Open Problems

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Abstract

Open problems from the 3rd Pingree Workshop on Dynamical Systems.

Mike Hochman

1. Let $X = \{0,1\}^{\mathbb{Z}}$ and $Y = \{y \in \{0,1,2\}^{\mathbb{Z}} : y_i \neq y_{i+1}\}$. Both are mixing shifts of finite type of entropy $\log 2$, but are not isomorphic since e.g. X contains fixed points and Y does not. Let $Per(X)$ denote the set of periodic points of X and similarly $Per(Y)$.

   Problem: Are $X \setminus Per(X)$ and $Y \setminus Per(Y)$ topologically conjugate?

   Remarks:

   (a) $X \setminus Per(X)$ is an invariant G_δ in X, and so the shift on it is a homeomorphism of a Polish space. Similarly for Y.

   (b) By a theorem of Adler and Marcus there are dense G_δ sets $X' \subseteq X$, $Y' \subseteq Y$ supporting all measures of entropy $> \log 2 - \varepsilon$ and containing all transitive points of the ambient system, such that the shifts on X' and Y' are topologically conjugate.

   (c) I recently showed that there are sets $X'' \subseteq X$, $Y'' \subseteq Y$ supporting *all* non-atomic invariant probability measures and such that with respect to the shift they are conjugate by a Borel map. More recently Mike Boyle and Jerome Buzzi showed this is true for the sets $X'' = X \setminus Per(X)$ and $Y'' = Y \setminus Pre(Y)$, but their conjugacy is still discontinuous.

2. Let $T : [0,1) \to [0,1)$ be the doubling map $x \mapsto 2x \bmod 1$ and let μ be an ergodic measure for T_a with $0 < h(\mu) < 1$. Let us say that a map $f : \mathbb{R} \to \mathbb{R}$, is non-singular for μ if there is a Borel set $A \subseteq \mathbb{R}$ suc that $\mu|_A \sim (f\mu)|_A$ (i.e. the measures μ and $f\mu$ are not mutually singular).

   Problem: Which affine maps $f(x) = ax + b$ are non-singular for such a measure μ?

   Remarks:

(a) There are many continuous non-singular maps so we should assume at least differentiability.

(b) The Rudolph-Johnson theorem shows that $f(x) = 3x$ cannot be non-singular (requires a little work).

(c) I recently proved that if f is non-singular on A that $f'(x) \in \{a^r : r \in \mathbb{Q}\}$ for μ-a.e. $x \in A$. This nearly characterizes the possible value of a for $f(x) = ax + b$, although I do not know if it is possible that a is not an integer.

(d) Characterizing the b part of $f(x) = ax + b$ is related to work of Bernard Host who has related results for actions of certain groups of translations on $\mathbb{R}/\mathbb{Z}$, specifically for $\mathbb{Z}[1/b]$ for $\gcd(2, b) = 1$ and groups of translations generated by $r \in \mathbb{R}/\mathbb{Z}$ such that the T-orbit of r is dense.

Karl Petersen

Let $A = \{0, 1, \ldots, d-1\}$, let σ be the shift on $A^{\mathbb{Z}}$. The tail fields are defined by

$$\begin{aligned} \mathcal{T}_n^+ &= \bigcap_{k \geq n} \mathcal{B}(x_k, x_{k+1} \ldots) \\ \mathcal{T}_n^- &= \bigcap_{k \geq n} \mathcal{B}(x_{-k}, x_{-k-1} \ldots) \end{aligned}$$

where $\mathcal{B}(\ldots)$ is the σ-algebra generated by the functions in the brackets. Clearly $\mathcal{T}_n^+ \supseteq \mathcal{T}_{n+1}^+$ and $\mathcal{T}_n^- \supseteq \mathcal{T}_{n+1}^-$. Let

$$\begin{aligned} \mathcal{T}^+ &= \bigcap_{n \geq 0} \mathcal{T}_n^+ \\ \mathcal{T}^- &= \bigcap_{n \geq 0} \mathcal{T}_n^- \end{aligned}$$

It is a classical fact (Pinsker) that $\mathcal{T}^+ = \mathcal{T}^-$, though the only known proof involves entropy. In particular, $\mathcal{T}^+ = \{\emptyset, A^{\mathbb{Z}}\}$ if and only if $\mathcal{T}^- = \{\emptyset, A^{\mathbb{Z}}\}$.

