $\{u\}$ denote the integer and fractional parts of u . Then $S = (s_t)_{t \in \mathbb{R}}$ is a measure-preserving flow called the *c-suspension* of $(\Omega, \mathcal{B}, P, s)$. Note that under s_c the flow decomposes into ergodic components of the form $P \times \delta_u$, $u \in [0, c)$ and the action of s_c on each ergodic component is by applying s to the first coordinate and fixing the second.

A more general construction is the *flow under a function* construction. Given a discrete time system $(\Omega, \mathcal{B}, P, s)$ and a positive measurable function $f : \Omega \rightarrow \mathbb{R}^+$. Let $\Omega' \subseteq \Omega \times \mathbb{R}^+$ denote the set

$$\Omega' = \{(\omega, r) : 0 \leq r < f(\omega)\} \subseteq \Omega \times \mathbb{R}$$

and on it put the probability measure $P' = (P \times \lambda)_{\Omega'}$ (this requires $\int f dP < \infty$ in order that $P \times \lambda(\Omega') < \infty$). Define a flow on this set by flowing vertically “up” from (ω, r) at unit speed until the second coordinate reaches $f(\omega)$, then jumping to $(s\omega, 0)$ and continuing to flow up. Formally, one considers Ω' to be a factor space of $\Omega \times \mathbb{R}$ by the equivalence relation \sim generated by $(\omega, r) \sim (s\omega, r - f(\omega))$. Define a flow on $\Omega \times \mathbb{R}$ by $s_t(\omega, r) = (\omega, t + r)$. This flow factors to the desired P' -preserving flow on Ω' . Note that the c -suspension is just a flow under the function $f \equiv c$.

2.4. Isomorphism, factors, natural extensions and processes. If $(\Omega, \mathcal{B}), (\Omega', \mathcal{B}')$ are measurable spaces and S is a semigroup acting measurably on both then a factor map

$\pi : \Omega \rightarrow \Omega$ is a measurable map which intertwines the actions, i.e.

$$s \circ \pi = \pi \circ s \quad \text{for all } s \in S$$

When the actions preserve measures P, P' respectively the factor map is also required to preserve measure, i.e. $P(\pi^{-1}A) = P'(A)$ for all $A \in \mathcal{B}'$.

If $(\Omega, \mathcal{B}, P, S)$ is a system with $S = \mathbb{Z}^+$ or \mathbb{R}^+ , the natural extension of the system is the \mathbb{Z} or \mathbb{R} system, respectively, obtained as follows. Let \overline{S} denote the (abstract) group generated by S and take

$$\overline{\Omega} = \{\omega \in \Omega^{\overline{S}} : \omega_{t+s} = s\omega_t \text{ for all } t \in \overline{S} \text{ and } s \in S\}$$

Let S act on $\overline{\Omega}$ by translation: $(s\omega)_t = \omega_{s+t}$. Then $\pi : \omega \mapsto \omega_0$ is a factor map from $\overline{\Omega}$ with the translation action and (Ω, S) , and there is a unique invariant measure \overline{P} on $\overline{\Omega}$ which projects to P . The system $(\overline{\Omega}, \overline{P}, \overline{S})$ is the natural extension of (Ω, P, S) and is characterized up to isomorphism by the property that if $(\Omega', P', \overline{S})$ is another \overline{S} -system and $\pi' : (\Omega', S) \rightarrow (\Omega, S)$ is a factor map (with respect to S) then there is a factor map $\varphi : (\Omega', \overline{S}) \rightarrow (\overline{\Omega}, \overline{S})$ such that $\pi' = \pi\varphi$.

It will sometimes be useful to identify a dynamical system with a process. Given $(\Omega, \mathcal{B}, P, S)$ we define the Ω -valued random variables $(X_s)_{s \in S}$ on (Ω, \mathcal{B}, P) by $X_s(\omega) = s\omega$. The family $(X_s)_{s \in S}$ is a stationary process in the sense that the joint distribution of any k -tuple $(X_{s_i})_{i=1}^k$ is the same as the joint distribution of $(X_{s_i+t})_{i=1}^k$ for every $t \in S$.

2.5. The ergodic theorem. If $(\Omega, \mathcal{B}, P, (S_t)_{t \in \mathbb{R}^+})$ is a semi-flow let $\mathcal{I} \subseteq \mathcal{B}$ be the σ -algebra of S -invariant sets. Then for $f \in L^1(\Omega, P)$,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f \circ S_t dt = \mathbb{E}(f | \mathcal{I})$$

P -a.e. and in L^1 . In particular for ergodic system \mathcal{I} is the trivial σ -algebra consisting of sets of measure 0 and 1, and the right hand side is then $\int f dP$. For discrete time systems the same result holds with the integral is replaced by a sum from 1 to T .

2.6. Generic points in topological systems. Let $S : X \rightarrow X$ be a continuous transformation of a compact metric space and let P be a Borel probability measure X . A point $x \in X$ is *generic* for P if

$$(2.1) \quad P_N = \frac{1}{N} \sum_{n=0}^{N-1} \delta_{S^n x} \rightarrow P$$

in the weak-* topology; that is, for every $f \in C(X)$,

$$(2.2) \quad \int f dP_N = \frac{1}{N} \sum_{n=0}^{N-1} f(S^n x) \rightarrow \int f dP$$

If P is an S -invariant distribution then P -a.e. x is generic for P . Indeed, in order for $x \in X$ to be generic for P it is enough that (2.2) hold for f in some fixed dense countable family $\mathcal{F} \subseteq C(X)$. Such a family exists because X is compact and metric. For $f \in \mathcal{F}$ and for P -a.e. x , (2.2) holds by the ergodic theorem.

Conversely, if x is generic for P then P is S invariant, and, more generally any accumulation point P of the averages (2.1) is S -invariant. Indeed, invariance is equivalent to the equality $\int f \circ S dP = \int f dP$ for every $f \in C(X)$. Since $S\delta_y = \delta_{Sy}$, for $f \in C(X)$ we have

$$\begin{aligned}
\int f \circ S dP - \int f dP &= \int (f \circ S) d\left(\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \delta_{S^n x}\right) - \int f d\left(\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \delta_{S^n x}\right) \\
&= \lim_{N \rightarrow \infty} \frac{1}{N} \left(\sum_{n=1}^N f(S(\delta_{S^n x})) - \sum_{n=1}^N f(\delta_{S^n x}) \right) \\
&= \lim_{N \rightarrow \infty} \frac{1}{N} \left(\sum_{n=1}^N f(\delta_{S^{n+1} x}) - \sum_{n=1}^N f(\delta_{S^n x}) \right) \\
&= \lim_{N \rightarrow \infty} \left(\frac{1}{N} f(\delta_{S^{N+1} x}) - \frac{1}{N} f(\delta_x) \right) \\
&= 0
\end{aligned}$$

Note that we used continuity of S to deduce that $f \circ S \in C(X)$, which was used in passing from the first to the second line.

Generic points for a continuous time action of \mathbb{R} on X are defined similarly.

3. BASIC PROPERTIES OF THE MODELS

This section contains some remarks on the definition of FDs and their relation to other models, and derivation of their some of their basic properties.

3.1. Remarks on normalization. If $\mu \in \mathcal{M}$ then, unless there is an atom at the origin, $S_t \mu \rightarrow 0$ as $t \rightarrow \infty$. Therefore some form of normalization is necessary if we wish to study the dynamics of rescaling. However, outside of some special cases (e.g. when there are second order densities, see Section 3.3) there is no natural way to do this.

The most mathematically straightforward approach is to bypass the issue of normalization altogether and work in the projective space \mathcal{M}/\mathbb{R}^+ , in which one identifies measures which are constant multiples of each other, or, better yet, the factor space in which equivalent measures are identified. However, this is somewhat inconvenient in practice.

Our choice of normalization, $\mu \mapsto \mu^*$, amounts to choosing a section over \mathcal{M}/\mathbb{R}^+ . We could of course choose to normalize some other neighborhood of the origin U using the normalization $\mu^U = \frac{1}{\mu(U)}\mu$, and associated operations. Although we stated in the introduction that FDs are defined in a coordinate-free way, our choice of B_1 as the set on which measures are normalized is coordinate dependent. However, if we use μ^U instead then the extended FDs with respect to U are in 1-1 correspondence with extended FDs as we have defined them since the maps $\mu \rightarrow \mu^U$ and $\nu \mapsto \nu^*$ are inverses of each other on \mathcal{M}^U and \mathcal{M}^* and induce a bijection of distributions on these sets which intertwine the respective scaling groups. Also, the normalization does not change the measure class of the measure and so modulo equivalence of measures, the choice of U is inconsequential.

Let us mention one more possibility for normalization. One can show that for every FD P there is a continuous positive function $f : \mathbb{R}^d \rightarrow \mathbb{R}^+$ such that $\int f d\mu < \infty$ for P -a.e. μ

(in fact any f with sufficiently fast decay is as $x \rightarrow \infty$ has this property). Let $\mu^f = \frac{1}{\int f d\mu} \mu$. Then P^f , the push-forward of P under $\mu \mapsto \mu^f$, is invariant under the appropriate scaling group $S^f = (S_t^f)_{t \in \mathbb{R}}$, and P^f satisfies an analogous quasi-Palm property. Furthermore, $\text{supp } P^f$ is weakly compact in $\mathcal{P}(\mathcal{M})$, and S^f acts on it continuously, two properties which are annoyingly absent in our normalization. However, this normalization is less convenient in applications, and we shall not have use for it.

3.2. Restricted and extended versions. The following claim holds generally for S^* -invariant distributions, not only for FDs.

Lemma 3.1. *The map $\mathcal{M} \rightarrow \mathcal{M}^\square$, $\mu \mapsto \mu^\square$ intertwines the \mathbb{R}^+ -actions of S^* and S^\square , i.e. $(S_t^* \mu)^\square = S_t^\square \mu$ for $t \geq 0$, and induces a one-to-one correspondence $P \mapsto P^\square$ between S^* -invariant distributions on \mathcal{M} and S^\square -invariant distributions on \mathcal{P} .*

Proof. The first claim is immediate.

Suppose that Q is an S^\square -invariant distribution; we construct a S^* -invariant distribution \tilde{Q} with $\tilde{Q}^\square = Q$, showing that restriction is surjective. Let $(\mu_t)_{t \in \mathbb{R}}$ be the unique stationary process with $\mu_0 \sim Q$ and

$$(3.1) \quad \mu_{t+s} = S_s^\square \mu_t$$

for all $t \in \mathbb{R}$ and $s \geq 0$. For each $t \geq 0$ define

$$\nu_t = S_t^* \mu_{-t}$$

Note that (3.1) implies that for $t \geq s \geq 0$,

$$(3.2) \quad \nu_t|_{B_{e^s}(0)} = \nu_s$$

Therefore we can define

$$\nu = \lim_{t \rightarrow \infty} \nu_t$$

i.e. $\nu(A) = \lim_{t \rightarrow \infty} \nu_t(A)$.

Let \tilde{Q} be the distribution of ν . Clearly $\nu^\square = \mu_0$, so $\tilde{Q}^\square = Q$. The property 3.2 ensures that \tilde{Q} is S^* invariant.

Finally, note that if R is any other S^* -invariant distribution with $R^\square = Q$ then by pushing R through the map $\tau \mapsto \tau|_{B_{e^t}(0)}$ for $t \geq 0$ we obtain the distributions of ν_t . It follows that $R = \tilde{Q}$, establishing injectivity of the restriction map $\mu \mapsto \mu^\square$. \square

Next, for a restricted CP-distribution Q we construct the extended version \tilde{Q} . This is an M_b^* -invariant distribution \tilde{Q} on $\mathcal{M}^* \times B_1$ whose push-forward via $(\mu, x) \mapsto (\mu^\square, x)$ is Q , but note that these two properties of Q do not characterize it as in the case of the extended version of an FD. An example is given below.

The construction is analogous to the construction of extended FDs. Let $(\mu_n, x_n)_{n \in \mathbb{Z}}$ the unique stationary process with marginal Q and such that

$$(\mu_{n+1}, x_{n+1}) = M_b^\square(\mu_n, x_n)$$

For $n \geq 0$ let

$$\nu_n = T_{\mathcal{D}_{b^n}(x_{-n})}^* \mu_{-n}$$

and

$$E_n = T_{\mathcal{D}_{b^n}(x_{-n})} B_1$$

then ν_n is supported on E_n and $B_1 = E_0 \subseteq E_1 \subseteq E_2 \subseteq \dots$. For $n \geq m \geq 0$ we have

$$(M_b^\square)^m(\mu_{-n}, x_{-n}) = (\mu_{-n+m}, x_{-n+m})$$

so

$$\nu_n|_{E_m} = \nu_m$$

and we may define

$$\nu = \lim_{n \rightarrow \infty} \nu_n$$

i.e. $\nu(A) = \lim_{n \rightarrow \infty} \nu_n(A)$. This measure is easily seen to be Radon. The distribution \tilde{Q} of ν is a distribution on $\mathcal{M}^* \times B_1$ and it is M_b^* -invariant. This is the extended version of Q . One may verify that the map $(\mu, x) \mapsto (\mu^\square, x)$ is a factor map between the measure preserving systems $(\mathcal{M}^* \times B_1, \tilde{Q}, M_b^*)$ and $(\mathcal{M}^\square \times B_1, Q, M_b^\square)$.

