PROBABILITY MEASURES ON $\mathbb{R}^d$ ARE AMENABLE

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Fix an integer $d \geq 0$ and a norm $\|\cdot\|$ on $\mathbb{R}^d$, and work throughout this note with the associated metric. In particular, $B_r(x)$ is the ball of radius r around $x \in \mathbb{R}^d$, with respect to this metric.

Theorem 1. *Let μ be a Borel probability measure on $\mathbb{R}^d$. Then*

$$\lim_{r \searrow 0} \frac{\mu(\partial B_r(x))}{\mu(B_r(x))} = 0$$

for μ-a.e. x.

This may look like result about measures in metric spaces, but it is not. To see this construct a metric space X and a measure on it as follows. Let X_t denote the metric space consisting of two points at distance t from each other, and put on X_t the measure giving each point probability $\frac{1}{2}$. Let

$$X = X_1 \times X_{1/2} \times X_{1/4} \times \ldots = \times_{n=01}^{\infty} X_{1/2^n}$$

and take the ultrametric

$$d((x_i), (y_i) = \max_i d(x_i, y_i)$$

which is compatible with the product topology,l and makes X compact. Notice that for $x \in X$ we have

$$B_{1/2^n}(x) = \{y \in X \,:\, x_i = y_i \text{ for } i = 1, \ldots, n\}$$

and

$$\partial B_{1/2^n(x)} = B_{1/2^n}(x) \setminus B_{1/2^{n+1}}(x)$$

Hence if μ denotes the product measure on X we have

$$\frac{\mu(\partial B_{1/2^n}(x))}{\mu(B_{1/2^n}(x))} = \frac{1}{2}$$

for every n. Thus the analogous theorem fails.

Proof. First, if the theorem fails for some measure μ then there is a set $A \subseteq \operatorname{supp}\mu$ of positive measure where the conclusion fails. Taking $\mu = \nu|_A$ the Besicovitch density theorem tells us that

$$\lim_{n \to \infty} \frac{\nu(B_r(x))}{\mu(B_r(x))} = 1$$

for ν-a.e. $x \in A$; therefore the conclusion of the theorem fails for ν at almost every point. We may thus assume that this was true of μ.

We now induct on d. For $d = 0$ there is nothing to prove since the space consists of a single point. Assume we have proved it for $d-1$ and consider the case d. We may choose x and r such that $\mu(\partial B_r(x))/\mu(B_r(x)) > 0$, and in particular if we set $A = \partial B_r(x)$ then $\mu(A) > 0$. Now consider $\nu = \mu|_A$. As before, ν is a counterexample to the theorem, but now ν is concentrated on a manifold of one dimension less, which by the induction hypothesis is impossible.

We have cheated slightly here, since $\partial B_R(x)$ is not a subset of $\mathbb{R}^{d-1}$. However, all that really matters for this induction is that the Besicovitch density theorem continue to hold as one passes to positive measure subsets, and that within the subsets we restrict to, which are smooth manifolds, every small enough ball around a point in the set has boundary which is a manifold of smaller dimension. Both these conditions hold. $\square$

The application I have for this result has nothing to do with probability measures. Rather, it is the key to proving the Chacon-Ornstein lemma for $\mathbb{Z}^d$-actions on infinite measure spaces, leading to a ratio ergodic theorem (Hopf's theorem)[1]. In this application, however, I need a version of this for measures on $\mathbb{Z}^d$, and furthermore I need a finitary (i.e. effective) version of it because rather than work with the infinite family of balls around a point the situation there involves a finite family of balls around each point, satisfying certain growth conditions. This means that the Besicovitch density theorem is unavailable (or at least must be quantified), making the proof much harder.

[1] A ratio ergodic theorem for multiparameter actions, to appear in Journal of the European Mathematical Society.