Now define $v_n : A^{\mathbb{Z}} \to \mathbb{N}^d$ by

$$(v_n(x))_i = \#\{0 \leq j \leq n : x_j = i\}$$

Notice that v_{n+1}, v_n determine x_{n+1}. Therefore is we set

$$\mathcal{F}_n^+ = \bigcap_{k \geq n} \mathcal{B}(v_n, v_{n+1}, \ldots)$$

then $\mathcal{F}_n^+ \supseteq \mathcal{T}_{n+1}^+$, so if we set

$$\mathcal{F}^+ = \bigcap_{n \geq 0} \mathcal{F}_n^+$$

we have $\mathcal{F}^+ \supseteq \mathcal{T}^+$. Similarly, we can define

$$(w_n(x))_i = \#\{0 \le j \le n \,:\, x_{-j} = i\}$$

and define $\mathcal{F}_n^-, \mathcal{F}^-$ analogously, so $\mathcal{T}^- \supseteq \mathcal{F}^-$.

Problem: Is it true that $\mathcal{F}^+ = \mathcal{F}^-$?

Problem: If not, is it true that $\mathcal{F}^+ = \{\emptyset, A^{\mathbb{Z}}\}$ if and only if $\mathcal{F}^- = \{\emptyset, A^{\mathbb{Z}}\}$?

Francois Ledrappier

1. Let M be a compact Riemannian manifold of constant negative curvature and $(g_t)_{t\in\mathbb{R}}$ the geodesic flow. Does there exist a probability measure μ which is invariant for the time-one map g_1 but not for the flow $(g_t)_{t\in\mathbb{R}}$?

2. (Thouvenot) Let (M,T) be a smooth map on a manifold having a good symbolic cover (i.e. there is a mixing shift of finite type factoring onto (M,T) which is injective on a set of full measure for every invariant probability measure), and let $f \in C^1(M)$ (or higher regularity). Let $(X,(T_t)_{t\in\mathbb{R}})$ denote the flow under the function f. Is it true that any ergodic measure preserving system of entropy $< h(X,T_1)$ can be realized as an invariant measure for on (M,T_1)?

Note that a positive answer to (2) would answer (1) positively, since the geodesic flow on a manifold as in (1) can be represented as a flow as in (2), and there are abstract ergodic systems of entropy zero which do not have a square root.

Mike Boyle

1. **Problem:** Characterize mixing SFTs up to topological orbit equivalence (TOE).

   Known invariants:

   (a) The zeta function (since OE systems have the same number of periodic points of each period).

   (b) The range of the measure of maximal entropy on the clopen sets (an OE must preserve the measure of maximal entropy, since it is obtained as the limit of uniform measure on the points of period n, and these are preserved by (1)).

   (c) The flow equivalence class (two SFTs X, Y are flow equivalent if their suspensions are homeomorphic by a map which preserves orbits and their orientations.

   Known (trivial) examples:

   (a) Conjugate SFTs are TOE.
   (b) Every SFT is TOE to its inverse.

2. **Problem**: Let S, T be subshifts, suppose S is a mixing SFT and T is topologically orbit equivalent (TOE) to S. Is T a mixing SFT?

   What is known:

   (a) There exist counterexamples if T is not assumed to be expansive.
   (b) There are subshifts S, T with S mixing sofic which are TOE but T is not sofic.

3. See "open problems in symbolic dynamics", downloadable from Mike Boyle's webpage.