As we remarked above, if Q is a restricted CP-distribution and R is a distribution on $\mathcal{M}^* \times B_1$ which is (i) invariant under the map M_b^* and (ii) factors onto \tilde{Q} via $(\mu, x) \mapsto (\mu^\square, x)$, it does not follow that $R = \tilde{Q}$. Indeed, consider the measure μ on \mathbb{R}^2 consisting of Lebesgue measure on the vertical line $x = 1$, and 2^{-n} times Lebesgue measure on the vertical line $x = 1 + 2^{-n}$, $n \in \mathbb{N}$. Let $I = \{1\} \times [-1, 1]$. Then the distribution $R = \int_I \delta_{(\mu, x)} d\mu(x)$ is adapted and invariant under M_b^* , and factors onto the CP-distribution Q , but R is not the extended version of Q . Rather, the extended version is $\tilde{Q} = \int_I \delta_{(\nu, x)} d\nu(x)$, where ν consists of Lebesgue measure on the line $x = 1$.

The example above is rather special and also typical of what can go wrong:

Lemma 3.2. *Let Q be a restricted ergodic CP-distribution and suppose that $\int \theta(B_1) dQ(\theta, x) < 1$. Then there is a unique CP-distribution \tilde{Q} factoring onto Q via $(\mu, x) \mapsto (\mu^\square, x)$.*

We do not use this, and omit the proof.

3.3. Relation to Zähle distributions. A class of models closely related to FDs are Zähle's notion of an α -scale-invariant distribution, see [30, 24]. For fixed $\alpha > 0$, Zähle's notion of scale invariance uses the additive \mathbb{R} -action $S^\alpha = (S_t^\alpha)_{t \in \mathbb{R}}$ on \mathcal{M} , defined by

$$S_t^\alpha \mu = e^{\alpha t} S_t \mu$$

An α -Zähle distribution is an S^α -invariant distribution P which in addition is a Palm distribution, i.e.

$$\int_B \delta_{T_x \mu} d\mu(x) = P$$

for every ball B centered at the origin. Note that the left hand side differs from $\langle \mu \rangle_B$ because we have not normalized the translated measure $T_x \mu$, and besides this the condition differs from the definition of quasi-Palm in that we require equality rather than equivalence of the distributions. We write \mathcal{Z}_α for the space of ergodic α -Zähle distributions.

Define the α -scenery of μ at x to be

$$\mu_{x,t}^\alpha = S_t^\alpha T_x \mu$$

and the scenery distribution

$$\langle \mu \rangle_{x,T}^\alpha = \frac{1}{T} \int_0^T \delta_{\mu_{x,t}^\alpha} dt$$

which is a distribution on $\mathcal{P}(\mathcal{M})$. In this space of distributions we consider weak convergence, i.e. $P_n \rightarrow P$ if $\int f dP_n \rightarrow \int f dP$ for every bounded continuous $f : \mathcal{M} \rightarrow \mathbb{R}$. Note that the weak topology is not compact, and it is entirely possible for $\mu_{x,t}^\alpha$ become unbounded as $t \rightarrow \infty$. It can also accumulate on the zero measure. Thus the α -scenery distributions may not have convergent subsequences, or may converge to the distribution on δ_0 .

Mörters and Preiss proved the following predecessor of Theorem 1.7:

Theorem 3.3. [Mörters and Preiss, [24]] *If $\mu \in \mathcal{M}$ and $\alpha > 0$ then for μ -a.e. x , every weak- $*$ accumulation point of the α -scenery at x is an α -Zähle distribution.*

Let us now make the analogy between EFDs and Zähle distributions precise. It is clear that if $P \in \mathcal{Z}_\alpha$ then P^* is an FD and $\mu \mapsto \mu^*$ is a factor map between (P, S^α) and (P^*, S^*) . This correspondence is clearly many-to-one for the following reason. Let us say that distributions P, Q are equivalent up to a constant if there is a $c > 0$ such that Q is the push-forward of P through the map $\mu \mapsto c\mu$. We denote this relation \approx . Note that if $P \in \mathcal{Z}_\alpha$ and $Q \approx P$ then $Q \in \mathcal{Z}_\alpha$, and also $P^* = Q^*$. Therefore the map $P \mapsto P^*$ from \mathcal{Z}_α to the space of EFDs is not injective: its fibers are saturated with respect to the \approx relation.

For a $\mu \in \mathcal{M}$ and $x \in \text{supp } \mu$, the α -dimensional *second-order density* of μ at x is

$$D_\alpha(\mu, x) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t e^{\alpha t} \mu(B_t(x)) dt$$

assuming the limit exists. Denote

$$\mathcal{A}_\alpha \in \{\mu \in \mathcal{M} : \mu \text{ has } \alpha\text{-dimensional second order densities a.e.}\}$$

Note that $D_\alpha(\mu, x)$ is homogeneous in μ , that is, $D_\alpha(c\mu, x) = cD_\alpha(\mu, x)$ for $c \in \mathbb{R}$. Therefore \mathcal{A}_α is saturated with respect to \approx .

Theorem 3.4. *For each $\alpha > 0$ the map $Q \mapsto Q^*$ is a bijection between $\mathcal{Z}_\alpha / \approx$ and the space of EFDs which are supported on \mathcal{A}_α .*

Proof. Let $P \in \mathcal{Z}_\alpha$. Note that $D_\alpha(\mu, 0)$ is an ergodic average with respect to S^α of the function $f(\mu) = \mu(B_1)$, so for P -a.e. μ the second-order density at the origin exists and by ergodicity is P -a.e. equal to $c' = \int f dP$. Since P is Palm, for P -typical μ and μ -typical x the α -dimensional second order density exists at x and $D_\alpha(\mu, x) = c'$.

Since $\mu^* = \frac{1}{\mu(B_1)}\mu$, it follows that at μ^* -a.e. point the α -dimensional second order densities exist and are equal to $\frac{c'}{\mu(B_1)}$. This shows that the map $P \mapsto P^*$ maps elements of \mathcal{Z}_α to EFDs supported on \mathcal{A}_α .

Let Q be an extended EFD supported on \mathcal{A}_α . For Q -typical μ it then follows by Theorem 3.3 that the distribution Q_x which is the weak limit of the α -sceneries of μ at x belongs to

\mathcal{Z}_α . However, it is simple to verify that

$$Q_x^\square = \lim_{T \rightarrow \infty} (\langle \mu \rangle_{x,T}^\alpha)^\square = \lim_{T \rightarrow \infty} \langle \mu \rangle_{x,T} = Q^\square$$

which implies that $Q_x^* = Q$. Hence the map $P \mapsto P^*$ maps \mathcal{Z}_α surjectively to the set of EFDs supported on \mathcal{A}_α .

Finally, to establish that the map is bijective modulo \approx , suppose that Q is an EFD supported on \mathcal{A}_α . For P^* -typical μ let $c(\mu) = D_\alpha(\mu, x)$. We claim that this is well defined, i.e. that the right hand side is μ -a.s. independent of x . Indeed, we already saw in the first paragraph of the proof that this is the case if $Q = P^*$ for some $P \in \mathcal{Z}_\alpha$, and in the previous paragraph we saw that such P exists. Next, define $\mu_* = \frac{1}{c(\mu)}\mu$. For $P \in \mathcal{Z}_\alpha$ with $P^* = Q$ the discussion in the first paragraph of the proof shows that for P -typical ν we have $(\nu^*)_* = \frac{1}{c'}\nu$, where c' is the a.s. α -dimensional second order density of P -typical measures. Hence $\mu \mapsto \mu_*$ is an inverse, modulo \approx , to the map $P \mapsto P^*$. \square

As a consequence of this characterization, we find that EFDs are a far broader model than Zähle distributions:

Corollary 3.5. *There exist EFDs which do not arise from a Zähle distribution.*

Proof. Any EFD supported on measures which a.s. do not have positive second order densities is such an example. To be concrete, we may take the EFD associated to any self-similar measure arising from similarities satisfying strong separation, and which is singular with respect to Hausdorff measure at the appropriate dimension. See Section 4. The fact that such a measure does not have second order densities follows from Patzschke and Zähle [25], and that this is true for the corresponding EFD follows from the fact that at a.e. point of a self similar measure the accumulation points of scenery all contain an affine copy of the original measure.

For a more concrete example consider the measure μ on $[0,1]$ which is the distribution of the random number x whose binary digits are chosen independently to be 0 with probability $0 < p < \frac{1}{2}$ and 1 with probability $1 - p$. The associated EFD is again not a Zähle distributions, for the same reason. \square

Finally, we remark that FDs do not necessarily have finite intensity; that is, if P is an FD then there may be compact $K \subseteq \mathbb{R}^d$ with $\int \nu(K) dP(\nu) = \infty$. This phenomenon does not occur for Zähle distributions.

3.4. Ergodic and spatial decompositions. Next, we prove the ergodic decomposition theorem for FDs. We require the following geometric tool:

Lemma 3.6 (Besicovitch). *If μ is a Radon measure on \mathbb{R}^d and $\mu(A) > 0$ then for μ -a.e. point in A ,*

$$\lim_{r \rightarrow 0} \frac{\mu(A \cap B_r(x))}{\mu(B_r(x))} = 1$$

Proof. See [22, Corollary 2.14]. \square

We can now prove the ergodic decomposition theorem for EFDs:

Proof of Theorem 1.3. Let P be a FD and let $\mathcal{U} \subseteq \mathcal{M}^*$ be an S^* -invariant measurable set with $P(\mathcal{U}) > 0$. We claim that for P -a.e. $\mu \in \mathcal{U}$ and μ -a.e. point x we have $T_x^* \mu \in \mathcal{U}$. Let

$$f(\mu) = \mu(\{x \in B_1 : T_x^* \mu \in \mathcal{U}\})$$

For $\mu \in \mathcal{M}$ and μ -a.e. x , Besicovitch's lemma implies that

$$\lim_{t \rightarrow \infty} f(\mu_{x,t}^*) \in \{0, 1\}$$

Hence by the quasi-Palm property, it follows that $\lim_{t \rightarrow \infty} f(S_t^* \mu) \in \{0, 1\}$ for P -a.e. μ . Hence by S^* invariance, $f(\mu) \in \{0, 1\}$ for P -a.e. μ , and from S^* -invariance again, for P -a.e. μ if $f(\mu) = 1$ then $T_x^* \mu \in \mathcal{U}$ for μ -a.e. x (not only for $x \in B_1$), and if $f(\mu) = 0$ then $T_x^* \mu \notin \mathcal{U}$ for μ -a.e. x . It is clear that up to P -null sets $\{f = 1\} \subseteq \mathcal{U}$ and $\{f = 0\} \subseteq \mathcal{M}^* \setminus \mathcal{U}$ and therefore these containments are equalities P -a.e.

From this discussion it is clear that

$$\int_{\mathcal{U}} \langle \mu \rangle_{B_1}^* dP(\mu) \ll P|_{\mathcal{U}}$$

and we have a similar relation using $\mathcal{U}' = \mathcal{M}^* \setminus \mathcal{U}$ instead of \mathcal{U} . From the fact that

$$\int \langle \mu \rangle_{B_1}^* (\mu) dP(\mu) \sim P$$

and $P = P|_{\mathcal{U}} + P|_{\mathcal{U}'}$ we conclude that in fact

$$(3.3) \quad \int_{\mathcal{U}} \langle \mu \rangle_{B_1}^* \mu dP(\mu) \sim P|_{\mathcal{U}}$$

so $P|_{\mathcal{U}}$ is an FD. Furthermore, we see that the Radon-Nikodym derivative of the left hand side with respect to the right in (3.3) is the same as the restriction to \mathcal{U} of the derivative of $\int \langle \mu \rangle_{B_1}^* dP$ with respect to P . It now follows (e.g. using the martingale theorem) that the conditional distributions of P on the σ -algebra of S^* -invariant sets satisfy a similar non-singularity relation and therefore consist a.s. of FDs. Since these are precisely the ergodic components, we are done. \square

It is natural to ask whether an ‘‘ergodic decomposition’’ exists for scaling measures on the spatial level. Two simple results of this type are the following.

Proposition 3.7. *If μ is a scaling measure, and $\mu(A) > 0$, then μ_A is a scaling measure and μ and μ_A generate the same distribution at μ_A -a.e. x .*

Proof. This is immediate from the Besicovitch density theorem (Lemma 3.6), which implies that for μ_A -a.e. x ,

$$S_t^\square T_x(\mu_A) - S_t^\square T_x \mu \rightarrow 0$$

and hence the sceneries at x for μ and μ_A are asymptotic. \square

More generally,

Proposition 3.8. *If μ is a scaling measure and $\emptyset \neq \nu \ll \mu$ then ν is a scaling measure and for ν -a.e. x the distributions generated by μ and ν at x are the same.*

The proof is the similar using the Besicovitch density theorem for functions.

One consequence is that if \mathcal{U} is a set of distributions and $A, B = \mathbb{R}^d \setminus A$ is the partition of \mathbb{R}^d according to whether $P_x \in \mathcal{U}$ or $P_x \notin \mathcal{U}$, respectively, then the conditional measures on the atoms of the partition behave as described above: μ_A -a.e. point generates a distribution in \mathcal{U} and μ_B -a.e. point generates a distribution in $\mathcal{M}^* \setminus \mathcal{U}$. It is now natural to ask if this phenomenon continues down to the partition according to the generated distributions. More precisely, fix a scaling measure μ and consider the partition of \mathbb{R}^d induced by the map $x \mapsto P_x$, where P_x is the distribution generated at x . We may disintegrate μ with respect to this partition, and one might expect that the induced measure on the atom $\{x : P_x = P\}$ will itself be a USM generating for P .

In general this is false. Consider the following example. Let μ be any USM, e.g. Hausdorff measure on the product $C \times C \subseteq [0, 1]^2$ of the usual middle- $\frac{1}{3}$ Cantor set (see Section 4 for a discussion of examples). Let P denote the FD which is generated at μ -a.e. point, and note that P is supported on measures which locally are homothetic copies of $C \times C$, hence $UP = P$ for a linear map U if and only if U is a rotation by an integer multiple of 90° or a reflection about one of the axes. Now consider (for example) the map $f : [0, 1]^2 \rightarrow \mathbb{R}^2$ given by

$$f(x, y) = (x, xy)$$

so that no two of the differentials $Df(x, y)$, $(x, y) \in (0, 1)^2$ are co-linear or related by a rotation or a reflection. Set $\nu = f\mu$. If $y = fx$ and μ generates P at x , then ν generates $Df(x)P$ at y , and so at a.e. point ν generates distinct distributions. Thus the partition according to generated distributions is the partition of \mathbb{R}^2 into points, and the conditional measures are atoms which all generate trivial distributions.

One may also ask the dual question concerning sums and integrals of SMs. Consider SMs μ, ν . Then $\mu + \nu$ is equivalent to the sum of three mutually singular measures, $\mu + \nu \sim \mu' + \theta + \nu'$, such that $\mu' \ll \mu$, $\nu' \ll \nu$, and θ absolutely continuous with respect to both μ and ν (the measure θ is obtained as the part of ν which is non-singular with respect to μ , and $\nu' = \nu - \theta$; then μ' is the part of μ singular with respect to θ). Using the propositions at the beginning of this section, we find that $\mu + \nu$ is a SM and at a typical point generates a distribution which is generated at some point by μ or ν (or both).

However, passing from finite sums to integrals of SMs we lose this behavior. Suppose Q is a distribution on SMs. Then $\int \mu dQ(\mu)$ need not be an SM. Indeed, any probability measure $\theta \in \mathcal{M}$ can be written as $\theta = \int \delta_x d\theta(x)$. Each δ_x is a USM but θ need not be.

3.5. From CP-distributions to FDs to USMs. In this section we show that centerings of CP-distributions are FDs, and typical measures for an FD are USMs.

We begin with the second of these. As discussed in Section 2.6, if P is an ergodic distribution for a transformation of a compact metric space, then P -a.e. point is generic for P . USMs are analogies of generic points and in our setting the analogue of the statement above is:

Theorem 3.9. *If P is an EFD then P -a.e. μ is a USM and generates P .*

Proof. We assume that P is in its extended version. For $f \in C(\mathcal{P}(\mathcal{M}^\square))$ let $\tilde{f} : \mathcal{M}^* \rightarrow \mathbb{R}$ denote the map $\mu \mapsto f(\mu^\square)$. Choose a countable norm-dense family $\mathcal{F} \subseteq C(\mathcal{P}(\mathcal{M}^\square))$. Then by the ergodic theorem, for $f \in \mathcal{F}$ and P -a.e. μ we have

$$(3.4) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \tilde{f}(S_t^\square \mu) dt = \int \tilde{f} dP$$

so, since \mathcal{F} is countable, for P -a.e. μ this holds simultaneously for all $f \in \mathcal{F}$. For P -a.e. μ the limit (3.4) in fact holds for all $f \in C(\mathcal{P}(\mathcal{M}^\square))$, because \mathcal{F} is dense in $C(\mathcal{P}(\mathcal{M}^\square))$ and the set of functions satisfying (3.4) is norm closed. Finally, by the quasi-Palm property for P -a.e. ν the limit (3.4) holds for $\mu = T_x^* \nu$ for ν -a.e. $x \in B_1$, and by S^* -invariance for ν -a.e. $x \in \mathbb{R}^d$, which is the desired result. \square

Next let Q be a base- b CP-distribution and recall the definition of the centering operation, Definition 1.13. Since it is immediate that $(\mu', x') = M_b^*(\mu, x)$ satisfies

$$S_{\log b}^*(T_x \mu) = T_{x'}^* \mu'$$

the map

$$\text{cent}_0 : (\mu, x) \mapsto T_x^* \mu$$

is a factor map from the discrete-time measure preserving system $(\mathcal{M}^* \times B_1, Q, M_b^*)$ to the discrete-time measure preserving system $(\mathcal{M}^*, \text{cent}_0 Q, S_{\log b}^*)$, and the continuous time system $(\mathcal{M}^*, \text{cent}_0 Q, S^*)$ is thus a factor of the suspension of $(\mathcal{M}^* \times B_1, Q, M_b)$ with roof function of height $\log b$.

Theorem 3.10. *The centering of an extended CP-distribution is a FD.*

Proof. Let Q be an extended CP-distribution with restricted version P . Adopting the notation from the construction of Q in Section 3.2, let $(\mu_n, x_n)_{n \in \mathbb{Z}}$ be the two-sided $\mathcal{M}^\square \times B_1$ -valued process with marginal P and $(\mu_{n+1}, x_{n+1}) = M_b^\square(\mu_n, x_n)$, and let π denote the map $(\mu_n, x_n)_{n \in \mathbb{Z}} \mapsto \nu$ constructed there, i.e.

$$(3.5) \quad \nu = \lim_{n \rightarrow \infty} T_{\mathcal{D}_{bn}(x_{-n})}^* \mu_{-n}$$

so $\nu \sim Q$.

Let $R = \text{cent}_0 Q$. Let $B \subseteq \mathbb{R}^d$ be a bounded neighborhood of 0 and let

$$(3.6) \quad R' = \int \langle \nu \rangle_B^* dR(\nu) = \int \left(\int_B \delta_{T_x^* \nu} d\nu(x) \right) dR(\nu)$$

we must show that $R \sim R'$.

Fix a measurable set $\mathcal{U} \subseteq \mathcal{M}^*$. Suppose that $R(\mathcal{U}) = 0$. In order to show that $R'(\mathcal{U}) = 0$ it suffices by (3.6) to show that

$$\int \int 1_{\mathcal{U}}(T_x^* \nu) d\nu(x) dR(\nu) = 0$$

For this it is enough to show that for a.e. realization $(\mu_n, x_n)_{n \in \mathbb{Z}}$ of the process and ν satisfying (3.5), we have

$$\int 1_{\mathcal{U}}(T_x^* \nu) d\nu(x) = 0$$

By (3.5), this will follow once we establish that for a.e. realization of the process and every $n \in \mathbb{N}$,

$$\int 1_{\mathcal{U}}(T_x^* \nu) dT_{\mathcal{D}_{b^n}(x-n)}^* \mu_{-n}(x) = 0$$

Fixing n , by stationarity of the process $(\mu_n, x_n)_{n \in \mathbb{Z}}$, and the fact that the map $(\mu_n, x_n) \rightarrow \nu$ intertwines the shift operation and $S_{\log b}^*$, we have

$$\begin{aligned} \int 1_{\mathcal{U}}(T_x^* \nu) dT_{\mathcal{D}_{b^n}(x-n)}^* \mu_{-n}(x) &= \int 1_{\mathcal{U}}(S_{n \log b}^* T_x \nu) d\mu_0(x) \\ &= \int 1_{S_{-n \log b}^* \mathcal{U}}(T_x \nu) d\mu_0(x) \\ &= R(S_{-n \log b}^* \mathcal{U}) \\ &= R(\mathcal{U}) \\ &= 0 \end{aligned}$$

because R is S^* -invariant. We have established the claim a.s. for each n and therefore a.s. for all $n \in \mathbb{N}$, as desired.

Conversely, for any measure ν , since $0 \in B$ we have

$$\langle \nu \rangle_{B_1}^* \ll \int \langle \theta \rangle_B^* d \langle \nu \rangle_{B_1}^* (\theta)$$

so since $R = \int \langle \nu \rangle_{B_1}^* dQ(\nu)$,

$$R = \int \langle \nu \rangle_{B_1}^* dQ(\nu) \ll \int \int \langle \nu \rangle_B^* d \langle \nu \rangle_{B_1}^* dQ(\nu) = \int \langle \nu \rangle_B^* dP(\nu) = R'$$

Finally, the fact that $\text{cent } Q = \int_0^{\log b} S_t^* \text{cent}_0 Q dt$ is quasi-Palm follows from the same property for $R = \text{cent}_0 Q$. Therefore $\text{cent } Q$ is an FD. \square

4. EXAMPLES

In this section we present a variety of examples EFDs and USMs. The proofs usually involve constructing a stationary process similar to a CP-process and centering. Many of our examples have appeared before in related contexts. For additional examples of similar constructions see [17, 5, 19].

4.1. Example: CP-processes arising from stationary processes. Recall that a stationary process is a sequence of random variables $(X_n)_{n \in \mathbb{Z}}$ such that for every $n \in \mathbb{Z}$ and $k \geq 0$ the k -tuples (X_0, \dots, X_k) and (X_n, \dots, X_{n+k}) have the same distribution.

The following example appears in [14]. Let process $(Y_n)_{n \in \mathbb{Z}}$ be a stationary process with values in $\{0, \dots, b-1\}$ and define a random point x and measure μ by¹⁴

$$(4.1) \quad \begin{aligned} x &= 1 + 2 \sum_{k=1}^{\infty} b^{-k} Y_k \\ \mu &= \text{distribution of } x \text{ given } (Y_n)_{n \leq 0} \end{aligned}$$

¹⁴If we had defined CP-processes on $[0, 1)^d$ using b -adic partitions, the definition of x_n would simplify to $x = \sum_{i=1}^{\infty} b^{-k} Y_k$.

Then one may verify that the distribution of (μ, x) is a CP-distribution.

With the same notation, another example is

$$(4.2) \quad \begin{aligned} x_n &= 1 + 2 \sum_{n=1}^{\infty} b^{-n} Y_n \\ \mu_n &= \delta_{x_n} \end{aligned}$$

Notice that the first construction generally gives rise to a non-trivial EFD, while the second gives rise to the EFD supported on the measure $\delta_0 \in \mathcal{P}(\mathcal{M})$.

As a concrete example, let $b = 3$ and consider the process (Y_n) in which each Y_n is chosen independently, equal to 0 or 2 with probability 1/2 each. This corresponds to the product measure on $\{0, 1, 2\}^{\mathbb{Z}}$ of the measure $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_2$. Because the Y_n are independent, given $Y_0, Y_{-1}, Y_{-2}, \dots$ the distribution of each $Y_k, k \geq 1$ is the same as before, and so $x = 1 + 2 \sum 3^{-i} Y_i$ is distributed according to the normalized Hausdorff measure μ an affine image of the standard Cantor set:

$$C = \{-1 + 2x : x \in [0, 1]\} \text{ can be written in base 3 using only the digits } 0, 2\}$$

The CP-distribution we get is $\delta_\mu \times \mu$. Note that the first component doesn't change when M_3 is applied.

As a by-product of this we have:

Proposition 4.1. *A given EFD may be the centering of many distinct CP-processes.*

Proof. The trivial EFD arises as the centering of the CP-distribution (δ_x, x) constructed as above from a process (Y_n) . These distributions are different for different initial processes because, from the distribution of (δ_x, x) we can recover the process, by setting $Y_n = n$ -th base- b digit of $\frac{1}{2}(x - 1)$. \square

4.2. Example: $\times m$ -invariant measures on the $[0, 1]$. Let $b \geq 2$ be an integer and let μ be an ergodic probability measure on $[0, 1]$ which is invariant under the b -to-1 map $f_b : x \mapsto bx \bmod 1$. Such a measure can be identified with a stationary process which is the sequence digits in the base- b expansion of $x \sim \mu$. As such, μ gives rise to a shift-invariant measure on $\{0, \dots, b - 1\}^{\mathbb{N}}$ and its natural extension may be realized as a shift-invariant measure $\tilde{\mu}$ on $\{0, \dots, b - 1\}^{\mathbb{Z}}$ which projects to μ on the positive coordinates. It is then not hard to see that $\mu = \int \nu_\omega d\tilde{\mu}(\omega)$, where ν_ω is the distribution of $\sum_{n=1}^{\infty} b^{-i} \omega_i$ given $(\omega_j)_{j \leq 0}$. As we saw above the distribution on ν_ω for $\omega \sim \tilde{\mu}$ is an ergodic restricted CP-distribution P (after we identify $[0, 1]$ with $B_1 = [-1, 1]$).

Proposition 4.2. *Every μ as above is a USM for an ergodic EFD P , supported on measures whose dimension is $\frac{1}{\log b} h(\mu, f_b)$, where $h(\cdot, \cdot)$ is the Kolmogorov-Sinai entropy.*

For a detailed the proof see [18].

4.3. Example: Self-similar measures. Let Λ be a finite set and $\{f_i\}_{i \in \Lambda}$ a system of contracting similarities of \mathbb{R}^d , i.e.

$$f_i(x) = r_i U_i(x) + v_i$$

for some $0 < r_i < 1$, $v_i \in \mathbb{R}^d$ and U_i an orthogonal transformation of \mathbb{R}^d . For $a = a_1 \dots a_n \in \Lambda^n$ write

$$f_a = f_{a_1} \circ f_{a_2} \circ \dots \circ f_{a_n}$$

It is well known that for every $x \in \mathbb{R}^d$ and $a \in \Lambda^{\mathbb{N}}$ the sequence $f_{a_1 \dots a_n}(x)$ converges, as $n \rightarrow \infty$, to a point $\varphi(a) \in \mathbb{R}^d$ which is independent of x . Furthermore, $\varphi : \Lambda^{\mathbb{N}} \rightarrow \mathbb{R}^d$ is continuous, and its image X is the maximal closed set with the property $X = \bigcup_{i \in \Lambda} f_i(X)$ (the attractor of the IFS). A set X which arises in this way is called a self similar set.

The IFS $\{f_i\}_{i \in \Lambda}$ satisfies the strong separation condition if the sets $f_i(X), i \in \Lambda$ are pairwise disjoint; this implies that φ is an injection. If strong separation holds then, by replacing Λ with Λ^k for some large k and the system $\{f_i\}_{i \in \Lambda}$ with $\{f_a\}_{a \in \Lambda^k}$, and then applying a compactness argument, one may assume that there is an open set $A \subseteq \mathbb{R}^d$ such that $X \subseteq A$ and $f_i(A), i \in \Lambda$ are pairwise disjoint and contained in A .

Let $\tilde{\mu}$ be a product measure on $\Lambda^{\mathbb{N}}$ with marginal $(p_i)_{i \in \Lambda}$. The image $\mu = \varphi \tilde{\mu}$ on X is called a self-similar measure. We claim that a self similar measure is uniformly scaling, and furthermore that the associated FD is supported on measures which on every bounded set are equivalent to a restriction of μ , up to a similarity.

Let $G < GL_n$ denote the group generated by the orthogonal maps $U_i, i \in \Lambda$ and let γ denote Haar measure on G . Consider the distribution P on $\mathcal{M} \times \mathbb{R}^d \times G$ obtained by selecting $y \sim \mu$ and $U \sim \gamma$, and forming the pair $(U\mu, Uy, U)$.

For $(\nu, y, V) \in \mathcal{M} \times \mathbb{R}^d \times G$ which satisfies $V^{-1}\nu = \mu$ and $V^{-1}y \in \text{supp } \mu$ define $M(\nu, y, U) = (\nu', y', V') \in \mathcal{M} \times \mathbb{R}^d \times G$ as follows. Let $i \in \Lambda$ is the unique index such that $V^{-1}y \in f_i A$. Let $A_i = f_i A$. Now define

$$\begin{aligned} \nu' &= \frac{1}{p_i} V f_i^{-1} V^{-1} (\nu|_{A_i}) \\ y' &= V f_i^{-1} V^{-1} y \\ V' &= U_i V \end{aligned}$$

One may then verify that (i) M is define P -a.e., (ii) P is invariant under M , (iii) P is adapted in the sense that the distribution of the second coordinate y given the first component ν is ν .

One can now repeat almost verbatim the construction of an FD from a CP-distribution to obtain an FD from P . One starts with a process $(\nu_n, y_n, V_n)_{n \in \mathbb{Z}}$ whose marginals are P and such that $M(\nu_n, y_n, V_n) = (\nu_{n+1}, y_{n+1}, V_{n+1})$. Let $i_n \in \Lambda$ be the index such that $V_n^{-1}y_n \in f_{i_n} A$ and define $g_n : \mathbb{R}^d \rightarrow \mathbb{R}^d$ by $g_n = V_n f_{i_n}^{-1} V_n^{-1}$. Then if for $n \geq 0$ we set

$$\theta_n = g_{-1} \dots g_{-n+1} g_{-n} \nu,$$

then $\theta_n, n \leq 0$ are a compatible sequence and the distribution of $T_{y_0}^*(\lim \theta_n)$ is an FD supported on affine images of μ in which the orthogonal part comes from G . In particular it follows that the original measure μ is uniformly scaling for this EFD. This argument is related to Theorem 1.37, although if G is infinite then μ is not homogeneous.

4.4. Example: Random fractals. As another demonstration of the versatility of EFDs we give an example of a fairly general random fractal construction that gives rise to an EFD.

Let $(I_n, J_n, w_n)_{n=1}^\infty$ be an ergodic stationary process in which each pair (I_n, J_n) consists of disjoint closed sub-intervals of $[-1, 1]$ with disjoint interiors and $0 < w_n < 1$. We assume that

$$(4.3) \quad \mathbb{E}(w_1 \log |I_1| + (1 - w_1) \log |J_1|) < \infty$$

where $|I|$ is the length of I .

Construct a random measure μ as follows: set $\mu(I_1) = w_1$ and $\mu(J_1) = 1 - w_1$, and continue recursively for each of the sub-intervals I_1, J_1 using (I_2, J_2, w_2) , i.e. in the next step we define $\mu(T_{I_1}^{-1}(I_2)) = w_1 w_2$, $\mu(T_{I_1}^{-1}(J_2)) = w_1(1 - w_2)$ and similarly $\mu(T_{J_1}^{-1}(I_2)) = (1 - w_1)w_2$, $\mu(T_{J_1}^{-1}(J_2)) = (1 - w_1)(1 - w_2)$. As before, we use the notation T_I for the homothety mapping I onto $[-1, 1]$.

Unlike in our previous examples, the measure μ and $x \in \text{supp } \mu$ does not necessarily determine the sequence of intervals $W_n \in \{I_n, J_n\}$ such that $x \in \bigcap W_n$. However the construction is very similar, except one must include this information in the points of the phase space explicitly.

Let \mathcal{U} denote the set of closed intervals of $[-1, 1]$ and consider the distribution P on triples $(\nu, y, (U_n)_{n=1}^\infty) \in \mathcal{M}^\square \times [-1, 1] \times \mathcal{U}^\mathbb{N}$ obtained by choosing a realization of the process (I_n, J_n, w_n) , constructing μ as above, choosing $x \sim \mu$, and selecting the sequence $U_n \in \{I_n, J_n\}$ in such a way that $T_{I_n} T_{I_{n-1}} \dots T_{I_1} x \in [-1, 1]$ (this defines the choice uniquely). Let M be the map given by

$$M(\nu, y, (U_n)_{n \in \mathbb{N}}) = (T_{U_1}^\square \nu, T_{U_1} y, (U_{n+1})_{n \in \mathbb{N}})$$

which is defined whenever $y \in \text{supp } \nu$ and $T_{I_n} T_{I_{n-1}} \dots T_{I_1} x \in [-1, 1]$ for all n , and hence is defined P -a.e. One then verifies that P is M -invariant. By definition, given ν , the distribution of y is ν .

One now proceeds as before to define an extended version of this distribution, and a corresponding EFD by centering. The only caveat here is that one does not introduce a discrete-time system first, because the amount of magnification at each step is different. Rather one constructs a continuous-time distribution directly as the suspension of the process by a function whose height at $(\nu, y, (U_n)_{n \in \mathbb{N}})$ is $\log |U_n|$. One must ensure that the mean of this height function is finite for the resulting distribution to be finite. This is the reason for the integrability condition (4.3) above, which ensures that the roof function is in L^1 . One may verify that when the integrability condition fails, μ has dimension 0 a.s.

5. EQUIVALENCE OF THE MODELS

So far we have shown that (a) the centering of a CP-distribution is an FD, and (b) a.e. measure of an FD is a USM. In this section we provide the remaining implications. In this section we prove the following statements:

Theorem 5.1. *Let $b \geq 2$ and $\mu \in \mathcal{M}$. For μ -a.e. x , every accumulation point P of $\langle \mu \rangle_{x,T}$ as $T \rightarrow \infty$ is the centering of a base- b CP-distribution Q , and in particular is an FD. Furthermore Q may be chosen so that $\int \theta^\square dQ(\theta) = \lambda^\square$.*

Theorem 5.2. *Given $b \geq 2$, every FD is the centering of a base- b CP-distribution Q which can be chosen so that $\int \theta^\square dQ(\theta) = \lambda^\square$.*

Throughout this section we fix an integer $b \geq 2$ and for brevity we write

$$\begin{aligned} M &= M_b^\square \\ S_t &= S_{t \log b}^\square \end{aligned}$$

Note that we have changed the time scale for S_t so that S_1 and M scale by the same factor b . In particular

$$\langle \mu \rangle_{x,T} = \frac{1}{T} \int_0^T \delta_{S_t(T_x \mu)} dt = \frac{1}{T \log b} \int_0^{T \log b} \delta_{S_t^\square(T_x \mu)} dt$$

We introduce analogous notation for CP-sceneries:

$$\langle \mu, x \rangle_N = \frac{1}{N} \sum_{n=1}^N \delta_{M^n(\mu, x)} \in \mathcal{P}(\mathcal{M}^\square \times B_1)$$

These operations are distinguished notationally by the number of arguments inside the brackets, which correspond to the space on which the resulting distribution is defined: $\langle \mu \rangle_{x,T}$ and $\langle \mu, x \rangle_N$ are distributions on \mathcal{M}^\square and on $\mathcal{M}^\square \times B_1$, respectively.

5.1. Outline of the argument. Let $\mu \in \mathcal{M}^\square$ and fix a μ -typical x . Let P be an accumulation point of the scenery distributions at x , i.e. for some sequence $N_k \rightarrow \infty$,

$$P = \lim_{k \rightarrow \infty} \langle \mu \rangle_{x, N_k}$$

We may assume without loss of generality that N_k are integers. Our strategy is to construct a CP-distribution whose centering is P . A first attempt, which fails only narrowly, is the most direct approach: pass to a subsequence $N_{k(i)}$ such that $\langle \mu, x \rangle_{N_{k(i)}} \rightarrow Q$ for a distribution Q on $\mathcal{M}^\square \times B_1$, and show that (i) Q is a CP-distribution, (ii) the extended version of P is the centering of the extended version of Q .

The argument is complicated by the fact that the transformation M is not continuous, and by some related issues. If M were continuous then (i) would follow as in Section 2.6, using the fact that the averages $\langle \mu, x \rangle_{N_{k(i)}}$ are nearly invariant under M and the continuity of M to obtain invariance of the limit. As for (ii), note that if it were not for the restriction involved we would have the identity

$$\text{cent } \langle \mu, x \rangle_{N_{k(i)}} = \langle \mu \rangle_{x, N_{k(i)}}$$

Indeed, if $(\mu', x') = M^n(\mu, n)$ and $\mu'' = T_x^* \mu'$ then $\mu'' = (S_n^* T_x \mu)|_B$, where B is a translate of B_1 and the restriction occurs because of the restriction in the definition of M . Thus the distribution on the left hand side is supported on measures which are restrictions of the measures supporting the distribution on the right. However, after taking the limiting distributions and passing to extended versions this difference should disappear.

Assuming that we can justify the previous steps, the main issue that remains is to show that Q is adapted, i.e. that conditioned on the first component ν , the distribution of the second component is ν . Let us prove this first.

In the following paragraphs we utilize the machinery of integration and martingales for measure-valued functions. To justify this one may appeal to the general theory of vector-valued integration. Alternatively, the definition of conditional expectation of measure-valued functions and the associated limit theorems can be obtained by integrating them against a dense countable family of continuous functions, noting that after integrating we have reduced to the usual theory of \mathbb{R} -valued expectations and martingales, and applying the Riesz representation theorem to recover a statement about measures.

Recall that a distribution Q on $\mathcal{M}^\square \times B_1$ is adapted if, given that the first component of a Q -realization is ν , the second component is distributed according to ν . One may verify that this is equivalent to the condition that for every $f \in C(\mathcal{M}^\square)$,

$$\int f(\nu) \cdot \nu dQ(\nu, x) = \int f(\nu) \cdot \delta_x dQ(\nu, x)$$

Proposition 5.3. *Let $\nu \in \mathcal{M}^\square$. Then for ν -a.e. y and any $f \in C(\mathcal{M}^\square)$, writing $Q_N = \langle \nu, y \rangle_N$,*

$$\lim_{N \rightarrow \infty} \left(\int f(\theta) \cdot (\theta - \delta_z) dQ_N(\theta, z) \right) = 0$$

in the weak- topology on \mathcal{M}^\square . In particular, any accumulation point of $\langle \nu, y \rangle_N$, $N = 1, 2, \dots$, is adapted.*

Proof. First, $C(\mathcal{M}^\square)$ is separable, so it suffices to prove this for a dense countable family of functions in $C(\mathcal{M}^\square)$. Hence it is enough to prove, with f fixed, that for ν -a.e. y the sequence $(\nu_n, y_n) = M^n(\nu, y)$ satisfies

$$(5.1) \quad \lim_{N \rightarrow \infty} \left(\frac{1}{N} \sum_{n=1}^N f(\nu_n) \delta_{\nu_n} - f(\nu_n) \delta_{y_n} \right) = 0$$

Define measure-valued functions $F_n, G_n : B_1 \rightarrow \mathcal{M}^\square$ by

$$F_n(y) = f(T_{\mathcal{D}_{b^n}(y)}^\square \nu) \cdot T_{\mathcal{D}_{b^n}(y)}^\square \nu$$

and

$$G_n(y) = f(T_{\mathcal{D}_{b^n}(y)}^\square \nu) \cdot \delta_{T_{\mathcal{D}_{b^n}(y)} y}$$

Now, proving (5.1) is equivalent to showing that for ν -a.e. y ,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N (F_n(y) - G_n(y)) = 0$$

in the weak-* sense. Since

$$\mathbb{E}(G_n | \mathcal{D}_{b^n}) = F_n$$

the desired limit follows from the law of large numbers for Martingale differences [10, Theorem 3 in Chapter 9].

Since the map $Q \mapsto \int f(\theta) \cdot (\theta - \delta_y) dQ(\theta, y)$ is continuous in Q for every $f \in C(\mathcal{M}^\square)$, we conclude from the comment preceding the proposition that any accumulation point Q of $\langle \nu, y \rangle_N$ is adapted. \square

Suppose we begin with a μ -typical point x , an accumulation point $P = \lim_{k \rightarrow \infty} \langle \mu \rangle_{x, N_k}$, and choose an accumulation points $Q = \lim_{i \rightarrow \infty} \langle \mu, x \rangle_{N_{k(i)}}$. We have just seen that Q is adapted and we would like to show that Q is M -invariant, and that the centering of the extended version of Q is the extended version of P . We shall not quite prove these statements for μ , but instead perturb μ by a random translation and prove them for the perturbed measure, replacing Q above with an accumulation point of the b -sceneries of the perturbed measure. We may first assume, without loss of generality, that μ is supported on $B_{1/2}(0)$, since it suffices to prove the result for an affine image of μ , and we may apply the map $x \mapsto \frac{1}{2}x$.

Proposition 5.4. *Let μ be a probability measure on $B_{1/2}$. Fix a λ -typical $y \in B_{1/2}$ and μ -typical x and let $\mu' = T_y \mu$ and $x' = x + y$. Suppose $N_k \rightarrow \infty$ and that $\langle \mu \rangle_{x, N_k} \rightarrow P$ and $\langle \mu', x' \rangle_{N_k} \rightarrow Q$. Then a.s. (with respect to the choice of y and x),*

- (1) $\int \theta dQ(\theta) = \lambda^\square$
- (2) Q is M -invariant.
- (3) $P = (\text{cent } \tilde{Q})^\square$, where \tilde{Q} is the extended versions of Q .

In particular, Q is a CP-distribution for a.e. choice of y and x and P is an EFD.

The final conclusion of the proposition follows by combining (2) and (3) above with the previous proposition, and implies Theorem 5.1. Indeed, let $y \in B_{1/2}$ be a λ -typical point. Fix a set A of full μ -measure so that for $x \in A$ the last proposition holds for $\mu' = T_y \mu$ and $x' = x + y$. Fix $x \in A$ and suppose that $P = \lim_{M_k \rightarrow \infty} \langle \mu \rangle_{x, M_k}$ for some sequence $M_k \rightarrow \infty$. Pass to a subsequence N_k of M_k such that $\langle \mu', x' \rangle_{N_k} \rightarrow Q$, and note that $\langle \mu' \rangle_{x', N_k} \rightarrow P$. So by the proposition Q is a CP-distribution with the desired properties and P is a centering of the extended version of Q .

We prove (1)–(3) in the sections below. Throughout the proof let y, x, μ', x', N_i, Q and P be as in the proposition. Write

$$Q_i = \langle \mu', x' \rangle_{N_i}$$

so $Q_i \rightarrow Q$. The statements the follow hold a.s. for the choice of x, y .

5.2. Proof of Proposition 5.4 (1). Note that the distributions of the second component of Q_i and Q are determined completely by x' and the sequence $(N_i)_{i=1}^\infty$, and do not depend on μ' . Since $x' = x + y$ where y is a λ -typical point, this is the same as the distribution of the second coordinate of $\langle \lambda_{B_1}, z \rangle_{N_i}$ for a λ_{B_1+x} -typical point z , and by Proposition 5.3 or the law of large numbers, these distributions converge to λ_{B_1} . Therefore we have

$$\int \delta_z dQ(\theta, z) = \lambda_{B_1}$$

But by adaptedness, we also have

$$\int \delta_z dQ(\theta, z) = \int \theta dQ(\theta, z)$$

as can be seen by conditioning on θ the left hand side on θ . (1) follows.

5.3. Proof of Proposition 5.4 (2). Let

$$E = \bigcup_{D \in \mathcal{D}_b} \partial D$$

where ∂ denotes topological boundary, and define

$$\begin{aligned} \mathcal{N}_1 &= \{(\theta, z) \in \mathcal{M}^\square \times B_1 : \theta(E) = 0\} \\ \mathcal{N}_2 &= \{(\theta, z) \in \mathcal{M}^\square \times B_1 : \theta(\mathcal{D}_b(z)) > 0\} \end{aligned}$$

and

$$\mathcal{N} = \mathcal{N}_1 \cap \mathcal{N}_2$$

These are Borel sets, and M is defined and continuous on \mathcal{N} .

Lemma 5.5. *Q is supported on \mathcal{N} .*

Proof. From Proposition 5.4 (1) it follows that Q is supported on \mathcal{N}_1 , since $\lambda(E) = 0$.

To prove that Q is supported on \mathcal{N}_2 we argue as follows. First, we claim that for a.e. choice of x, y in the definition of μ', x' ,

$$(5.2) \quad \liminf_{n \rightarrow \infty} \left(-\frac{1}{n \log b} \log \mu'(\mathcal{D}_{b^n}(x')) \right) \leq d$$

Indeed, given y and $\mu' = T_y \mu$, for μ' -a.e. x' (equivalently, μ -a.e. x and $x' = x + y$), this bound follows from the fact that the Hausdorff dimension of \mathbb{R}^d is d , and hence any measure supported on it has upper pointwise dimension at most d . See e.g. Section 6.1.

Next note that, writing again $(\mu'_n, x'_n) = M_n(\mu', x')$

$$\mu'(\mathcal{D}_{b^n}(x')) = \prod_{j=0}^{n-1} \frac{\mu'(\mathcal{D}_{b^{j+1}}(x'))}{\mu'(\mathcal{D}_{b^j}(x'))} = \prod_{j=1}^{n-1} \mu'_j(\mathcal{D}_b(x'_j))$$

Taking logarithms and using (5.2) we see that

$$\frac{1}{n \log b} \sum_{j=0}^{n-1} (-\log \mu'_j(\mathcal{D}_b(x'_j))) \leq d + 1$$

for all large enough n . For $n = N_i$ and Q_i defined as above, this can be written as

$$\int (-\log \theta(\mathcal{D}_b(z))) dQ_i(\theta, z) \leq (d + 1) \log b$$

Since the integrand is non-negative, we conclude e.g. by Chebychev that for every $r > 0$,

$$Q_i \{(\theta, z) : \theta(\mathcal{D}_b(z)) < e^{-r}\} \leq \frac{(d + 1) \log b}{e^r}$$

From this we conclude (by approximating $\theta(\mathcal{D}_b(z))$ by a continuous function) that a similar bound holds for Q , and hence

$$Q \{(\theta, z) : \theta(\mathcal{D}_b(z)) > 0\} = 1$$

so Q is supported on \mathcal{N}_2 . □

Let \widehat{Q}_i denote the distribution obtained by replacing each summand $\delta_{(\theta,z)}$ in the average Q_i with $\delta_{(\widehat{\theta},z)}$ where $\widehat{\theta}$ is the measure

$$\widehat{\theta} = \theta_{B_1 \setminus E}$$

Then $\widehat{Q}_i \in \mathcal{N}$. Clearly the fact that Q is supported on \mathcal{N}_1 implies that

$$\widehat{Q}_i - Q_i \rightarrow 0$$

Since $Q_i = \frac{1}{N_i} \sum_{j=1}^{N_i} M^j \langle \mu', x' \rangle$ it is also clear that

$$MQ_i - Q_i \rightarrow 0$$

Writing

$$M\widehat{Q}_i - \widehat{Q}_i = (M\widehat{Q}_i - MQ_i) + (MQ_i - Q_i) + (Q_i - \widehat{Q}_i)$$

we see that the last two terms on the right tend to 0 as $i \rightarrow \infty$. The first term on the right does also, again using (1), and we conclude that

$$M\widehat{Q}_i - \widehat{Q}_i \rightarrow 0$$

Since we have already shown that $\widehat{Q}_i \rightarrow Q$, that $\widehat{Q}_i, Q \in \mathcal{N}$, and that M is continuous on \mathcal{N} , we conclude that $M\widehat{Q}_i \rightarrow MQ$, and so by the above

$$MQ - Q = 0$$

as desired.

5.4. Proof of Proposition 5.4 (3). We continue with the previous notation. We wish to show that the extended version of P is the centering of the extended version \widetilde{Q} of Q , or equivalently, that $(\text{cent } \widetilde{Q})^\square = P$. We continue to write $M = M_b^\square$ and $S_t = S_{t \log b}^\square$.

For $m \in \mathbb{N}$ let $P_m = S_m \text{cent } Q$, or more explicitly, the push-forward of Q through the map

$$(5.3) \quad (\theta, z) \mapsto \int_0^1 \delta_{S_{m+t}(T_z \theta)} dt$$

Lemma 5.6. $P_m \rightarrow P$.

It is easy to see from the definition of the extended version of Q and the centering operation that $P_m \rightarrow (\text{cent } \widetilde{Q})^\square$, so the lemma implies $P = (\text{cent } Q)^\square$.

Proof of the Lemma. Let $P_{m,i} = S_m \text{cent } Q_i$. Our work will be completed by showing that

$$\lim_{i \rightarrow \infty} P_{m,i} = P_m$$

and

$$\lim_{m \rightarrow \infty} \lim_{i \rightarrow \infty} P_{m,i} = P$$

Indeed, the first of these follows from the fact that the map (5.3) is continuous wherever it is defined (and it is defined Q -a.e.). Note that this continuity relies partly on the integration in (5.3) since the maps $S_t = S_{t \log b}^\square$ are discontinuous when there is mass on the boundary

of $B_{e^{-t \log b}}$. However this can occur only for countable many t and from this one can deduce continuity. We omit the details.

As for the second statement, we make the following observation. With m fixed and $n \in \mathbb{N}$, write $(\mu'_n, x'_n) = M^n(\mu', x')$, and notice that as long as the distance of x'_n from E is at least b^{-m} , we have

$$S_{n+m}(T_{x'}\mu') = S_m(T_{x'_n}\mu'_n)$$

(note that if the operators S_t were defined using the $*$ -variant instead of \square this would be an identity without any assumption on x'_n). Thus under these assumptions,

$$\int_0^1 \delta_{S_{n+m+t}(T_{x'}\mu')} dt = S_m \text{ cent } \delta_{(\mu'_n, x'_n)}$$

Therefore, we can write

$$\begin{aligned} P_{m,i} - \langle \mu' \rangle_{x', N_i} &= \frac{1}{N_i} \sum_{n=1}^{N_i} S_m \text{ cent } \delta_{(\mu'_n, x'_n)} - \frac{1}{N_i} \sum_{n=0}^{N_i-1} \int_0^1 \delta_{S_t(T_{x'}\mu')} dt \\ &= \frac{1}{N_i} \sum_{n=1}^{N_i-m} \left(S_m \text{ cent } \delta_{(\mu'_n, x'_n)} - \int_0^1 \delta_{S_{n+m+t}(T_{x'}\mu')} dt \right) \\ &\quad + \frac{m}{N_i} \theta_{m,i} \end{aligned}$$

where $\theta_{m,i}$ is a probability measure, so that with m fixed, we have $\frac{m}{N_i} \theta_{m,i} \rightarrow 0$ as $i \rightarrow \infty$. On the other hand, as we have seen, in the sum above the summand vanishes whenever the distance of x'_n from E is at least b^{-m} . Therefore, as $i \rightarrow \infty$ the right hand side is a probability measure whose total mass is asymptotic to

$$(5.4) \quad \frac{1}{N_i} \#\{1 \leq n \leq N_i : d(x_n, E) > b^{-m}\}$$

Using again the fact that $\frac{1}{N_i} \sum_{n=1}^{N_i} \delta_{x_n} \rightarrow \lambda_{B_1}$ as $i \rightarrow \infty$, we see that (5.4) is bounded by c_m with $\lim_{m \rightarrow \infty} c_m = 0$. This completes the proof. \square

5.5. CP-distribution from FDs.

Proof of Theorem ??. We show that if P is an FD then there is a CP-process Q with $\text{cent } Q = P$. If P is ergodic choose a P -typical μ . By Theorem 1.6 we know that μ generates P , so by Proposition 5.4 (3), P is the centering of a CP-distribution with the desired properties.

In the non-ergodic case let \mathcal{ECP} denote the set of ergodic CP-distributions and \mathcal{EFD} the set of EFDs. One may verify that both sets are measurable. The map $\text{cent} : \mathcal{ECP} \rightarrow \mathcal{EFD}$ is measurable and the paragraph above shows that it is onto. Thus any probability measure τ on \mathcal{EFD} lifts to a probability measure τ' on \mathcal{ECP} with $\text{cent } \tau' = \tau$. If P is an FD we may identify it with a probability measure $\tau \in \mathcal{P}(\mathcal{EFD})$ corresponding to its ergodic decomposition. Let $\tau' \in \mathcal{P}(\mathcal{ECP})$ be the lift of τ . Then $Q = \int R d\tau'(R)$ is a CP-distribution and $\text{cent } Q = P$.

In order to obtain the additional property of Q replace \mathcal{EFD} in the argument above with the set of ergodic CP-distributions Q with $\int \theta dQ(\theta) = \lambda^\square$. We have seen that $Q \mapsto \text{cent } Q$ is still onto \mathcal{EFD} , and we proceed as before. \square

6. GEOMETRY OF FDS AND USMS

6.1. Dimension and entropy. Recall that \mathcal{D}_b is the partition of \mathbb{R}^d into cubes $\times_{i=1}^d I_i$ for intervals of the form $I_i = [\frac{k}{b}, \frac{k+2}{b})$, for $k \in \mathbb{Z}$ with $k = b \pmod 2$. This partitions B_1 into b^d homothetic cubes. For a partition \mathcal{D} of $E \subseteq \mathbb{R}^d$ and $x \in E$, we write $\mathcal{D}(x)$ for the unique partition element containing x . The proof of the following can be found in [28, Theorem 15.3]:

Lemma 6.1. *Let μ be a measure on \mathbb{R}^d and b an integer. Then for μ -a.e. x ,*

$$\begin{aligned} \overline{D}_\mu(x) &= \limsup_{n \rightarrow \infty} \frac{\log \mu(\mathcal{D}_{b^n}(x))}{\log(1/b^n)}, \\ \underline{D}_\mu(x) &= \liminf_{n \rightarrow \infty} \frac{\log \mu(\mathcal{D}_{b^n}(x))}{\log(1/b^n)}. \end{aligned}$$

In particular, $\dim \mu = \alpha$ if and only if

$$\lim_{n \rightarrow \infty} \frac{\log \mu(\mathcal{D}_{b^n}(x))}{n \log(1/b)} = \alpha \quad \text{for } \mu\text{-a.e. } x$$

If μ is a probability measure and \mathcal{Q} is a finite or countable partition, then the Shannon entropy of \mathcal{Q} with respect to μ is

$$H(\mu, \mathcal{Q}) = - \sum_{Q \in \mathcal{Q}} \mu(Q) \log \mu(Q),$$

with the convention that $0 \log 0 = 0$. This quantity measures how spread out μ is among the atoms of \mathcal{Q} . For the basic properties of Shannon entropy see [6].

Lemma 6.2. *If $\underline{\dim} \mu \geq \beta$ then*

$$\liminf_{b \rightarrow \infty} \frac{1}{\log b} H(\mu, \mathcal{D}_b) \geq \beta$$

Proof. Immediate from Lemma 6.1 and the basic properties of $H(\cdot, \cdot)$. See also [9]. \square

With b fixed, the function $H(\cdot, \mathcal{D}_b)$ is discontinuous on the space of probability measures on \mathbb{R} , but only mildly so:

Lemma 6.3. *There is a constant C (depending only on the dimension d) such that, for every b , there is a continuous function f_b on the space of probability measures on \mathbb{R}^d , such that for any probability measure μ on \mathbb{R}^d ,*

$$|f_b(\mu) - H(\mu, \mathcal{D}_b)| < C$$

Proof. Choose a countable partition unity φ_u , $u \in \mathbb{N}^d$ such that each φ_u is continuous and supported on a cube $B_{2/b}(\frac{1}{b}u)$; it is not hard to see such a partition exists. We claim that

$$f_b(\mu) = - \sum_{u \in \mathbb{Z}^d} \left(\int \varphi_u d\mu \right) \log \left(\int \varphi_u d\mu \right)$$

has the desired properties.

Given μ define a probability measure ν on $\mathbb{R}^d \times \mathbb{N}^d$ by

$$\nu(A \times \{i\}) = \int_A \varphi_i(x) d\mu(x)$$

Let \mathcal{F}_1 be the partition of $\mathbb{R}^d \times \mathbb{N}^d$ induced from the first coordinate by \mathcal{D}_b , that is $\mathcal{F}_1 = \{I \times \mathbb{N} : I \in \mathcal{D}_b\}$. Then

$$H(\mu, \mathcal{D}_b) = H(\nu, \mathcal{F}_1)$$

Also, let \mathcal{F}_2 be the partition of $\mathbb{R}^d \times \mathbb{N}^d$ according to the second coordinate. Then

$$H(\nu, \mathcal{F}_2) = f_b(\mu)$$

Let \mathcal{E} denote the partition $\mathcal{E} = \{I \times \{j\} : I \in \mathcal{D}_b, j \in \mathbb{N}\}$. Notice that \mathcal{E} refines \mathcal{F}_1 and each $A \in \mathcal{F}_1$ contains at most 3^d atoms $A' \in \mathcal{E}$. Therefore,

$$H(\nu, \mathcal{F}_1) \leq H(\nu, \mathcal{E}) = H(\nu, \mathcal{F}_1) + H(\nu, \mathcal{E}|\mathcal{F}_1) \leq H(\nu, \mathcal{F}_1) + d \log 3$$

Similarly \mathcal{E} refines \mathcal{F}_2 and each atom $A \in \mathcal{F}$ contains at most 3^d -atoms of \mathcal{E} , so

$$H(\nu, \mathcal{F}_2) \leq H(\nu, \mathcal{E}) = H(\nu, \mathcal{F}_2) + H(\nu, \mathcal{E}|\mathcal{F}_2) \leq H(\nu, \mathcal{F}_2) + d \log 3$$

Combining the last four equations we have

$$|H(\mu, \mathcal{D}_b) - f_b(\mu)| = |H(\nu, \mathcal{F}_1) - H(\nu, \mathcal{F}_2)| \leq d \log 9$$

□

6.2. Dimension of FDs.

Proof of Lemma 1.18. Let P be an EFD and $0 < r < 1$. Set

$$F(\mu) = \frac{\log \mu(B_r(0))}{\log r}$$

and notice that

$$\begin{aligned} \frac{\log \mu(B_{r^N})}{\log(r^N)} &= \frac{1}{N \log r} \sum_{n=1}^N \log (\mu(B_{r^n}(0)) / \mu(B_{r^{n-1}}(0))) \\ &= \frac{1}{N} \sum_{n=1}^N \frac{\log(S_{-n \log r}^* \mu(B_r(0)))}{\log r} \\ &= \frac{1}{N} \sum_{i=1}^n F(S_{-n \log r}^* \mu) \end{aligned}$$

This is an ergodic average for the transformation $S_{-\log r}^*$ (note that $-\log r > 0$), so the limit exists P -a.e. Although $S_{-\log r}^*$ may not be ergodic, it is easy to see directly that the limit is invariant under S_t^* for every t (e.g. because it is the local dimension at 0, which is invariant under staling), so it is P -a.e. constant and equal to $\alpha = \int \frac{\log \mu(B_r(0))}{\log r} dP(\mu)$. Thus the local dimension at 0 satisfies $D_\mu(0) = \alpha$ for P -a.e. μ and by the quasi-Palm property, for P -a.e. μ we have $D_\mu(x) = \alpha$ for μ -a.e. x , so $\dim \mu = \alpha$. □

Setting

$$\tilde{F}(\mu) = \int_0^{-\log r} F(S_t^* \mu) dt$$

it follows from the lemma above that $\dim P = \int \tilde{F} dP$, and the same formula holds for non-ergodic FDs. The advantage of \tilde{F} is that, as a function $\tilde{F} : \mathcal{M}^\square \rightarrow \mathbb{R}$, it is continuous.

Proof of Proposition 1.19. Let $\mu \in \mathcal{M}$. Recall that the set of accumulation points of $\langle \mu \rangle_{x,t}$ is denoted \mathcal{V}_x .

Let us show for example that for μ -a.e. x ,

$$\underline{D}_\mu(x) \geq \operatorname{ess\,inf}_{y \sim \mu} \inf_{Q \in \mathcal{V}_y} \dim Q$$

Denote the right hand side by β . By Theorem 1.7 there is a set of full μ -measure of points x such that every accumulation point of $\langle \mu \rangle_{x,T}$ is an FD. For such a point x , for every $Q \in \mathcal{V}_x$ we have $\int \tilde{F} dQ = \dim Q \geq \beta$, and so, since \tilde{F} is continuous we must have

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T \tilde{F}(\mu_{x,T}^\square) dt \geq \beta$$

since otherwise there would be accumulation points $Q \in \mathcal{V}_x$ of $\mu_{x,T}$ with mean \tilde{F} -value smaller than β . Since $\tilde{F} \geq 0$ it follows that

$$\liminf_{N \rightarrow \infty} -\frac{1}{N \log b} \sum_{n=1}^N F(\mu_{x,-n \log b}^\square) dt \geq \beta$$

and we conclude that $\underline{D}_\mu(x) \geq \beta$ as in the proof above. \square

We next show that there is a unique FD of maximal dimension. A similar statement for Zähler distributions was proved in [30].

Proposition 6.4. *The only d -dimensional FD on \mathbb{R}^d is the point mass at λ^\square .*

Proof. It suffices to prove this for EFDs. Let P be a d -dimensional EFD on \mathbb{R}^d , and let Q be an ergodic CP-distribution with cent $Q = P$. By [14], the dimension of Q (defined as the Q -a.s. dimension of a measure) is given by

$$d = \int \frac{1}{\log b} H(\mu, \mathcal{D}_b) dP(\mu)$$

Since the maximal entropy of any measure on $[-1, 1]^d$ with respect to \mathcal{D}_b is $d \log b$, we conclude that P -a.e. μ gives equal mass to each of the \mathcal{D}_b -cells of $[-1, 1]^d$. Iterating M_b for a μ -typical point we find that every cell $D \in \mathcal{D}_b^n$ of positive μ -mass has mass b^{-n} . It follows that $\mu_{B_1} = \lambda_{B_1}$ and hence $P = \delta_{\lambda^*}$. \square

USMs do not enjoy this property. In fact any measure μ on \mathbb{R}^d which has exact dimension d is a USM and generates the unique d -dimensional FD. Indeed at μ -a.e. x every $Q \in \mathcal{V}_x$ has dimension d (Proposition 1.19), and hence $\mathcal{V}_x = \{\delta_{\lambda^*}\}$. Since there is only one accumulation point it is in fact the limit of the scenery distribution, so μ generates δ_{λ^*} at x . Hence there are USMs μ of dimension 1 with $\mu \perp \lambda$.

6.3. Dimension of projected measures. In this section we prove Theorem 1.23. We first establish the case of an arbitrary measure and a linear map.

Let $\mu \in \mathcal{M}$ be a probability measure and $\pi \in \Pi_{d,k}$ and suppose $\underline{\dim} \pi\mu < \alpha$. Let \mathcal{V}_x denote the accumulation points of scenery distributions at x . We aim to show that there is a set of positive measure $A \subseteq \mathbb{R}^d$ such that for $x \in A$ there is some $Q \in \mathcal{V}_x$ such that $E_Q(\pi) < \alpha$. Let $C \subseteq \mathbb{R}^k$ denote the set of y with $\underline{D}_{\pi\mu}(y) < \alpha - \varepsilon$ where $\varepsilon > 0$ is chosen so that $\pi\mu(C) > 0$. By Proposition 3.8 it suffices to prove our claim for $\mu|_{\pi^{-1}C}$. So we may assume from the outset that $\underline{D}_{\pi\mu} < \alpha - \varepsilon$ at $\pi\mu$ -a.e. point.

Also, we may assume for convenience that π is the coordinate projection $\pi(x) = (x_1, \dots, x_k)$, that $\mu \in \mathcal{M}^\square$, that a base $b \geq 2$ has been fixed and that a random translation applied to μ so that the conclusions of Section 5.1 hold, i.e. for μ -a.e. x , every accumulation point Q of the b -scenery distribution $\langle \mu, x \rangle_N$ as $N \rightarrow \infty$ is a CP-distribution and its centering is an FD in \mathcal{V}_x .

For brevity denote $M_b = M_b^\square$, and when there is ambiguity write \mathcal{D}_b^d or \mathcal{D}_b^k to indicate whether the partition is of \mathbb{R}^d or \mathbb{R}^k . For each N define $\tau_N : \mathbb{R}^k \rightarrow \mathbb{N}$ by

$$\tau_N(y) = \min\{n \geq N : \pi\mu(\mathcal{D}_{b^n}^k(y)) > 2^{-(\alpha-\varepsilon)n}\}$$

For $x \in \mathbb{R}^d$ we write $\tau_N(x) = \tau_N(\pi x)$. Note that the level sets of τ_N in \mathbb{R}^k belong to $\bigcup_{n \geq N} \mathcal{D}_{b^n}^k$, and if $D \in \mathcal{D}_{b^n}^k(y)$ is one of these sets then $\pi^{-1}D$ decomposes into a disjoint union of $\mathcal{D}_{b^n}^d$ -cells, because π is a coordinate projection.

Lemma 6.5. *For each N ,*

$$\int \frac{1}{\tau_N(y)} \sum_{n=1}^{\tau_N(y)} \frac{1}{\log b} H(T_{\mathcal{D}_{b^{n-1}}^\square(y)}^\square \pi\mu, \mathcal{D}_{b^n}^k) d\pi\mu(y) < \alpha - \varepsilon$$

Proof. Write $\nu = \pi\mu$. Fix N and a ν -typical $y \in \mathbb{R}^k$ and write $\tau = \tau_N(y)$. For $1 \leq n \leq \tau$ let $(\nu_n, y_n) = M_b^n(\nu, y)$, and let $D_n = \mathcal{D}_{b^n}^k(y)$. By definition,

$$\nu(D_\tau) > b^{-(\alpha-\varepsilon)\tau}$$

Taking logarithms we have

$$\begin{aligned} \alpha - \varepsilon &> -\frac{1}{\tau \log b} \log \nu(D_\tau) \\ &= -\frac{1}{\tau \log b} \sum_{n=1}^{\tau} \log \frac{\nu(D_n)}{\nu(D_{n-1})} \\ &= -\frac{1}{\tau \log b} \sum_{n=1}^{\tau} \log \nu_{n-1}(D_b(y_{n-1})) \end{aligned}$$

Integrating over $y \sim \nu$ gives the lemma. \square

Define the distribution $Q_N \in \mathcal{P}(\mathcal{M}^\square(\mathbb{R}^d) \times B_1)$ by

$$Q_N = \int \langle \mu, x \rangle_{\tau_N(x)-1} d\mu(x)$$

Proposition 6.6. $\int \frac{1}{\log b} H(\pi\theta, \mathcal{D}_b) dQ_N(\theta) < \alpha - \varepsilon$ for all large enough N .

Proof. This is essentially the previous lemma together with concavity of entropy. By definition of Q_N , the quantity we wish to bound is

$$(6.1) \quad \int \frac{1}{\tau_N(x)} \sum_{n=1}^{\tau_N(x)} \frac{1}{\log b} H(\pi T_{\mathcal{D}_{b^n}(x)}^\square \mu, \mathcal{D}_b^k) d\mu(x)$$

Fix a $\pi\mu$ -typical y and $1 \leq n \leq \tau_N(y)$. Let $D = \mathcal{D}_{b^n}^k(y)$ and $E = \pi^{-1}(D)$. Since E is the union of the cells $\mathcal{D}_{b^n}(x)$, $x \in E$ we have

$$\int_E \pi T_{\mathcal{D}_{b^n}(x)}^\square \mu d\mu(x) = T_D^\square \pi\mu$$

therefore by concavity of entropy

$$\int_E H(\pi T_{\mathcal{D}_{b^n}(x)}^\square \mu, \mathcal{D}_b) d\mu(x) \leq H(T_D^\square \pi\mu, \mathcal{D}_b)$$

inserting this into 6.1, the desired inequality follows from the previous lemma. \square

Using Markov's inequality, it follows that there exists an $\varepsilon' > 0$ such that for each N there is a set $A_N \subseteq \mathbb{R}^d$ with $\mu(A_N) > \varepsilon'$ and such that for $x \in A_N$ we have

$$\frac{1}{\tau_N(x)} \sum_{n=1}^{\tau_N(x)} \frac{1}{\log b} H(\pi T_{\mathcal{D}_{b^n}(x)}^\square \mu, \mathcal{D}_b^k) < \alpha - \varepsilon'$$

Therefore there is a set $A \subseteq \mathbb{R}^d$ satisfying $\mu(A) \geq \varepsilon'$ and such that for each $x \in A$ the above holds for some sequence $N_i \rightarrow \infty$. By the same argument with b^m in place of b , $m = 1, 2, \dots$ we may assume that as $N_i \rightarrow \infty$ this holds for all b^m . Replacing $H(\cdot, \mathcal{D}_{b^m}^k)$ with a continuous approximation as in Lemma 6.3, we see that for $x \in A$, any accumulation point Q of $\langle \mu, x \rangle_{\tau_{N_i}(x)}$ satisfies

$$\int \frac{1}{m \log b} H(\pi\theta, \mathcal{D}_{b^m}) dQ(\theta) < \alpha - \varepsilon' + \frac{C}{m \log b}$$

Since $\pi\mu$ is exact dimensional Q -a.s. we have, using bounded convergence,

$$\int \frac{1}{m \log b} H(\pi\theta, \mathcal{D}_{b^m}) dQ(\theta) \rightarrow \int \dim \pi\theta dQ(\theta) = \dim Q$$

(Q may not be ergodic, but recall that in that case the dimension is by definition the integral of the dimension of measures). Since this holds for all m and the error term tends to zero as $m \rightarrow \infty$, the inequality above implies $E_Q(\pi) \leq \alpha - \varepsilon'$. The centering of Q is in \mathcal{V}_x , and this is what we set out to prove.

The bound for $\underline{\dim} \varphi\mu$ when φ is a regular map $\varphi \in C^1(\mathbb{R}^d, \mathbb{R}^k)$ can be reduced to the linear one by “straightening out” the map at the expense of “twisting” the measure. First, we note that it suffices to prove the claim locally, i.e. that for every point $x \in \text{supp } \mu$ there is a neighborhood in which the theorem holds for the restricted measure. For a small enough neighborhood U of x we can find a C^1 local diffeomorphism $U \rightarrow \mathbb{R}^d$, so that $\varphi = \pi\psi$ where π is a fixed linear map (e.g. projection onto the first k coordinates). The existence of such a ψ is a simple consequence of the implicit function theorem. Thus we can apply the linear theorem to $\psi\mu$ and what remains is to understand the relation between the accumulation

points of scenery distributions of $\psi\mu$ at ψx and the analogous accumulation points for μ at x . But this is a simple matter using 1.9. We leave the details to the reader.

The result is valid under the weaker assumption of differentiability of φ rather than C^1 . For this one may repeat the argument in the proof of the linear case, using directly the nearly-linear nature of φ at small scales. We omit the details.

6.4. A USM whose projections misbehave. In this section we show that when μ is a USM generating an EFD P , it may still happen that $\underline{\dim} \pi\mu > E_P(\pi)$, i.e. the inequality in Theorem 1.23 cannot in general be replaced by an equality, or that $\pi\mu$ is not exact dimensional.

We first introduce a general method for constructing USMs. Let μ, ν be probability measures on B with $\nu(\partial B) = 0$, and N a large integer. The (ν, N) -discretization of μ is the measure η defined by

$$\eta = \sum_{A \in \mathcal{D}_N} \mu(A) \cdot T_A^{-1}\nu$$

That is, on every cell $A \in \mathcal{D}_N$ we replace $\mu|_A$ with a scaled copy of ν of total mass as $\mu(A)$.

Suppose we are given USMs ν_i generating EFDs P_i . Consider the following construction, which produces a sequence of non-atomic measures μ_1, μ_2, \dots . Begin with $\mu_1 = \nu_1$. Assuming we have constructed μ_{i-1} , choose a large integer N_i and let μ_i be the (ν_i, N_i) -discretization of μ_{i-1} .

Lemma 6.7. *For $\mu_1, \mu_2 \dots$ constructed as above, if N_i is chosen sufficiently large at each stage then $\mu_n \rightarrow \mu$, and for every $\varepsilon > 0$, for μ -a.e. x and large enough i we have for all $N_i \leq T < N_{i+1}$ that*

$$d(\langle \mu \rangle_{x,T}, \text{conv}(P_i, P_{i+1})) < \varepsilon$$

where $d(\cdot, \cdot)$ is a compatible metric on $\mathcal{P}(\mathcal{M}^\square)$. In particular, if $P_i^\square \rightarrow P$ then μ is a USM for P .

Proof. We do not give a full proof but point out the main consideration. Consider the measure μ_2 which is the (ν_2, N_2) -discretization of ν_1 . If one examines a scenery, then up to time close to $\log N_2$ the scenery “looks like” the scenery of ν_1 . Also, on most of the space (away from the boundaries of the cells of \mathcal{D}_{N_2}), from time slightly larger than $\log(1 + \varepsilon)N_2$ the scenery looks like that of ν_2 . There is a small part of the space where this is not so, but this part can be made arbitrarily small by choosing N_2 large (because by assumption $\nu_2(B_1) = 0$). Now for $T > (1 + \varepsilon)N_2$ the scenery at a good point will look like a convex combination of the scenery of ν_1 at that point up to time N_2 , some “mixed” scenery from time N_2 to $(1 + \varepsilon)N_2$, and the scenery of ν_2 at an associated point up to time $T - (1 + \varepsilon)N_2$. If we make N_2 large enough then again with very high probability (on the choice of the initial point) the first and last of these will be close in distribution to P_1 and P_2 . Therefore overall the scenery up to time T will be close to a convex combination of P_1 and P_2 . Also, assuming that $N_n, n \geq 3$ are chosen large enough this will continue to hold when μ_2 is replaced by μ_3 , for $T \leq N_3$. This argument can now be repeated incrementally, with a choice of a sequence of $\varepsilon \rightarrow 0$. The set of “bad” points at each stage can be made summable

so that by Borel-Cantelli, for μ -a.e. point we will be bad at only finitely many scales, and the lemma follows. \square

We now turn to the first construction, a USM whose projection is larger than predicted by Theorem 1.23. Let ν' be Hausdorff measure on the set of $x \in [0, 1]$ whose base-10 expansion contains only the digits 0 and 9. This is a USM, e.g. since the measure ν' is homogeneous (Definition 1.35). Let $\nu = \nu' \times \nu'$, which is also homogeneous and is a USM for an EFD P which is supported on product measures. Note that $\dim \nu = \dim P = 2 \dim \nu' < 1$. Let $\pi(x, y) = x$, and note that $\dim \pi\nu = \dim \nu' < \dim \nu$.

Let $R_\theta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denote rotation by angle θ . Choose a sequence $\theta_n \rightarrow 0$ such that $\dim \pi(R_{\theta_n} \nu) = \alpha(P)$. This is possible since the θ 's for which this holds have full Lebesgue measure by Theorem 1.20. Let $\nu_n = R_{\theta_n} \nu$ and $P_n = R_{\theta_n}^* P$, so ν_n generated P_n and $P_n \rightarrow P$. Finally, construct μ from the sequence ν_n as above using a sequence N_1, N_2, \dots for which the conclusion of the last lemma holds.

By the lemma μ generates P . We claim that if the N_i grow quickly enough, $\dim \pi\mu = \alpha(P) > E_P(\pi)$. In order not to keep the presentation short, let us show only the weaker claim that there is a sequence $b_n \rightarrow \infty$ such that $\frac{1}{\log b_n} H(\pi\mu, \mathcal{D}_{b_n}) \rightarrow \alpha(P)$. This is enough to deduce that the exact dimension of $\pi\mu$, if it exists, is not equal to $\dim \pi\mu$. The point is that for each n the projection $\pi\nu_n$ has dimension $\alpha(P)$, and hence for large enough b'_n we have $\frac{1}{\log b'_n} H(\pi\nu_n, \mathcal{D}_{b'_n}) > \alpha(P) - \frac{1}{n}$. It follows that the same is true (with slightly worse error term) for μ . Letting $N_{n+1} \gg b'_n + N_n$ gives the desired result.

Our second construction is of a USM μ whose projection is not exact dimensional. The idea is the same, except that for even n we define θ_n as before and for odd n we take $\theta_n = 0$. If N_i grow quickly enough we will have that the entropy of μ at certain scales is like that of a measure of dimension $\dim \nu'$ and at others like a measure of dimension $\dim \nu = 2 \dim \nu'$, and hence the measure is not exact dimensional.

6.5. Conditional measures. Let us first discuss in a more detail conditional measures and the remark after Theorem 1.27. For a measure ν and set E write $\nu^E = \frac{1}{\nu(E)}\nu$. Let $\mu \in \mathcal{M}$ and $\pi \in \Pi_{d,k}$ and recall that $[x]_\pi = \pi^{-1}(\pi x)$. For $x \in \text{supp } \mu$ define

$$\mu_{[x]_\pi} = \lim_{r \rightarrow \infty} \left(\lim_{\varepsilon \rightarrow 0} \mu_{B_r \cap \pi^{-1}(B_\varepsilon(\pi x))} \right)^{B_1(x)}$$

where in this formula $B_r \subseteq \mathbb{R}^d$ and $B_\varepsilon(y) \subseteq \mathbb{R}^k$. There is then a set of full μ -measure of x such that for each fixed r the inner limit exists and is equal to the conditional measure of μ_{B_r} on $\pi^{-1}(y)$. Furthermore these measures are consistent, up to multiplication by a scalar, on each B_r , so after normalization the limit as $r \rightarrow \infty$ of these measures exists. This may be taken as the definition of the ‘‘conditional measure’’ of μ on $[x]_\pi$ when μ is infinite.

These conditional fiber measure exists a.e. but may not exist for a particular x . However, if P is an EFD then by the quasi-Palm property, since for P -typical μ the conditional measures are defined a.e., for P -typical μ it is also defined for $x = 0$. Also it is clear that

$$(S_t^* \mu)_{\pi^{-1}(0)} = S_t^*(\mu_{\pi^{-1}(0)})$$

Hence the push-forward $P_{\pi^{-1}(0)}$ of μ via $\mu \mapsto \mu_{\pi^{-1}(0)}$ is an S^* -invariant distribution (which is defined on \mathbb{R}^d but its measures are supported on $\pi^{-1}(0)$, which may be identified with \mathbb{R}^k).

Finally, $P_{\pi^{-1}(0)}$ may be seen to be an EFD. We omit the proof of this. Since EFDs are exact dimensional this establishes Proposition 1.28. As noted in the introduction, Theorem 1.30 on dimension conservation follows from Furstenberg's corresponding result for coordinate projections and CP-processes, and from our correspondence between EFDs and CP-distributions. What remains to prove is Theorem 1.31 on the conditional measures of USMs, which follows by standard arguments from the dual version for projections:

Proof of Theorem 1.31. It is a general fact that if μ is a finite measure with exact dimension, then for every $\pi \in \Pi_{d,k}$ and μ -a.e. x ,

$$\underline{\dim} \pi\mu + \overline{\dim} \mu_{[x]_\pi} = \dim \mu$$

See for example Theorem ?? of Matilla [22], which may be adapted to prove this. Our theorem now follows from Theorem 1.23 in the linear case and in the non-linear case it follows by "straightening the map out", as explained at the end of Section 6.3. \square

6.6. A USM without dimension conservation. In this section we construct a self-similar Cantor set $C \subseteq \mathbb{R}$ and a C^∞ diffeomorphism $f_0 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with $Df|_{C \times C} = \text{id}$, such that the coordinate projection $\pi(x, y) = x$ is injective on the set $E = f_0(C \times C)$ and $\dim \pi E = \dim C$.

Taking μ to be the natural (normalized Hausdorff) measure on $C \times C$ we know that μ is a USM for an EFD P of the same dimension as μ , i.e. $2 \dim C$. Because of the condition on the derivative of f_0 the measure $\nu = f_0\mu$ is also a USM for P (Proposition 1.9). ν is supported on E and $\pi\nu$ is supported on πE so its dimension is $\leq \dim C$. Since the conditional measure $\nu_{[x]_\pi}$ are ν -a.s. supported on singletons, we find that for ν -a.e. x ,

$$\dim \pi\nu + \dim \nu_{[x]_\pi} \leq \dim C + 0 < 2 \dim C = \dim \nu$$

Thus dimension conservation fails for the USM ν . Similarly, taking $f = \pi f_0$ we find that for every $y \in f(C \times C)$,

$$\dim f(C \times C) + \dim((C \times C) \cap f^{-1}(y)) < \dim C \times C$$

Since $C \times C$ is a homogeneous set in the sense of Furstenberg [14] this shows that dimension conservation for homogeneous sets does not hold for C^∞ maps.

We turn to the construction. Let C be the regular Cantor set constructed on $[0, 1]$ by removing the middle $\frac{8}{20}$ of the interval and iterating.¹⁵ C is the set of points $x \in [0, 1]$ whose base-20 representation $x = 0.x_1x_2\dots$ does not contain the eight digits 6, 7, \dots , 12, 13. We also associate to each $x = 0.x_1x_2\dots \in C$ a sequence $(c_n(x))_{n=1}^\infty \in \{0, 1\}^\mathbb{N}$ defined by $c_n(x) = 1_{\{x_n \geq 6\}}$. This gives a homeomorphism of C and the product space $\{0, 1\}^\mathbb{N}$.

We first define f_0 on $C \times C$ and later extend it to $[0, 1]^2$. Fix a rapidly increasing sequence of integers $n_k \rightarrow \infty$ (we will specify its properties later) and define

¹⁵One can carry out the construction for any Cantor set. We have chosen to work in base 20 for convenience.

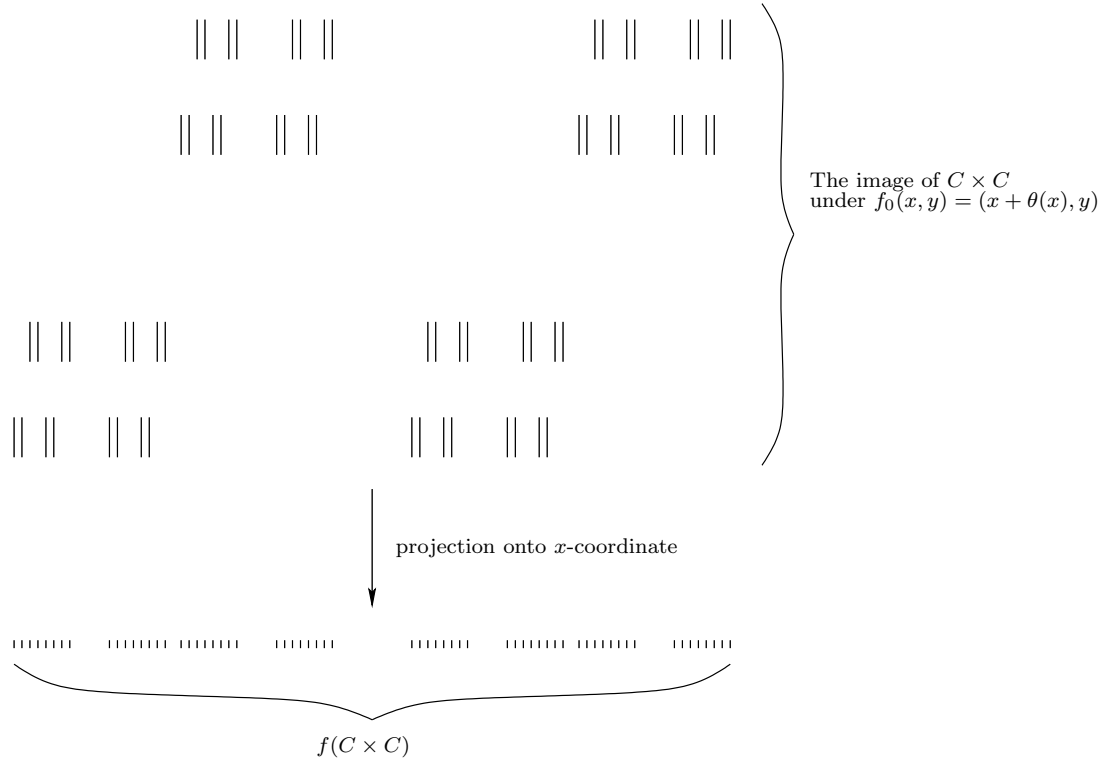


FIGURE 6.1. Schematic representation of f as composition of a plane map and a projection.

$$\theta_k(y) = c_k(y) \cdot \frac{7}{20} \cdot \left(\frac{6}{20}\right)^{-n_k+1}$$

and

$$\theta(y) = \sum_{k=1}^{\infty} \theta_k(y)$$

finally,

$$f_0(x, y) = (x + \theta(y), y)$$

see figure 6.1.

Lemma 6.8. *If n_k grows rapidly enough then $x + \theta(y)$ determines both x and y , i.e. $f = \pi f_0$ is injective.*

Proof. We prove by induction on k that $x + \theta(x)$ determines x_1, \dots, x_{n_k} and $c_1(y), \dots, c_k(y)$.

For $k = 1$, we can cover C (uniquely) by 2^{n_1} disjoint intervals $I_1, \dots, I_{2^{n_1}}$ of length $(\frac{6}{20})^{-n_1}$ (ordered from left to right). The gap between each two intervals is at least $\frac{8}{20}(\frac{6}{20})^{-n_1+1}$, which is longer than any of the intervals I_j (this is why C is slightly easier to work with than the usual middle-third Cantor set). Given one of these intervals I_j , note that $I_j + \theta_1(y)$ either is equal to I_j (if $c_1(y) = 0$) or else is a proper subinterval of the gap between I_j, I_{j+1} , centered at the midpoint between them (if $j = 2^{n_1}$ it is contained in the ray $(1, \infty)$ to the right of I_j). Thus if we know which of the intervals I_j or $I'_j = I_j + \frac{7}{20} \cdot (\frac{6}{10})^{-n_1+1}$ the point

$x + \theta_1(y)$ belongs to we can determine j (which is the same as determining x_1, \dots, x_{n_1}) and determine $c_1(y)$. Since all these intervals are disjoint, if $x + \theta(y)$ is close enough to $x + \theta_1(y)$ then we can determine the interval that the latter belongs to by choosing the interval that the former is closest to. More precisely, the gaps between I_j, I'_j, I_{j+1} are of size $\frac{1}{20}(\frac{6}{20})^{-n_1+1}$, so we can recover the interval from $x + \theta(y)$ if

$$\sup_y \sum_{k=2}^{\infty} \theta_k(y) < \frac{1}{40}(\frac{6}{20})^{-n_1+1}$$

and this holds as long as n_{k+1}/n_k is large enough. Assuming this, $x + \theta(x)$ determines $x_1 \dots x_{n_1}, c_1(y)$.

Now fix k and suppose we know $x_1, \dots, x_{n_{k-1}}$ and $c_1(y), \dots, c_{k-1}(y)$. Let $Q_1, \dots, Q_N \subseteq C \times C$ rectangles whose intersection with $C \times C$ partition it according to these digits. Each Q_i projects to an interval under f and these intervals are disjoint. It suffices to show that we can recover $x_{n_{k-1}+1} \dots x_{n_k}$ and $c_k(y)$ for each $(x, y) \in Q_i$. The argument for this is identical to the base of the induction above. \square

Lemma 6.9. *If $n_k \rightarrow \infty$ rapidly enough then $\dim \pi f_0(C \times C) = \dim C$.*

Proof. First note that $C \subseteq \pi f_0(C^2)$ giving the lower bound.

For the other direction, fix k and let $I_1, \dots, I_{2^{n_k}}$ be the intervals of length $(\frac{6}{20})^{n_k-1}$ covering C . Then $f(C \times C)$ is covered by

$$\{(I_i)' + \sum_{j=1}^k \theta_j(y) : 1 \leq i \leq 2^{n_k}, y \in C\}$$

where I' denotes the interval with the same center as I but of length $1 + \frac{2}{20}$ the length of I . There are altogether $2^k \cdot 2^{n_k}$ intervals in this collection, having length $(1 + \frac{2}{20})(\frac{6}{20})^{-n_k}$, so the box dimension of $f(C^2)$ is bounded above by

$$\liminf_{k \rightarrow \infty} \frac{\log(2^k 2^{n_k})}{\log((1 + \frac{2}{20})(\frac{6}{20})^{-n_k})}$$

if n_k grows rapidly enough this equals $1/\log_2(10/3) = \dim C$. \square

Each $\theta_k(\cdot)$ is currently defined on C but can be extended to a smooth function on $[0, 1]$, and if n_k is large enough then the extension θ_k can be made arbitrarily C^k -close to the constant function and its derivatives on $C \times C$ can be made equal to the identity, since the original θ_k is just a translation. Therefore, if $n_k \rightarrow \infty$ rapidly enough $\theta = \sum \theta_k$ will be smooth, and therefore so will f_0 , and the derivatives will be the identity on $C \times C$.

7. FRACTAL DISTRIBUTIONS WITH ADDITIONAL INVARIANCE

7.1. Homogeneous measures.

Proof of Proposition 1.36. Let μ be a homogeneous measure. Then we may choose a point x which is both homogeneous and so that every accumulation point of the scenery distributions at x are FDs. Let P be such an accumulation point (we think of P as a restricted FD) and choose a P -typical ν . Then ν is an accumulation point of $\mu_{x,t}$ as $t \rightarrow \infty$, and so by

homogeneity $\mu \ll T_B^\square \nu$ for some ball B . Since ν is a USM generating the ergodic component P_ν of P we find that μ is a USM. In particular $P = P_\nu$ and μ -a.s is independent of x .

It remains to show that a P -typical ν is homogeneous. Let \tilde{P} denote the extended version of P . By our previous remark a P -typical ν contain a positive-measure ball $E \subseteq B_1$ on which ν is equivalent, after re-scaling, to μ . It follows from S^* -invariance of \tilde{P} that for \tilde{P} -typical $\tilde{\nu}$ there is a ball $E \supseteq B_1$ on which $\tilde{\nu}$ is equivalent, after re-scaling, to μ . Since $\tilde{\nu}^\square$ is equivalent to $\tilde{\nu}$ on B_1 , we find that $\tilde{\nu}^\square$ is \tilde{P} -a.s. equivalent to μ on some ball A , so the same is true for P -typical ν instead of \tilde{P} -typical $\tilde{\nu}$. Now, if ν is fixed and ν' is an accumulation point of ν at a ν -typical point y , we know by homogeneity of μ that $\mu \ll T_C \nu'$ for a ball C and hence $\nu \ll T_{A\nu} T_C \nu'$. This shows that ν is homogeneous. \square

Theorem 1.37 is now immediate from our results for typical measures of EFDs.

7.2. EFDs invariant under groups of linear transformations. Next we turn to Proposition 1.38. In fact this is an immediate consequence of lower semi-continuity of the function $E_P(\cdot)$, and the last part uses also Marstrand's theorem (Theorem 1.20).

This proposition sheds light on the main results of [19]. Let us recall these two of these:

Theorem 7.1 ([19]). *Let μ, ν be measures on $[0, 1]$ which are invariant, respectively, under $x \mapsto 2x \bmod 1$ and $y \mapsto 3y \bmod 1$. Then for every $x \in \Pi_{2,1}$ except the coordinate projections, we have*

$$\dim \pi(\mu \times \nu) = \min\{1, \dim \mu + \dim \nu\}$$

Theorem 7.2 ([19]). *Let f_1, \dots, f_r be contracting similarities of \mathbb{R}^d . Suppose the orthogonal parts of f_i generate a dense subgroup of the orthogonal group. Let X be the attractor of the IFS $\{f_i\}$ and μ the Hausdorff measure on X . Then for every $\pi \in \Pi_{d,k}$ we have*

$$\dim \pi \mu = \min\{k, \dim \mu\}$$

In both these cases, one can show that the measures of interest are USMs generating an EFD. Furthermore, the invariance of the original measure leads to invariance of the EFD. Indeed, in Theorem 7.1, $\mu \times \nu$ is invariant under $(x, y) \mapsto (2x \bmod 1, y)$, which leads to invariance of the generated EFD under $(x, y) \mapsto (2x, y)$ (this follows from considering the action on sceneries). It is similarly invariant under $(x, y) \mapsto (x, 3y)$. Let G denote the linear group generated by these two maps. It is not hard to show that the orbit orbits of $GL_k(\mathbb{R}) \times G$ on $\Pi_{d,k}$ consist of a dense open set whose complement corresponds to the coordinate projections. Theorem 7.1 now follows from Proposition 1.38. Similarly, for μ as in Theorem 7.2, self-similarity of μ leads to invariance of the generated EFD under the dense subgroup H of the orthogonal group generated by the linear parts of the contractions. Thus the orbit of $GL(\mathbb{R}^k) \times H$ on $\Pi_{d,k}$ is the entire space $\Pi_{d,k}$ leading to Theorem 7.2.

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